IMPROPER INTEGRALS

THERE ARE FOUR TYPES OF IMPROPER INTEGRALS:

DEFINITION

(i) IF $f(x)$ IS CONTINUOUS ON $[a,\infty)$, THEN
$$\int_a^\infty f(x)\,dx = \lim_{L\to\infty}\int_a^L f(x)\,dx.$$

(ii) IF $f(x)$ IS CONTINUOUS ON $(a,b]$ THEN
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

(iii) IF $f(x)$ IS CONTINUOUS ON $[a,b)$ THEN
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

(iv) IF $f(x)$ IS CONTINUOUS ON $[a,c) \cup (c,b]$ THEN
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_b^c f(x)\,dx.$$

— TREATED THEN AS IN (ii) AND (iii) BY LIMITS —

KEY RESULT 1 CONSIDER $I = \int_1^\infty \frac{1}{x^p}\,dx$. THEN I IS FINITE IFF $p>1$.

PROOF DEFINE $I_L \equiv \int_1^L x^{-p}\,dx$. WE CALCULATE $I_L = \begin{cases} \dfrac{x^{1-p}}{1-p}\Big|_1^L, & p\neq 1 \\ \ln x\Big|_1^L, & p=1 \end{cases}$.

THIS GIVES
$$I_L = \begin{cases} \dfrac{L^{1-p}}{1-p} - \dfrac{1}{1-p}, & \text{IF } p\neq 1 \\ \ln L, & \text{IF } p=1 \end{cases}$$

NOW LET $L\to\infty$. WE HAVE $I_L \to$ FINITE VALUE IFF $p>1$ FOR THEN $L^{1-p}\to 0$. THUS, IF $p>1$, $I = \lim_{L\to\infty} I_L = \frac{1}{p-1}$.

QUALITATIVELY, A LIMIT EXISTS IFF $f(x) = x^{-p}$ DECAYS FAST ENOUGH (i.e. $p>1$) AS $x\to+\infty$.

KEY RESULT 2 CONSIDER AN EXAMPLE OF (ii) WHERE $I = \int_0^1 \frac{1}{x^p}\,dx$. THEN I IS FINITE IFF $p<1$, i.e. IF x^{-p} BLOWS UP "SLOW ENOUGH" AS $x\to 0^+$. WE DEFINE $I_c \equiv \int_c^1 \frac{1}{x^p}\,dx$, AND ARE INTERESTED IN $\lim_{c\to 0^+} I_c$.

WE CALCULATE $I_C = \begin{cases} \frac{1}{1-p} x^{1-p}\Big|_c^1 = \frac{1}{1-p}\left(1 - c^{1-p}\right) & \text{IF } p \neq 1 \\ \ln x \Big|_c^1 = -\ln c & \text{IF } p = 1. \end{cases}$ ②

IN ORDER FOR $\lim_{c \to 0^+} I_c$ TO EXIST WHEN $p \neq 1$ WE NEED $\lim_{c \to 0} c^{1-p} = 0.$

THIS ONLY OCCURS IF $p < 1$. THUS INTEGRAL CONVERGES IFF $p < 1$ AND IN THIS CASE

$$I = \lim_{c \to 0} I_c = \frac{1}{1-p} \quad \text{FOR} \quad p < 1.$$

WITH THESE TWO BASIC KEY RESULTS WE CAN USE THEM IN CONJUNCTION WITH A STANDARD COMPARISON TEST TO PROVE CONVERGENCE OR DIVERGENCE OF INTEGRALS.

THEOREM (COMPARISON TEST)

SUPPOSE $f(x)$ AND $g(x)$ ARE CONTINUOUS ON $[a, \infty)$ WITH $0 \leq f(x) \leq g(x)$ FOR $x \geq a$. THEN

(I) IF $\int_a^\infty g(x)\,dx < \infty \implies \int_a^\infty f(x)\,dx < \infty.$

(II) IF $\int_a^\infty f(x)\,dx$ diverges THEN SO DOES $\int_a^\infty g(x)\,dx$.

PROOF OF (I) SINCE $0 \leq f(x) \leq g(x)$ FOR $x \geq a$ WE HAVE

$$\int_a^L f(x)\,dx \leq \int_a^L g(x)\,dx \quad \text{FOR ANY } L > a.$$

NOW IF $\int_a^\infty g(x)\,dx$ IS FINITE WE HAVE SINCE $g(x) \geq 0$ THAT

$$\int_a^L f(x)\,dx \leq \int_a^L g(x)\,dx \leq \int_a^\infty g(x)\,dx < \infty.$$

LETTING $L \to \infty$ GIVES THE RESULT. (II) IS PROVED THE SAME WAY.

