COMPARISON TEST

RECALL FROM OUR INTEGRAL TEST THAT THE "P-SERIES" GIVEN BY

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

CONVERGES PROVIDED THAT $p > 1$. IT DIVERGES IF $p \leq 1$. WE NOW CONSIDER SERIES THAT CAN BE COMPARED TO A P-SERIES TO SEE IF IT CONVERGES OR DIVERGES.

COMPARISON TESTS FOR SERIES ARE VERY SIMILAR TO THAT FOR IMPROPER INTEGRALS WHERE A COMPARISON TEST WAS USED. THE FIRST RESULT IS:

THEOREM 1 (CLP THEOREM 3.3.8). LET $N_0 > 0$ BE AN INTEGER. THEN

(I) IF $|a_n| < c_n$ FOR ALL $n \geq N_0$, AND $\sum_{n=0}^{\infty} c_n$ CONVERGES, THEN $\sum_{n=0}^{\infty} a_n$ converges

(II) IF $a_n > d_n$ FOR ALL $n \geq N_0$ AND $\sum_{n=0}^{\infty} d_n$ DIVERGES, THEN $\sum_{n=0}^{\infty} a_n$ ALSO DIVERGES.

WE WILL NOT GIVE A FORMAL PROOF HERE.

REMARK IN (I) WE HAVE $|\ |$ AND SO a_n COULD HAVE DIFFERENT SIGNS. WE OBSERVE THAT IF $\sum_{n=0}^{\infty} |a_n|$ CONVERGES THEN SO MUST $\sum_{n=0}^{\infty} a_n$. THIS IS BECAUSE

$$-|a_n| \leq a_n \leq |a_n| \quad \text{FOR ALL } n.$$

AND SO $-\sum_{n=0}^{\infty} |a_n| \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} |a_n|$. THUS $-S \leq \sum_{n=0}^{\infty} a_n \leq S$

WITH $S \equiv \sum_{n=0}^{\infty} |a_n|$, AND SO $\sum_{n=0}^{\infty} a_n$ CONVERGES.

LET'S USE THIS TEST IN A FEW EXAMPLES:

EXAMPLE 1 DOES $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converge or diverge?

SOLUTION **INTUITION** FOR LARGE n WE HAVE $a_n = \frac{1}{n^2+2n+3} \approx \frac{1}{n^2}$ AND SO WE EXPECT CONVERGENCE SINCE $p = 2$ FOR P-SERIES. WE NOW CONFIRM THE INTUITION: WE WANT $a_n < c_n$ WITH $\sum_{n=1}^{\infty} c_n$ converge.

NOTICE THAT $n^2 + 2n + 3 \geq n^2$ FOR ALL $n \geq 1$.

SO FLIPPING THIS: $\dfrac{1}{n^2+2n+3} \leq \dfrac{1}{n^2}$ FOR $n \geq 1$.

THUS $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} c_n$ WITH $c_n = \dfrac{1}{n^2}$ AND $\sum_{n=1}^{\infty} c_n$ converges.

BY PART (I) OF THEOREM I WE HAVE $\sum_{n=1}^{\infty} \dfrac{1}{n^2+2n+3}$ converges.

EXAMPLE 2 DOES $\sum_{n=1}^{\infty} \dfrac{1}{3n^2-5}$ converge or diverge?

SOLUTION LET $a_n = \dfrac{1}{3n^2-5}$ THEN FOR n LARGE $a_n \cong \dfrac{1}{3n^2}$ AND

SINCE $\frac{1}{3}\sum_{n=1}^{\infty} 1/n^2$ CONVERGES, THIS INTUITION PREDICTS $\sum_{n=1}^{\infty} a_n$ CONVERGES.

WE NOW CONFIRM INTUITION WITH A PROOF.

WE WANT $a_n \leq c_n$ FOR ALL $n \geq N_0$ AND $\sum c_n$ converges.

SO WE NEED AN INEQUALITY $3n^2 - 5 \geq$ SOMETHING

WE WRITE $3n^2 - 5 = 2n^2 + \underbrace{(n^2-5)}_{>0 \text{ if } n \geq 3} \geq 2n^2$ IF $n \geq 3$.

THUS $\dfrac{1}{3n^2-5} \leq \dfrac{1}{2n^2}$ IF $n \geq 3$.

