NUMERICS SUMMARY

MIDPOINT RULE:

$$M_n = \Delta x \left[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right]$$

WHERE $x_i^* = \frac{1}{2}\left[x_{i-1} + x_i \right]$ FOR $i = 1, \ldots, n$

AND $x_i = a + i\,\Delta x$.

ERROR ESTIMATE:

$$E_M(n) = \left| \int_a^b f(x)\,dx - M_n \right| \leq \frac{K(b-a)^3}{24 n^2}, \qquad K = \max_{a \leq x \leq b} |f''|.$$

TRAPEZOIDAL RULE:

$$T_n = \Delta x \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right]$$

$$E_T(n) = \left| \int_a^b f(x)\,dx - T_n \right| \leq \frac{K(b-a)^3}{12 n^2}.$$

SIMPSON'S RULE:

NEED n EVEN

$$S_n = \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

$$E_S(n) = \left| \int_a^b f(x)\,dx - S_n \right| \leq \frac{L(b-a)^5}{180 n^4} \quad \text{WITH} \quad L = \max_{a \leq x \leq b} |f''''|.$$

REMARK TO OBTAIN AN ERROR OF 1×10^{-6} CHOOSE n SO $\frac{L(b-a)^5}{180 n^4} \leq 1 \times 10^{-6}$.

FOR SIX DIGITS OF ACCURACY SET ERROR AT 5×10^{-7}.

T_n $|E_T(n)|$

exact	trap-rule	error	h = Δx	n
0.22488	0.51235	0.28747E+00	0.31416E+01	2
0.22488	0.30218	0.77296E-01	0.15708E+01	4
0.22488	0.23396	0.90805E-02	0.78540E+00	8
0.22488	0.22499	0.10522E-03	0.39270E+00	16
0.22488	0.22486	0.15619E-04	0.19635E+00	32
0.22488	0.22488	0.39209E-05	0.98175E-01	64
0.22488	0.22488	0.98035E-06	0.49087E-01	128
0.22488	0.22488	0.24510E-06	0.24544E-01	256
0.22488	0.22488	0.61275E-07	0.12272E-01	512
0.22488	0.22488	0.15319E-07	0.61359E-02	1024
0.22488	0.22488	0.38297E-08	0.30680E-02	2048
0.22488	0.22488	0.95742E-09	0.15340E-02	4096

$h = \frac{b-a}{n} = \Delta x$

Table 1: The error is absolute value of exact minus trapezoidal sum. Observe that for small h, if we decrease h by a factor of 2 the error decreases by a factor of 4 in accordance with the error estimate of $\mathcal{O}(h^2)$. $= O(1/n^2)$

Example 1: Trapezoidal integration of a Simple Non-Periodic function

$$I = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+x^2}\,dx \qquad \text{exact value is } I = \frac{1}{2\pi}\arctan(2\pi) \approx 0.22488\,.$$

T_n $|E_T(n)|$

exact	trap-rule	error	h = Δx	n
0.27344	1.00000	0.72656E+00	0.31416E+01	2
0.27344	1.00000	0.72656E+00	0.15708E+01	4
0.27344	0.50000	0.22656E+00	0.78540E+00	8
0.27344	0.28125	0.78125E-02	0.39270E+00	16
0.27344	0.27344	0.55511E-16	0.19635E+00	32
0.27344	0.27344	0.16653E-15	0.98175E-01	64
0.27344	0.27344	0.33307E-15	0.49087E-01	128
0.27344	0.27344	0.16653E-15	0.24544E-01	256
0.27344	0.27344	0.77716E-15	0.12272E-01	512
0.27344	0.27344	0.15543E-14	0.61359E-02	1024
0.27344	0.27344	0.34417E-14	0.30680E-02	2048
0.27344	0.27344	0.17208E-14	0.15340E-02	4096

Table 2: Observe that the trapezoid rule in this case converges **much faster** than the expected error estimate of $\mathcal{O}(h^2)$. For integrating functions that are periodic, the trapezoidal rule is said to be super-convergent.

Example 2: Integrating a Periodic function

$$I = \frac{1}{2\pi}\int_0^{2\pi} \cos^8(x)\,dx \qquad \underline{\text{exact}} \text{ value is } I = \frac{(1)(3)(5)(7)}{(2)(4)(6)(8)} \approx 0.2734.$$ (SEE PAGE 4)

<u>DEEP</u>: IF YOU NEED TO INTEGRATE A PERIODIC FUNCTION $f(x)$ WITH $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$, .. ETC, i.e. $I = \int_0^{2\pi} f(x)\,dx$ USE TRAPEZOIDAL RULE AS IT WILL BE <u>VERY ACCURATE</u> (MUCH BETTER THAN C/n^2)

INTEGRATION VIA RECURRENCE RELATIONS

WE WILL DERIVE (SLICKLY) A FORMULA FOR $I_m \equiv \int_0^{2\pi} (\cos(x))^m \, dx$

WHERE m IS AN EVEN INTEGER.

WE WRITE $I_m = \int_0^{2\pi} (\cos x)^{m-1} \cos x \, dx$ AND USE IBP.

let $u = [\cos x]^{m-1}$ so $du = (m-1)(\cos x)^{m-2}(-\sin x)\,dx$

$dv = \cos x$ so $v = \sin x$

so $I_m = (\cos x)^{m-1} \sin x \Big|_0^{2\pi} + (m-1)\int_0^{2\pi} (\cos x)^{m-2} \sin^2 x \, dx$

BUT $\sin(2\pi) = \sin(0) = 0$. USE $\sin^2 x = 1 - \cos^2 x$.

SO $I_m = (m-1)\int_0^{2\pi} (\cos x)^{m-2}[1 - \cos^2 x]\,dx = (m-1)\int_0^{2\pi} (\cos x)^{m-2} dx - (m-1)\int_0^{2\pi} (\cos x)^m dx$

THIS GIVES $I_m = (m-1) I_{m-2} - (m-1) I_m$.

SOLVE FOR I_m: $I_m + [m-1] I_m = (m-1) I_{m-2}$.

WE GET $m I_m = (m-1) I_{m-2}$ OR EQUIVALENTLY

$$(*) \quad I_m = \frac{(m-1)}{m} I_{m-2} \quad \text{FOR } m \geq 2.$$

- SUPPOSE <u>m IS EVEN</u>. USE $I_0 = \int_0^{2\pi} 1 \, dx = 2\pi$ TO GET

$$I_2 = \frac{1}{2} I_0, \quad I_4 = \frac{3}{4} I_2 = \frac{3 \cdot 1}{4 \cdot 2} I_0, \quad I_6 = \frac{5}{6} I_4 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} I_0.$$

BY INDUCTION

$$(+) \quad I_m = \int_0^{2\pi} (\cos x)^m dx = \left(\frac{1 \cdot 3 \cdot 5 \cdots (m-1)}{2 \cdot 4 \cdot 6 \cdots m}\right) 2\pi \quad \text{IF } m \text{ EVEN}.$$

- SUPPOSE m IS ODD. THEN $I_1 = \int_0^{2\pi} \cos x \, dx = 0$ AND $(*)$ YIELDS $I_3 = I_5 = I_7 = \cdots = 0$.

THEREFORE

$$\frac{1}{2\pi}\int_0^{2\pi} (\cos x)^8 dx = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cong 0.2734.$$