NEXT, WE DO SOME EXAMPLES WITH THE COMPARISON TEST AND OUR TWO KEY RESULTS 1 AND 2.

EXAMPLE 1 LET $f(x) = \frac{\sin^2 x}{x^2}$ FOR $x \geq 1$.

DOES $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ CONVERGE? IF SO, GIVE A PROOF.

SOLUTION WE MUST FIND A COMPARISON FUNCTION. WE OBSERVE THAT SINCE $|\sin^2 x| \leq 1$ $\forall x$ THAT $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ FOR $x \geq 1$.

LET $g(x) = \frac{1}{x^2}$. THEN $\int_1^\infty \frac{1}{x^2}\,dx < \infty$ SINCE $p = 2 > 1$ (KEY RESULT 1).

THUS, BY (I) OF COMPARISON THEOREM $\int_1^\infty f(x)\,dx \leq \int_1^\infty g(x) < \infty$ AND $\int_1^\infty f(x)dx < \infty$.

EXAMPLE 2 CONSIDER $\int_1^\infty \frac{dx}{\sqrt{x^2-.5}}$. DOES THIS INTEGRAL CONVERGE OR DIVERGE?

WE FIRST GET SOME INTUITION: FOR x LARGE $\frac{1}{\sqrt{x^2-.5}} \cong \frac{1}{x}$ AND $\int_1^\infty \frac{dx}{x}$ DIVERGES SO WE EXPECT DIVERGENCE. WE TRY TO IMPLEMENT (II) OF COMPARISON TEST.

NOTICE THAT $\sqrt{x^2-.5} < \sqrt{x^2}$, $\forall$ $x \geq 1$

THUS,

$$\frac{1}{\sqrt{x^2-.5}} > \frac{1}{\sqrt{x^2}} = \frac{1}{x}.$$

DEFINE $g(x) = \frac{1}{x}$ AND $f(x) = \frac{1}{\sqrt{x^2-.5}}$. WE HAVE $\int_1^\infty f(x)dx > \int_1^\infty g(x)\,dx$

AND $\int_1^\infty g(x)\,dx$ DIVERGES, BY (II) OF COMPARISON TEST $\int_1^\infty f(x)\,dx$ DIVERGES.

EXAMPLE 3 CONSIDER $\int_2^\infty \frac{x\ dx}{x^3 + x^2 + 1}$. DOES THE INTEGRAL CONVERGE OR DIVERGE? JUSTIFY YOUR ANSWER. INTUITION: $x^3 + x^2 \cong x^3$ FOR x LARGE, SO THAT

$\frac{x}{x^3 + x^2 + 1} \cong \frac{x}{x^3} = \frac{1}{x^2}$ AND $\int_2^\infty \frac{1}{x^2}\,dx$ CONVERGES. SO WE EXPECT CONVERGENCE.

WE NOW MAKE A PROOF, USING (I) OF COMPARISON TEST.

WE OBSERVE $x^3 + x^2 + 1 \geq x^3$ ON $x \geq 2$. THUS $\frac{1}{x^3 + x^2 + 1} \leq \frac{1}{x^3}$ ON $x \geq 2$,

AND SO $\int_2^\infty \frac{x}{x^3 + x^2 + 1}\,dx \leq \int_2^\infty \frac{x}{x^3}\,dx = \int_2^\infty \frac{1}{x^2}\,dx$. LET $f(x) = \frac{x}{x^3 + x^2 + 1}$, $g(x) = \frac{1}{x^2}$

SINCE $\displaystyle\int_2^\infty g(x)\,dx < \infty \qquad (p = 2 > 1)$

THEN $\displaystyle\int_2^\infty f(x)\,dx < \infty.$

NOW CONSIDER IMPROPER INTEGRALS WITH AN INTERIOR SINGULARITY.

EXAMPLE 1 $I = \displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$. NOW $f(x) = \dfrac{1}{x^2}$ IS NOT CONTINUOUS AT $x = 0$ AND SO

WE CAN'T SIMPLY FIND ANTI-DERIVATIVE: i.e. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx \neq -\frac{1}{x}\Big|_{-1}^1 = -2.$

INSTEAD WE WRITE

$$I = \underbrace{\lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}\,dx}_{\text{DIVERGES SINCE } p=2>1} + \underbrace{\lim_{a\to 0^-}\int_{-1}^a \frac{1}{x^2}\,dx}_{\text{diverges since } p=2>1.}$$

SO $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx$ IS DIVERGENT.