BUT $\frac{1}{2}\sum_{n=3}^{\infty} \dfrac{1}{n^2}$ converges SO $\sum_{n=3}^{\infty} \dfrac{1}{3n^2-5}$ converges BY PART I OF theorem

$\rightarrow \sum_{n=1}^{\infty} \dfrac{1}{3n^2-5}$ converges.

EXAMPLE 3 DOES $\sum_{n=1}^{\infty} \dfrac{2n+1}{6n^3-5}$ CONVERGE OR DIVERGE?

SOLUTION $a_n = \dfrac{2n+1}{6n^3-5} \cong \dfrac{1}{3n^2}$ FOR n LARGE SO OUR INTUITION EXPECTS

CONVERGENCE. HOWEVER, IT IS NOW A BIT MORE TEDIOUS TO MAKE A

PROOF. WE NEED $\dfrac{2n+1}{6n^3-5} \leq c_n$ WITH $\sum c_n$ converging.

WE OBSERVE $6n^3 - 5 = 5n^3 + (n^3-5) \geq 5n^3$ IF $n^3 - 5 > 0$, i.e. $n \geq 2$

SO $6n^3 - 5 \geq 5n^3$ IF $n \geq 2$

AND $2n+1 = 3n + \underbrace{(1-n)}_{-\leq 0} \leq 3n$ IF $n \geq 1$.

COMBINING THESE together, $\dfrac{2n+1}{6n^3-5} \leq \dfrac{3n}{5n^3} = \dfrac{3}{5n^2}$ IF $n \geq 2$.

SINCE $\dfrac{3}{5} \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges SO DOES $\sum_{n=1}^{\infty} \dfrac{2n+1}{6n^3-5}$.

<u>EXAMPLE 4</u> NOW MODIFY THIS TO $\sum_{n=1}^{\infty} \dfrac{2n+1}{6n^2-5}$. converge or diverge?

<u>SOLUTION</u> FOR LARGE n, $a_n = \dfrac{2n+1}{6n^2-5} \approx \dfrac{1}{3n}$ BUT $\dfrac{1}{3} \sum_{n=1}^{\infty} 1/n$ <u>diverges</u>.

SO INTUITION PREDICTS DIVERGENCE. WE NOW MAKE A PROOF.

WE WANT $a_n \geq d_n$ WITH $\sum d_n$ diverging.

SO $6n^2 - 5 \leq 6n^2$ FOR ALL $n = 1, 2, \ldots \longrightarrow \dfrac{1}{6n^2-5} \geq \dfrac{1}{6n^2}$.

$2n+1 \geq 2n$ FOR ALL n

COMBINING WE GET $\dfrac{2n+1}{6n^2-5} \geq \dfrac{2n}{6n^2} = \dfrac{1}{3n}$ FOR ALL $n = 1, 2, \ldots$

SINCE $1/3 \sum_{n=1}^{\infty} 1/n$ diverges, THEN SO DOES $\sum_{n=1}^{\infty} \dfrac{2n+1}{6n^2-5}$ BY (II) OF THEOREM 3

IN MORE COMPLICATED EXAMPLES IT GETS <u>REALLY</u> <u>TEDIOUS</u> TO FIND EXPLICIT BOUNDS ON n TO ENSURE THAT EITHER

$a_n \leq c_n$ FOR $n \geq N_0$

OR $a_n \geq d_n$ FOR $n \geq N_0$.

THE NEXT THEOREM AVOIDS THIS DETAIL AND FOCUSES <u>ONLY</u> ON COMPARING THE LARGE n BEHAVIOR OF THE SERIES

THEOREM 2 (LIMIT COMPARISON THEOREM CLP 3.3.11)

LET $\sum_{n=1}^{\infty} a_n$ AND $\sum_{n=1}^{\infty} b_n$ BE TWO SERIES WITH $b_n > 0$ FOR ALL n.

ASSUME THAT $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ EXISTS.

THEN (I) IF $\sum_{n=1}^{\infty} b_n$ converges, we have that $\sum_{n=1}^{\infty} a_n$ converges.

(II) IF $\underline{L \neq 0}$ AND $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

PROOF (SEE CLP BOOK). THE IDEA IS THAT IF a_n HAS THE SAME BEHAVIOR AS $n \to \infty$ AS DOES b_n THEN BOTH SERIES EITHER CONVERGE OR DIVERGE TOGETHER.

REMARK IN (II), $L \neq 0$ IS ESSENTIAL. TO SEE THIS SUPPOSE THAT $a_n = 1/n^2$ AND $b_n = 1/n$. WE KNOW $\sum a_n$ converges AND $\sum b_n$ diverges AND WE calculate $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/n} = 0$. i.e. $\underline{L = 0}$. THE LIMIT COMPARISON PRINCIPLE DOES NOT APPLY. (SINCE IF WE IGNORED THE CONDITION $L \neq 0$ WE GET THE WRONG CONCLUSION!)

LET'S USE THIS LIMIT COMPARISON TEST FOR A FEW NONTRIVIAL EXAMPLES.

EXAMPLE 1 LET $a_n = \frac{2n+1}{6n^3-5}$. DOES $\sum_{n=1}^{\infty} a_n$ converge or diverge?

SOLUTION THIS WAS EXAMPLE 3 EARLIER WHERE MAKING BOUNDS WAS REAL TEDIOUS. NOW IT IS EASY! FOR n LARGE $a_n \cong \frac{1}{3n^2}$. SO CHOOSE

$$b_n = \frac{1}{n^2}.$$

CALCULATE $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)\, n^2}{6n^3-5} = \frac{1}{3} = L.$$

SINCE $L = 1/3$ AND $\sum 1/n^2$ converges, BY (I) OF THEOREM 2 $\Longrightarrow$ $\sum a_n$ converges.

EXAMPLE 2 DOES $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{2n^2+1}}{n^2-2}$ CONVERGE OR DIVERGE?

SOLUTION LET $a_n = \dfrac{\sqrt{2n^2+1}}{n^2-2}$. FOR n LARGE, $a_n \approx \dfrac{\sqrt{2}}{n}$ SO WE EXPECT DIVERGENCE. SO COMPARE WITH $\sum b_n$ AND $b_n = \dfrac{1}{n}$.

WE CALCULATE $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{2n^2+1}}{n^2-2} = \sqrt{2}$.

SINCE $L = \sqrt{2} \neq 0$, WE HAVE BY (II) OF THEOREM 2 (LIMIT COMPARISON TEST) THAT $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{2n^2+1}}{n^2-2}$ DIVERGES.

EXAMPLE 3 DOES $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+2\sin(n)}{\sqrt{9n^8+1}}$ CONVERGE OR DIVERGE?

SOLUTION LET $a_n = \dfrac{n^2+2\sin(n)}{\sqrt{9n^8+1}}$.

NOW SINCE $|\sin(n)| \leq 1$, FOR ALL n, THEN $a_n \approx \dfrac{n^2}{3n^4} = \dfrac{1}{3n^2}$ FOR LARGE n. SINCE $\sum 1/n^2$ CONVERGES WE EXPECT CONVERGENCE.

WE NOW COMPARE WITH $b_n = 1/n^2$ IN LIMIT COMPARISON TEST.

WE CALCULATE $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n^2+2\sin n)}{\sqrt{9n^8+1}} = \lim_{n\to\infty} \frac{n^4\left(1+\frac{2}{n^2}\sin n\right)}{n^4\sqrt{9+1/n^8}}$.

THE LIMIT IS SIMPLY $L = 1/3$. SINCE $\sum 1/n^2$ CONVERGES THEN BY (I) OF LIMIT COMPARISON TEST (THEOREM 2) WE HAVE THAT

$$\sum_{n=2}^{\infty} \frac{n^2+2\sin n}{\sqrt{9n^8+1}}.$$

NOW OUR FINAL EXAMPLES ARE A BIT MORE <u>SUBTLE</u> AND COMBINE THE LIMIT COMPARISON TEST <u>TOGETHER</u> WITH THE INTEGRAL TEST FROM THE EARLIER NOTES.

<u>EXAMPLE 4</u> DOES THE SERIES $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n\sqrt{3\log n+2}}$ converge or diverge?