EXAMPLE 2 LET $I = \displaystyle\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$. SINCE $f(x) = (x-1)^{-2/3}$ IS NOT CONTINUOUS AT $x = 1$

WE MUST WRITE

$$I = \lim_{a\to 1^-}\int_0^a \frac{1}{(x-1)^{2/3}}\,dx + \lim_{a\to 1^+}\int_a^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{a\to 1^-} 3(x-1)^{1/3}\Big|_0^a + \lim_{a\to 1^+} 3(x-1)^{1/3}\Big|_a^3$$

$$= 3 + 3\cdot 2^{1/3} = 3(2^{1/3}+1)$$ FINITE. $\rightarrow$ INTEGRAL CONVERGES.

EXAMPLE 3 $I = \displaystyle\int_1^2 \frac{x}{\sqrt{x^2-1}}\,dx$. IS IT CONVERGENT OR DIVERGENT.

SOLUTION WE WRITE $I = \displaystyle\int_1^2 \frac{x}{\sqrt{(x-1)(x+1)}}\,dx$. LET $f(x) = \dfrac{x}{\sqrt{(x-1)(x+1)}}$

NEAR $x = 1$, $f(x) \approx \dfrac{1}{2\sqrt{x-1}}$ AND $\displaystyle\int_1^b \frac{dx}{2\sqrt{x-1}}$ EXISTS SINCE $p = \tfrac{1}{2} < 1$.

INTEGRAL CONVERGES!

IMPROPER INTEGRALS

1) WE KNOW THAT $\lim_{L\to\infty}\int_2^L \frac{1}{x}\,dx$ IS INFINITE. WHAT HAPPENS IF WE CHOOSE AN $f(x)$ THAT DECAYS SLIGHTLY MORE RAPIDLY AS $x\to\infty$. LET $p>0$ AND CONSIDER $f(x)=\frac{1}{x[\ln x]^p}$.

THEN USING SUBSTITUTION RULE $u=\ln x$

$$\int_2^L \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln L} u^{-p}\,du = \begin{cases} u^{1-p}/_{1-p}\Big|_{\ln 2}^{\ln L} & \text{if } p\neq 1 \\ \ln u\Big|_{\ln 2}^{\ln L} & \text{if } p=1 \end{cases}$$

($u>0$ so $|\;|$ NOT needed.)

THUS $I_L \equiv \int_2^L \frac{1}{x(\ln x)^p}\,dx = \begin{cases} \frac{(\ln x)^{1-p}}{1-p}\Big|_2^L & \text{if } p\neq 1 \\ \ln(\ln x)\Big|_2^L & \text{if } p=1 \end{cases}$

WE CONCLUDE THAT $\lim_{L\to\infty} I_L$ IS FINITE ONLY IF $p>1$.

SIMILARLY WE KNOW THAT IF $f(x)=\frac{1}{x}$ THAT $I_\varepsilon \equiv \int_\varepsilon^{1/2} \frac{1}{x}\,dx$ DIVERGES AS $\varepsilon\to 0^+$ SINCE $\frac{1}{x}$ BLOWS UP TOO FAST AS $x\to 0^+$.

WHAT IF WE MODIFY $f(x)$ SLIGHTLY TO $f(x)=\frac{1}{x[\ln x]^p}$, FOR $p>0$.

WE STILL HAVE $|f(x)|\to\infty$ AS $x\to 0^+$ (SINCE $\lim_{x\to 0^+} x[\ln x]^p = 0$)

BUT NOW IF WE DEFINE

$$I_\varepsilon \equiv \int_\varepsilon^{1/2} \frac{1}{x[\ln x]^p}\,dx$$

WE CALCULATE AS ABOVE THAT

$$I_\varepsilon = \begin{cases} \frac{(\ln x)^{1-p}}{1-p}\Big|_\varepsilon^{1/2} & \text{if } p\neq 1 \\ \ln(\ln x)\Big|_\varepsilon^{1/2} & \text{if } p=1 \end{cases}$$

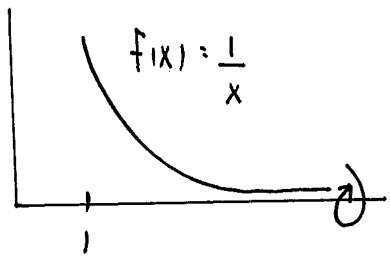
SINCE $(\ln\varepsilon)^{1-p}\to 0$ AS $\varepsilon\to 0^+$ IFF $p>1$ WE HAVE THAT

$\lim_{\varepsilon\to 0^+} I_\varepsilon$ IS FINITE IFF $p>1$.

2) <u>VOLUMES OF REVOLUTION</u>

WE LET $f(x) = \frac{1}{x}$ AND WE KNOW THAT $\int_1^\infty \frac{1}{x}\,dx$ IS INFINITE.