<u>SOLUTION</u>

<u>METHOD I</u> $$a_n = \frac{1}{n\sqrt{3\log n+2}} \simeq \frac{1}{n\sqrt{3\log n}} = \frac{1}{\sqrt{3}\, n(\log n)^{1/2}} \quad \text{FOR LARGE } n.$$

WE MIGHT BE <u>WRONGLY</u> INCLINED TO COMPARE WITH $b_n = 1/n$. IF WE DID THIS WE WOULD CALCULATE $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n\sqrt{3\log n+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{3\log n+2}} = 0.$

SO $L=0$. <u>OOPS</u>. (II) OF THEOREM 2 (LIMIT COMPARISON TEST) HAS NOTHING TO SAY. LET'S INSTEAD COMPARE WITH $b_n = \dfrac{1}{n(\log n)^{1/2}}$.

WE NOW CALCULATE $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(\log n)^{1/2}}{\sqrt{3}\, n(\log n)^{1/2}\left(1+ 2/_{3\log n}\right)^{1/2}} = \frac{1}{\sqrt{3}} = L.$$

SO $L = 1/\sqrt{3}$. ALL WE NEED TO DO IS DETERMINE WHETHER $\sum b_n$ converges OR diverges AND THEN WE CAN USE EITHER (I) OR (II) OF THE LIMIT COMPARISON TEST TO ESTABLISH THAT FOR $\sum a_n$.

TO STUDY $\displaystyle\sum_{n=6}^{\infty} b_n = \sum_{n=6}^{\infty} \frac{1}{n(\log n)^{1/2}}$ USE INTEGRAL TEST.

LET $f(x) = \dfrac{1}{x\sqrt{\log x}}$. CLEARLY $f(x) > 0$ FOR $x>1$ AND IS DECREASING.

WE CALCULATE $$I = \int_6^{\infty} \frac{1}{x\sqrt{\log x}}\,dx = \lim_{L\to\infty} \int_6^{L} \frac{1}{x\sqrt{\log x}}\,dx = \lim_{L\to\infty} \int_{\log 6}^{\log L} u^{-1/2}\,du.$$

$u = \log x$ SUBSTITUTION

SO $\quad I = \lim_{L\to\infty} 2u^{1/2}\Big|_{\log 6}^{\log L} = \lim_{L\to\infty}\left(2(\log L)^{1/2} - 2(\log 6)^{1/2}\right) = \infty.$

THUS BY INTEGRAL TEST $\Longrightarrow \displaystyle\sum_{n=6}^{\infty} \frac{1}{n(\log n)^{1/2}}$ diverges.

THEN BY (II) OF LIMIT COMPARISON TEST $\Rightarrow \displaystyle\sum_{n=6}^{\infty} \frac{1}{n(3\log n+2)^{1/2}}$ diverges.

<u>METHOD 2</u> WE COULD HAVE USED INTEGRAL TEST <u>DIRECTLY</u> ON $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n\sqrt{3\log n+2}}$. TO DO SO, LET $f(x) = \dfrac{1}{x\sqrt{3\log x+2}}$.

BY USING SUBSTITUTION $u = 3\log x + 2$ IN $I = \displaystyle\int_6^{\infty} \frac{1}{x\sqrt{3\log x+2}}\,dx$

WE GET $du = \dfrac{3}{x}\,dx$ SO $I = \dfrac{1}{3}\displaystyle\int_{3\log 6+2}^{\infty} u^{-1/2}\,du \leftarrow$ diverges.

SO $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n(3\log n+2)^{1/2}}$ diverges BY integral test.

<u>REMARK</u> IF instead we asked whether $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n(3\log n+2)^{3/2}}$ converges OR diverges, we could Repeat same procedure WITH $\displaystyle\sum_{n=6}^{\infty} b_n$ AND $b_n = \dfrac{1}{n(\log n)^{3/2}}$. NOW, WE WOULD FIND THAT $\displaystyle\sum_{n=6}^{\infty} b_n$ converges BY integral test $\Longrightarrow \displaystyle\sum_{n=6}^{\infty} \frac{1}{n(3\log n+2)^{3/2}}$ converges.

ALTERNATING SERIES

THE INTEGRAL TEST ONLY WORKS FOR SERIES $\sum_{n=1}^{\infty} a_n$ WITH $a_n > 0$, AND a_n decreasing

FREQUENTLY WE HAVE TO DEAL WITH ALTERNATING SERIES WHERE a_n ALTERNATES BETWEEN BEING + AND -. FOR INSTANCE $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

THERE IS A SIMPLE CRITERIA TO SEE IF ALTERNATING SERIES CONVERGE.

THEOREM 1 (ALTERNATING SERIES TEST - CLP 3.3.14). CONSIDER

THE ALTERNATING SERIES

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 \ldots \quad \text{WITH } b_n > 0 \text{ FOR } n=1,2,\ldots$$

THEN, IF

- $b_n \geq b_{n+1}$ FOR ALL $n \geq N_0$ (FOR SOME N_0)

AND

- $\lim_{n \to \infty} b_n = 0$,

THE INFINITE SERIES IS CONVERGENT.

REMARK (i) A PROOF OF THIS IS GIVEN IN THE APPENDIX; SEE ALSO THE CLP II SECTION 3.3.10 (OPTIONAL)

(ii) WE ONLY NEED THAT $b_{n+1} - b_n \leq 0$ FOR n Large enough SINCE WE CAN ALWAYS WRITE

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \underbrace{\sum_{n=1}^{N_0} (-1)^{n-1} b_n}_{\text{FINITE SUM}} + \sum_{n=N_0}^{\infty} (-1)^{n-1} b_n$$

(iii) A NICE WAY TO CHECK IF b_n IS A DECREASING SEQUENCE IS TO DEFINE $b_n = f(n)$ AND SEE IF $f'(x) < 0$ WITH x A CONTINUOUS VARIABLE.

<u>EXAMPLE 1</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. converges?

<u>SOLUTION</u> let $b_n = \frac{1}{n}$. CLEARLY $\lim_{n\to\infty} b_n = 0$ AND $b_{n+1} \le b_n$ FOR <u>ALL</u> $n = 1, 2, \ldots$

THUS, BY alternating series test the series converges.

<u>EXAMPLE 2</u> $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$. converges?

<u>SOLUTION</u> LET $b_n = \frac{\log n}{n^2}$. WE HAVE $b_n > 0$ FOR $n = 2, \ldots$ WITH $b_1 = 0$.

SO WE CAN WRITE $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$ SINCE $b_1 = 0$.

NOW $b_n = f(n)$ FOR $n \ge 2$ WHERE $f(x) = \frac{\log x}{x^2} = x^{-2} \log x$.

WE CALCULATE $f'(x) = x^{-3} - 2x^{-3} \log x = x^{-3}(1 - 2\log x) < 0$ IF $x > e^{1/2} = (2.718\ldots)^{1/2}$

SO $f'(x) < 0$ IF $x \ge 2$ WORKS. THUS, $b_{n+1} \le b_n$ FOR $n \ge 2$. $\rightarrow$ b_n DECREASING SEQUENCE

BY Alternating series test, $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\log n}{n^2}$ converges.

<u>EXAMPLE 3</u> $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$ converges?

<u>SOLUTION</u> DEFINE $b_n = \frac{\sqrt{n}}{n+4}$. CLEARLY $b_n \to 0$ AS $n \to \infty$ SINCE

WE HAVE $b_n \cong \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ AS $n \to \infty$.

NOW DEFINE $f(x) = \frac{\sqrt{x}}{x+4}$. WE CALCULATE $f'(x) = \frac{1}{2\sqrt{x}} (x+4)^{-2} (4-x) < 0$

IF $x > 4$ SO THAT b_n IS A DECREASING SEQUENCE FOR $n \ge 4$, i.e. $b_{n+1} \le b_n$

FOR $n \ge 4$. BY alternating series test we have that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+4}$

converges.

EXAMPLE 4 INVESTIGATE THE CONVERGENCE | DIVERGENCE OF

(I) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+5}$ (II) $\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{\pi}{n}\right)$

SOLUTION FOR (I) WE HAVE $b_n \equiv \frac{n^2}{n^2+5}$. SINCE $\lim_{n\to\infty} b_n = 1 \neq 0$ THE ALTERNATING SERIES TEST DOES NOT APPLY AND THIS TEST GIVES NO INFORMATION ABOUT CONVERGENCE OR DIVERGENCE.