NOW SUPPOSE WE CALCULATED THE VOLUME OF REVOLUTION



WE GET $V = \pi \int_1^\infty (f(x))^2\,dx = \pi \int_1^\infty x^{-2}\,dx$

SO $V = \pi \left(-x^{-1}\right)\Big|_1^\infty = \pi$, WHICH IS FINITE.

(MORE PROPERLY $V_L = \pi \int_1^L (f(x))^2\,dx \to \pi$ AS $L \to \infty$.

THUS WE EXPECT THAT IF $f(x) \to 0$ AS $x \to \infty$ MORE slowly than $1/x$ WE CAN GET A FINITE INTEGRAL FOR THE VOLUME.

SUPPOSE $f(x) = 1/x^p$ FOR $p > 0$.

THEN $V(x) = \pi \lim_{L\to\infty} \int_1^L [f(x)]^2\,dx = \pi \lim_{L\to\infty} \int_1^L x^{-2p}\,dx = \pi \begin{cases} \dfrac{x^{1-2p}}{1-2p}\Big|_1^L, & \text{IF } p \neq 1/2 \\ \ln x \Big|_1^L, & \text{IF } p = \frac{1}{2} \end{cases}$

WE CONCLUDE THAT IFF $p > 1/2$ WE HAVE A FINITE INTEGRAL.

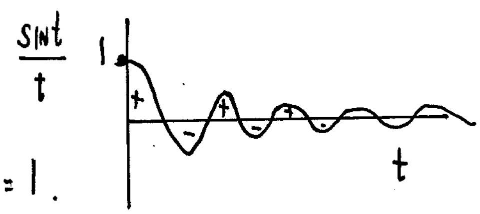
HENCE WE GET A <u>FINITE</u> VOLUME IF $f(x) \approx \frac{1}{x^p}$ WITH $p > 1/2$ AS $x \to \infty$

(slower decay is allowed than with calculating areas).

3) ONE MIGHT ASK whether WE CAN get A finite integral through "AREA CANCELLATION".

A FAMOUS SPECIAL FUNCTION IS

$$\mathcal{F}(x) = \int_0^x \frac{\sin t}{t}\,dt$$



NOTICE $t = 0$ IS FINE SINCE $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

HOWEVER, DO WE GET A FINITE LIMIT AS $x \to \infty$ (IF NO $\sin t$ TERM THEN $\int_1^x \frac{1}{t}\,dt = \ln x \to +\infty$.).

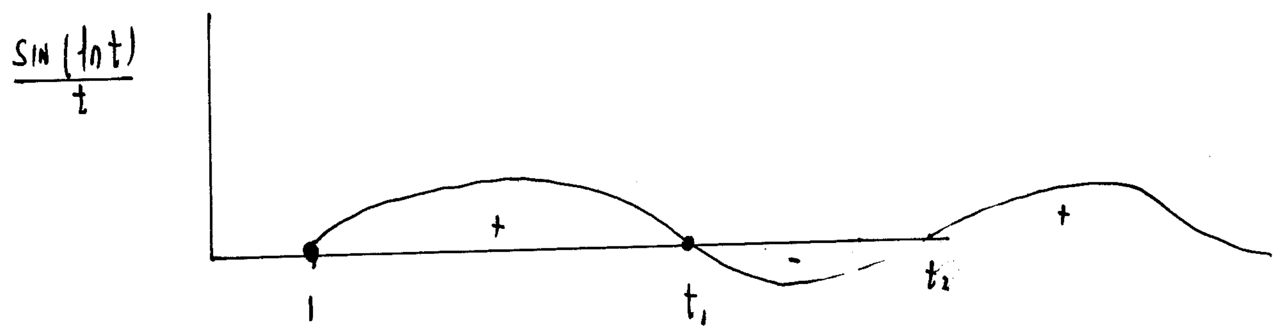
THIS IS A DIFFICULT QUESTION (M300) TO EVALUATE $\lim_{x\to\infty} \mathcal{F}(x)$ BUT WOLFRAM ALPHA GIVES $\lim_{x\to\infty} \mathcal{F}(x) = \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

SO AN AREA CANCELLATION PROCEDURE MUST BE AT PLAY HERE.

TO SHOW THE SUBTLETY IN THIS CONSIDER MODIFYING $f(x)$ TO

$$\hat{f}(x) = \int_1^x \frac{\sin(\ln t)}{t}\, dt$$

THEN $\sin(\ln t) = 0$ WHEN $\ln t = n\pi \rightarrow t_n = e^{n\pi}$ AND SO WE STILL GET SOME AREA CANCELLATION, BUT $\sin(\ln t)$ VARIES REALLY SLOWLY IN t.