HOWEVER, IF WE PROCEED ANOTHER WAY AND DEFINE $a_n \equiv (-1)^{n-1} \frac{n^2}{n^2+5}$ THEN FOR LARGE n WE HAVE $a_n \simeq (-1)^{n-1}$ WHICH DOES NOT TEND TO ZERO AS $n \to \infty$. SINCE $a_n \not\to 0$ AS $n \to \infty$ WE HAVE FROM OUR BASIC DIVERGENCE TEST THAT $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+5}$ is divergent.

REGARDING (II) WE LET $b_n \equiv \cos\left(\frac{\pi}{n}\right)$. SINCE $\lim_{n\to\infty} b_n = 1 \neq 0$ THE ALTERNATING SERIES TEST DOES NOT APPLY AGAIN. NOTICE THAT $a_n = (-1)^{n-1} \cos\left(\frac{\pi}{n}\right) \approx (-1)^{n-1}$ AS $n \to \infty$ SO, $\lim_{n\to\infty} a_n \neq 0$. THUS, BY OUR BASIC DIVERGENCE TEST $\sum_{n=1}^{\infty} (-1)^{n-1} \cos\left(\frac{\pi}{n}\right)$ DOES NOT converge.

EXAMPLE 5 (OPTIONAL) SHOW THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$.

PROOF BY ALTERNATING SERIES TEST WE HAVE FROM EXAMPLE 1 THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES. THE DIFFICULT PART IS TO DETERMINE THE FINITE NUMBER THAT IT CONVERGES TO. (i.e. $\log 2$). TO SHOW THIS REQUIRES A LITTLE CREATIVITY. WE BEGIN WITH THE FINITE GEOMETRIC SUM

$$1 + r + r^2 + \dots + r^{N-1} = \frac{1 - r^N}{1 - r}.$$

WE NOW INTEGRATE IN r BOTH SIDES FROM $-1 \leq r \leq 0$:

$$\int_{-1}^{0} \left(1 + r + r^2 + \dots + r^{N-1}\right) dr = \int_{-1}^{0} \frac{1}{1-r}\, dr - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

THIS GIVES:
$$\left(r + \frac{r^2}{2} + \frac{r^3}{3} + \dots + \frac{r^N}{N}\right)\Bigg|_{-1}^{0} = -\log(1-r)\Big|_{-1}^{0} - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

WE EVALUATE
$$-\left((-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{3} \dots + \frac{(-1)^N}{N}\right) = \log 2 - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

THIS GIVES
$$1 - \frac{1}{2} + \frac{1}{3} \dots \frac{(-1)^{N-1}}{N} = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} = \log 2 - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

THUS, WE HAVE FOR ANY INTEGER $N > 1$ THAT
$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} = \log 2 - \int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

THIS IS <u>EXACT</u>! WE NOW WANT TO TAKE THE LIMIT $N \to \infty$ ON BOTH SIDES AND ESTIMATE $E_N \equiv -\int_{-1}^{0} \frac{r^N}{1-r}\, dr$ (ERROR TERM).

SINCE
$$\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} = \log 2 + E_N \qquad (*)$$

IF WE CAN SHOW THAT $E_N \to 0$ AS $N \to \infty$, THEN $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$ AND WE ARE DONE. SO LET'S FIGHT WITH E_N:

$$E_N \equiv -\int_{-1}^{0} \frac{r^N}{1-r}\, dr.$$

let $u = -r$. THEN $E_N = (-1)^N \int_{1}^{0} \frac{u^N}{1+u}\, du = -(-1)^N \int_{0}^{1} \frac{u^N}{1+u}$

WE TAKE $|\ \ |$ TO GET $|E_N| = \int_{0}^{1} \frac{u^N}{1+u}\, du.$

ALTHOUGH THIS INTEGRAL IS IMPOSSIBLE TO CALCULATE NICELY, WE CAN SIMPLY USE $1 + u \geq 1$ ON $0 \leq u \leq 1$ SO THAT $\frac{u^N}{1+u} \leq u^N$ ON $0 \leq u \leq 1$.

THIS MEANS $|E_N| \leq \int_{0}^{1} u^N\, du = \frac{1}{N+1}$ AND SO $E_N \to 0$ AS $N \to \infty$.

$(*)$ THEN GIVES $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2.$

ESTIMATING REMAINDERS WITH ALTERNATING SERIES

THEOREM 2 (REMAINDER). IF $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ IS A CONVERGENT ALTERNATING SERIES WITH $b_n \geq 0$, $b_{n+1} \leq b_n$, AND $\lim_{n \to \infty} b_n = 0$, THEN THE REMAINDER R_N DEFINED BY

$$R_N \equiv |S - S_N| \quad \text{WITH} \quad \sum_{n=1}^{N} (-1)^{n-1} b_n$$

SATISFIES THE BOUND

$$R_N = |S - S_N| \leq b_{N+1}.$$

i.e. THE REMAINDER IS BOUNDED BY THE FIRST NEGLECTED TERM AFTER TAKING NTH PARTIAL SUM. $\square$

REMARK THE PROOF OF THIS RESULT IS GIVEN IN THE APPENDIX (OPTIONAL).

EXAMPLE WE RECALL THAT $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. HOW MANY TERMS IN THE INFINITE SUM ARE NEEDED TO GET AN ERROR OF 2.5×10^{-8} FOR e^{-1}.

SOLUTION LET $x = -1$. $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ $b_n = \frac{1}{n!}$ is decreasing sequence, positive, and $\lim_{n \to \infty} b_n = 0$.

THE REMAINDER ESTIMATE GIVES

$$\left| e^{-1} - S_N \right| \leq \frac{1}{(N+1)!}$$

THUS IF $N = 10$, WE CALCULATE $\frac{1}{11!} \simeq 2.5 \times 10^{-8}$.

SO $e^{-1} \simeq 1 - \frac{1}{1!} + \frac{1}{2!} \cdots + \frac{1}{10!} = .3678794642 \pm 2.5 \times 10^{-8}$.

A REALLY GOOD ESTIMATE WITH 10 TERMS.

EXAMPLE HOW MANY TERMS IN $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ IS NEEDED TO GET AN APPROXIMATION TO THE INFINITE SUM WITHIN AN ERROR OF 10^{-5}?

SOLUTION LET $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^6}$ AND BY ALTERNATING SERIES TEST $S_N \to S$ AS $N \to \infty$ (INFINITE SERIES converges). THEN, BY REMAINDER THEOREM,

$$|S - S_N| \leq \frac{1}{(N+1)^6}.$$

SO WANT $\frac{1}{(N+1)^6} \leq 1 \times 10^{-5}$ OR $(N+1) \geq (1 \times 10^5)^{1/6} \simeq 6.812$

THUS $N \geq 5.812$.

SO TAKE $N = 6$ TO get this eRROR.

EXAMPLE IF WE HAVE THE INFINITE SERIES $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (WHICH WE KNOW IS $\log 2$ BY EXAMPLE 5) WE HAVE

$$|\log 2 - S_N| \leq \frac{1}{N+1} \quad \text{WITH} \quad S_N \equiv \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n}.$$

SO TO get $|\log 2 - S_N| \leq 1 \times 10^{-5}$ WE NEED

$N+1 \geq 1 \times 10^5$ TERMS.

AN ENORMOUS NUMBER OF TERMS SINCE the series converges <u>so very slowly</u>.

APPENDIX (OPTIONAL)

PROOF OF THEOREM I (ALTERNATING SERIES TEST)

DEFINE $S_N \equiv \sum_{n=1}^{N} (-1)^{n-1} b_n$. ASSUME $b_n \geq 0$ AND $b_n - b_{n+1} \geq 0$ FOR $n = 1, 2, \ldots$ (I.E. b_n IS A POSITIVE, DECREASING SEQUENCE) WITH $b_n \to 0$ AS $n \to \infty$.

WE CALCULATE S_N WHEN N IS EVEN AS FOLLOWS:

$$S_2 = b_1 - b_2 \geq 0$$

$$S_4 = (b_1 - b_2) + (b_3 - b_4) = S_2 + \underbrace{(b_3 - b_4)}_{\geq 0} \geq S_2$$

$$S_6 = S_4 + \underbrace{(b_5 - b_6)}_{\geq 0} \geq S_4.$$

CONTINUING ON, WE GET $S_{2n} = S_{2n-2} + \underbrace{b_{2n-1} - b_{2n}}_{\geq 0} \geq S_{2n-2}$ FOR $n = 2, 3, 4, \ldots$

SINCE $S_{2n} \geq S_{2n-2}$ FOR $n = 2, 3, 4, \ldots$ WE HAVE THAT $\{S_{2n}\}$ IS AN <u>INCREASING SEQUENCE</u> OF NUMBERS. NOW WE SHOW THIS SEQUENCE HAS AN UPPER BOUND.