WE NOW CALCULATE USING $u = \ln t \quad du = 1/t\, dt$

$$\int_1^x \frac{\sin(\ln t)}{t}\, dt = \int_0^{\ln x} \sin(u)\, du = -\cos u \Big|_0^{\ln x} = 1 - \cos(\ln x).$$

NOTICE THEN THAT $\hat{f}(x) \equiv 1 - \cos(\ln x)$.

AS $x \to \infty$ $\hat{f}(x)$ OSCILLATES between 0 AND 2

AND DOES NOT APPROACH A LIMITING VALUE.

HENCE IMPROPER INTEGRALS CAN BE FINITE, INFINITE, OR OSCILLATORY DEPENDING ON SPECIFICS OF THE PROBLEM.

4) <u>REMARK</u> CONSIDER $I = \int_a^b \frac{g(x)}{f(x)}\, dx$.

IF $f(c) = 0$ FOR $a < c < b$ WITH $f'(c) \neq 0$ AND $g(c) \neq 0$ THE INTEGRAL DIVERGES SINCE BY TANGENT LINE APPROXIMATION

$$\frac{g(x)}{f(x)} \simeq \frac{g(c)}{f'(c)(x-c)} \quad \text{NEAR } x = c.$$

5) FINALLY CONSIDER

$$I = \int_1^L \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx .$$

WE KNOW THAT SEPARATELY $\int_1^\infty \frac{1}{x}$ AND $\int_1^\infty \frac{1}{\sqrt{x^2+1}}$ ARE INFINITE,

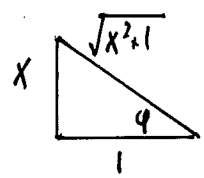
BUT CAN WE GET A FINITE INTEGRAL THROUGH CANCELLATION.

(WHEN WE DO TAYLOR SERIES WE NOTE THAT FOR $x \to +\infty$

$$\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} = \frac{1}{x(1+1/x^2)^{1/2}} - \frac{1}{x} = \frac{1}{x}\left(1+\frac{1}{x^2}\right)^{-1/2} - \frac{1}{x} \hat{=} \frac{1}{x}\left(1 - \frac{1}{2x^2} + \cdots\right) - \frac{1}{x}$$

$$\approx 1/_{2x^3} \quad \text{AS} \quad x \to \infty \quad \text{AND} \quad \int_A^\infty \frac{1}{2x^3}\, dx \text{ IS FINITE.}$$

SO WE EXPECT THAT $\lim\limits_{L \to \infty} I_L$ IS FINITE.



$\sec\varphi = \sqrt{x^2+1}$

$\tan\varphi = x$

WE CALCULATE: $x = \tan\varphi \qquad dx = \sec^2\varphi \, d\varphi$ SO

$$\int \frac{1}{\sqrt{x^2+1}}\, dx = \int \frac{\sec^2\varphi}{\sec\varphi}\, d\varphi = \int \sec\varphi = \log\left[\sec\varphi + \tan\varphi\right] + C$$

SO $$\int \frac{1}{\sqrt{x^2+1}}\, dx = \log\left[\sqrt{x^2+1} + x\right] + C .$$

THUS $$I_L \equiv \int_1^L \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx = \left(\log\left(\sqrt{x^2+1} + x\right) - \log x \right)\Big|_1^L$$

$$= \log\left(\frac{x + \sqrt{x^2+1}}{x} \right)\Big|_1^L = \log\left(1 + \sqrt{1 + \frac{1}{x^2}} \right)\Big|_1^L$$

SO $$I_L = \log\left(1 + \sqrt{1 + \frac{1}{L^2}} \right) - \log\left(1 + \sqrt{2}\right)$$

NOW let $L \to \infty$ SO $$I_L \to \log\left(\frac{2}{1+\sqrt{2}} \right) = \int_1^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx .$$

SINCE $\sqrt{x^2+1} > x$ ON $1 < x \to \frac{1}{\sqrt{x^2+1}} - \frac{1}{x} < 0$ ON $x > 1$ SO $I_L < 0$.

INDEED $\frac{2}{1+\sqrt{2}} < 1$ SO $\log\left(\frac{2}{1+\sqrt{2}} \right) < 0$.

INTERESTINGLY WOLFRAM FAILS TO CALCULATE $\int_1^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} \right) dx$.

6) CONSIDER AN IMPROPER INTEGRAL BUT ONE FOR WHICH WE GET A FINITE VALUE; i.e.

$$I = \int_0^1 \frac{f(x)}{\sqrt{x}}\, dx \quad \text{WHERE } f(x) \text{ IS CONTINUOUS ON } 0 \le x \le 1 \text{ WITH } f(0) \neq 0.$$

SINCE THE INTEGRAND BLOWS UP AS $x \to 0^+ \to$ STANDARD RIEMANN SUMS ARE NOT SO GOOD.

LET $u = x^{1/2}$ $\quad du = \frac{1}{2} x^{-1/2}\, dx = \frac{1}{2u}\, dx$

$$\frac{dx}{\sqrt{x}} = \frac{2u}{u}\, du.$$

$x = 0 \to u = 0$

$x = 1 \to u = 1$

SO $$I = \int_0^1 2 f(u^2)\, du$$

$$I = 2\int_0^1 f(u^2)\, du$$

$\Uparrow$ equivalent integral BUT better FOR numerical quadrature.

WORK

(W1)

ENERGY EXPENDED ACTING AGAINST A FORCE, i.e. ENERGY EXPENDED MOVING A WEIGHT AGAINST GRAVITY.

- t : SECONDS
- MASS $m \to$ Kg
- position $s \to$ metres
- NEWTON'S SECOND LAW: FORCE = MASS x Acceleration $\quad F = m \dfrac{d^2 s}{dt^2}$

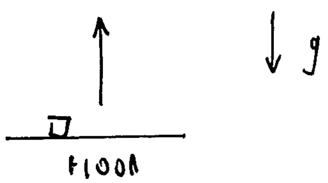
  NEWTONS $\to$ Kg - m/SEC^2

- WORK AT CONSTANT FORCE IS W = FORCE x DISPLACEMENT = $F \times d$

  FORCE MEASURED IN NEWTONS. WORK IN Joules (Newton-metre).

QUESTION (CONSTANT FORCE)

HOW MUCH WORK DONE MOVING A 1Kg BOOK FROM FLOOR TO TOP OF A 2m HEIGH shelf?



DISPLACEMENT = 2 m

FORCE = $9.8 \times 1 = 9.8$ NEWTONS

WORK = 19.6 Joules

FOR A VARIABLE FORCE $\quad W = \displaystyle\int_a^b F(x)\, dx.$

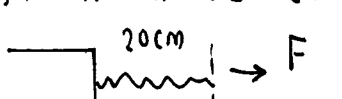
HOOKE'S LAW

$F = KX \qquad$ K: SPRING CONSTANT

X: AMOUNT OF STRETCHING.

A SPRING HAS A NATURAL LENGTH OF 20cm. IF A 25N FORCE IS REQUIRED TO KEEP IT STRETCHED AT A LENGTH OF 30cm HOW MUCH WORK IS REQUIRED TO STRETCH IT FROM 20cm TO 25cm.

(i) FIRST WORK OUT SPRING CONSTANT

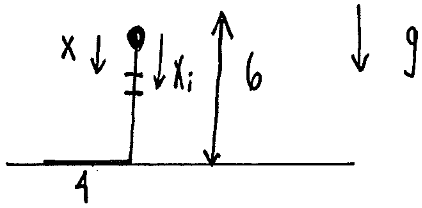


$F = K(.30 - .20) = 25N \to K = 250 N/m$

(iii) NOW X RANGES FROM $X=0$ TO $X=.05$. WE HAVE

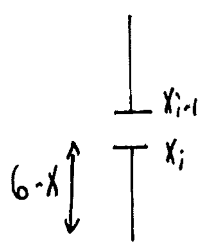
$$W = \int_0^{0.05} F(x)\,dx = \int_0^{0.05} 250x\,dx = 125x^2\Big|_0^{0.05} = 125 \times .0025 = .3125\text{J}.$$

EXAMPLE A CHAIN LYING ON THE GROUND IS 10 m LONG AND WEIGHS 80 kg. HOW MUCH WORK IS REQUIRED TO RAISE ONE END OF THE CHAIN TO A HEIGHT OF 6 m? THE CONSTANT DENSITY OF THE CHAIN IS 8 kg/m.



$0 < X < 6.$

SPLIT CHAIN INTO SEGMENTS $[X_{i-1}, X_i]$ AND FIGURE OUT HOW MUCH WORK IS DONE IN LIFTING EACH SEGMENT.