WE WRITE $S_{2n} = b_1 - b_2 + b_3 - b_4 \ldots - b_{2n-2} + b_{2n-1} - b_{2n}$.

GROUP TERMS AS SHOWN: $S_{2n} = b_1 - \underbrace{(b_2 - b_3)}_{\geq 0} - \underbrace{(b_4 - b_5)}_{\geq 0} - \ldots - \underbrace{(b_{2n-2} - b_{2n-1})}_{\geq 0} - \underbrace{b_{2n}}_{\geq 0}$.

THIS SHOWS THAT $S_{2n} \leq b_1$ FOR ALL $n = 2, 3, 4, \ldots$

WE CONCLUDE THAT $\{S_{2n}\}$ IS AN INCREASING SEQUENCE THAT IS BOUNDED ABOVE BY b_1. AS SUCH $\lim_{n \to \infty} S_{2n}$ EXISTS AND WE DENOTE IT BY $S = \lim_{n \to \infty} S_{2n}$.

NOW OBSERVE THAT $S_{2n+1} = S_{2n} + b_{2n+1}$.
SINCE $b_n \to 0$ AS $n \to \infty$ WE HAVE $\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} \Rightarrow S = \lim_{n \to \infty} S_{2n+1}$.

SO, FINALLY, $\{S_{2n}\}$ AND $\{S_{2n+1}\}$ ARE CONVERGENT SEQUENCES WITH THE SAME LIMITING VALUE $\longrightarrow$ $\{S_n\}$ IS CONVERGENT SEQUENCE WITH SUM S.

PROOF OF THEOREM 2 (REMAINDER ESTIMATE)

WE WRITE $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ AND THEN SUBTRACT TO GET

$$S - S_N = \sum_{n=N+1}^{\infty} (-1)^{n-1} b_n = \begin{cases} -b_{N+1} + \underbrace{(b_{N+2} - b_{N+3})}_{\geq 0} + \underbrace{(b_{N+4} - b_{N+5})}_{\geq 0} \ldots & \text{IF } N \text{ ODD} \\ b_{N+1} - \underbrace{(b_{N+2} - b_{N+3})}_{\geq 0} - \underbrace{(b_{N+4} - b_{N+5})}_{\geq 0} \ldots & \text{IF } N \text{ EVEN} \end{cases}$$

THIS SHOWS THAT

$$\left.\begin{array}{ll} S - S_N > -b_{N+1} & \text{IF } N \text{ ODD} \\ S - S_N < b_{N+1} & \text{IF } N \text{ EVEN.} \end{array}\right\} (*)$$

NOW GROUP THE TERMS IN A DIFFERENT WAY:

$$S - S_N = -\underbrace{(b_{N+1} - b_{N+2})}_{\geq 0} - \underbrace{(b_{N+3} - b_{N+4})}_{\geq 0} \ldots \quad \text{IF } N \text{ IS ODD}$$

THIS GIVES $\quad S - S_N < 0$ IF N ODD. $\}$ $(+)$

SIMILARLY, IF N IS EVEN,

$$S - S_N = \underbrace{(b_{N+1} - b_{N+2})}_{\geq 0} + \underbrace{(b_{N+3} - b_{N+4})}_{\geq 0} \ldots$$

THIS GIVES $\quad S - S_N \geq 0$ IF N EVEN. $(++)$

FINALLY COMBINE $(*)$, $(+)$, AND $(++)$ TO CONCLUDE THAT

$$0 < S - S_N < b_{N+1} \quad \text{IF } N \text{ EVEN}$$

$$-b_{N+1} < S - S_N < 0 \quad \text{IF } N \text{ ODD.}$$

WE CAN COMBINE THESE together TO CONCLUDE THAT

$$|S - S_N| \leq b_{N+1} \quad \text{FOR ALL } N \text{ (EVEN OR ODD).}$$

THEREFORE THE ERROR IN APPROXIMATING BY N TERMS IS SIMPLY THE FIRST NEGLECTED TERM.