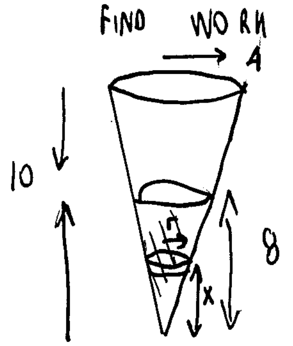
CONSIDER SEGMENT $[X_{i-1}, X_i]$

- LIFTED $6-X$ m AGAINST FORCE OF GRAVITY
- MASS = DENSITY $\cdot\, \Delta X = 8\,\Delta X$
- SEGMENT WEIGHS MASS $\cdot\, g = 8(9.8)\Delta X = \Delta F$
- $\Delta W = (6-X)\,8(9.8)\,\Delta X$

SO $$W = \int_0^6 78.4(6-x)\,dx = 78.4\left(6x - x^2/2\right)\Big|_0^6 = 18 \times 78.4 = 1411.2\text{J}.$$

EXAMPLE (WORK DONE IN PUMPING WATER OUT OF A TANK)

TANK IS SHAPED LIKE INVERTED CONE HEIGHT = 10 m RADIUS = 4 m FILLED TO HEIGHT OF 8 m. ASSUME DENSITY OF WATER IS 1000 kg/m^3. FIND WORK INVOLVED IN PUMPING ALL WATER OUT OF TANK



HOW MUCH WORK FOR EACH SLICE OF WATER.

MASS = VOLUME × density.

Δm = MASS IN SLICE $= \left(\pi (r(x))^2\,\Delta x\right)1000.$

NOW WEIGHT $\Delta F = 9.8\,\Delta m$ AND $r = \frac{2}{5} x$ BY SIMILAR TRIANGLES

WE HAVE TO MOVE IT $10 - x$ METRES

$$\Delta W = (10-x)\,9.8\,\Delta m = (10-x)(9.8)\,\pi \left(\frac{4}{25}x^2\right)\Delta x\ (1000)$$

SO WORK

$$W = (1000)\left(\frac{4}{25}\pi\right)(9.8)\int_0^8 (10-x)\,x^2\,dx$$

$$W = 1568\,\pi \int_0^8 (10-x)\,x^2\,dx$$

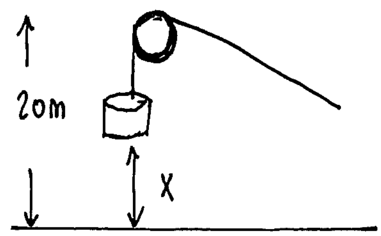
$$W = 1568\,\pi\left[\frac{10}{3}(8)^3 - \frac{1}{4}(8)^4\right]$$

$$W = (1568\,\pi)(8^3)\left(\frac{10}{3} - 2\right)$$

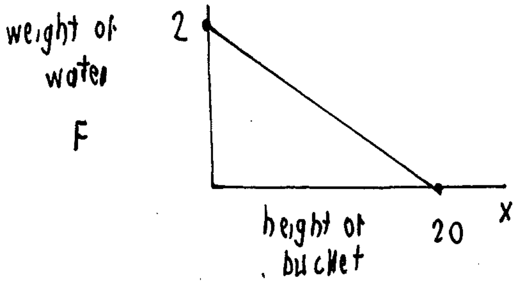
$$= (1568)\,\pi\,(512)\left(\frac{4}{3}\right) \mathrel{\hat{=}} 3.36 \times 10^6 \text{ Joules.}$$

EXAMPLE A LEAKY BUCKET WEIGHING 5 N IS LIFTED 20 m INTO THE AIR AT CONSTANT SPEED. THE BUCKET STARTS WITH 2N OF WATER AND LEAKS AT A CONSTANT RATE. IT FINISHES DRAINING JUST AS IT REACHES THE TOP. HOW MUCH WORK WAS DONE LIFTING THE WATER ALONE?

SOLUTION



SINCE WATER DRAINS OUT AT A CONSTANT RATE WE HAVE



THUS $F = -\frac{1}{10}x + 2$ FOR WEIGHT OF WATER.

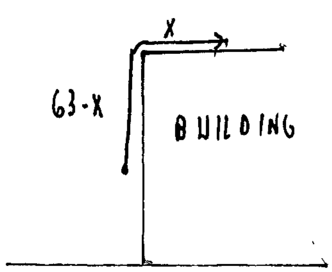
THE TOTAL WORK NEEDED IS

$$W = \int_0^{20} F(x)\,dx = \int_0^{20} \left(2 - x/10\right) dx$$

SO $W = 40 - \frac{1}{20}x^2 \Big|_0^{20} = 20$ JOULES.

EXAMPLE A CHAIN 63 metres LONG WHOSE MASS IS 27 kilograms IS HANGING OVER THE EDGE OF A TALL BUILDING AND DOES NOT TOUCH THE GROUND. HOW MUCH WORK IS REQUIRED TO LIFT THE TOP 11 METRES OF THE CHAIN TO THE TOP OF THE BUILDING. (HINT: DON'T FORGET THAT WHEN YOU LIFT THE TOP 11 METRES OF THE CABLE YOU ARE ALSO LIFTING THE BOTTOM 52 metres OF THE CABLE, JUST NOT ALL THE WAY TO THE TOP).

SOLUTION



AFTER x METRES OF CHAIN PULLED UP, 63 - X METRES REMAIN THAT ARE SUBJECT TO GRAVITY.

$$\text{MASS HANGING BELOW} = (63 - x) \cdot \underbrace{\frac{27}{63}}_{\text{DENSITY}}$$

WEIGHT HANGING BELOW IS

$$9.8\,(63 - x)\,27/63.$$

THUS $\text{WORK} = 9.8 \int_0^{11} \left(27 - \frac{3x}{7}\right) dx$

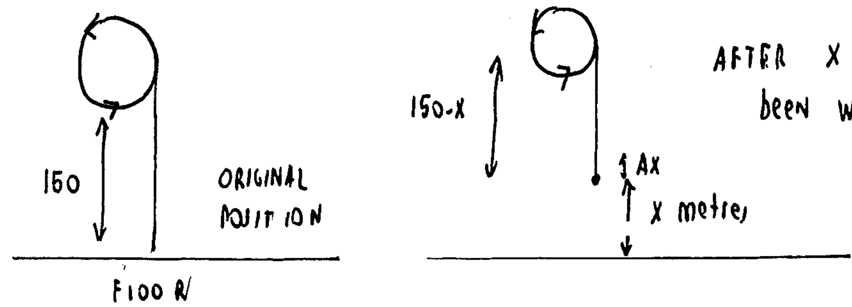
$$= 9.8\left[(27)(11) - \frac{3}{14}(11)^2\right]$$

$$\text{WORK} = \frac{3795}{14} \doteq 2656.5 \text{ Joules}.$$

EXAMPLE

WE HAVE A FULLY EXTENDED CABLE OF 150 metres WEIGHING 2 KG/metre. HOW MUCH WORK IS DONE AFTER WINDING 50 m OF CABLE?

SOLUTION



AFTER X metres have been WOUND UP.

AS CABLE IS WOUND UP IT BECOMES SHORTER AND SHORTER WEIGHING LESS AND LESS. THE FORCE IS $\Delta F = \underbrace{(150-x)}\ \underbrace{\frac{2\text{ KG}}{\text{METRE}}}_{\text{density}} \times \underbrace{9.8}_{\text{gravity}} \times \Delta x$

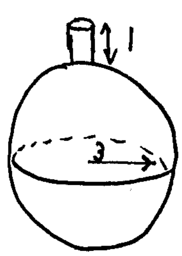
THUS $\quad W = 9.8 \int_0^{50} 2(150-x)\,dx = 9.8\left[300(50) - 50^2\right]$

$W = (9.8) \times 1.25 \times 10^4$ JOULES

EXAMPLE

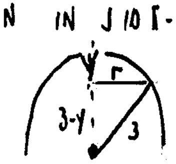
A TANK OF DIMENSIONS SHOWN IS INITIALLY FULL OF WATER THAT HAS DENSITY OF $1000\text{ kg}/m^3$. FIND THE WORK REQUIRED TO PUMP THE WATER OUT OF THE SPOUT.

SOLUTION



THE TANK IS A SPHERE OF RADIUS 3 AND THE SPOUT IS 1 m HIGH. GRAVITY IS ACTING DOWNWARDS.

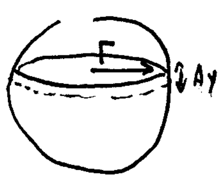
TAKE A HORIZONTAL SLICE AT DEPTH y FROM SPOUT AS SHOWN IN SIDE-VIEW



SO $r^2 = 9-(3-y)^2 = 6y - y^2$ BY PYTHAGORAS

$r = \sqrt{6y-y^2}$

THE MASS IN VOLUME OF WIDTH Δy IS THEN



$$\Delta V = \pi r^2 \Delta y = \pi(6y-y^2)\Delta y$$

MASS $= \Delta V \times$ density $= 1000\pi(6y-y^2)\Delta y$ IN SLICE

GRAVITY FORCE $= 9.8(1000\pi)(6y-y^2)\Delta y.$

(W6)

NOW SINCE WATER MUST BE PUMPED OUT OF TANK'S SPOUT WHICH IS A DISTANCE OF $y+1$ (metres), THE WORK DONE IS

$$W = (9.8)(1000\pi)\int_0^6 (y+1)(6y-y^2)\,dy$$

$$= 9800\pi \int_0^6 (6y-y^2+6y^2-y^3)\,dy$$

$$= 9800\pi\left(-\frac{6^4}{4}+\frac{5\cdot 6^3}{3}+3\cdot 6^2\right)$$

$$W = 9800\pi\left(\frac{-1296}{4}+\frac{1080}{3}+108\right)$$

OR $W \cong 4.4\times 10^6$ Joules.