First Name: ___________________________ Last Name: ___________________________

Student-No: ___________________________ Section: ___________________________

Grade:

The remainder of this page has been left blank for your workings.
Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral \( \int \frac{\sin(x)}{\sqrt{\cos(x)}} \, dx \) for \( 0 < x < \pi/2 \).

| Answer: \( I = -2\sqrt{\cos(x)} + C \) |

| Solution: Let \( u = \cos(x) \), so that \( \sin(x) \, dx = -du \). Then, \( I = - \int u^{-1/2} \, du = -2\sqrt{u} + C \). Using \( u = \cos(x) \) we get \( I = -2\sqrt{\cos(x)} + C \). |

(b) Calculate the indefinite integral \( \int \frac{x+1}{x^2+3x} \, dx \) for \( x > 0 \).

| Answer: \( \frac{\ln(|x|)}{3} + \frac{2\ln(|x+3|)}{3} + C \) |

| Solution: The denominator \( x^2 + 3x \) factorizes as \( x(x+3) \). Thus, the partial fraction decomposition is \( \frac{x+1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} \). Multiplying everything by \( x(x+3) \) we get \( x + 1 = A(x+3) + Bx \), and by plugging in the values \( x = 0 \) and \( x = -3 \) we obtain \( A = \frac{1}{3} \) and \( B = \frac{2}{3} \). Then \( \int \frac{x+1}{x^2+3x} \, dx = \int \left( \frac{1}{3x} + \frac{2}{3(x+3)} \right) \, dx = \frac{\ln(|x|)}{3} + \frac{2\ln(|x+3|)}{3} + C \). |
(c) (A Little Harder): Calculate the indefinite integral \( \int x^2 e^{-x} \, dx \).

Answer: \( e^{-x}(-x^2 - 2x - 2) + C \)

Solution: Let \( u = x^2 \) and \( dv/dx = e^{-x} \). We calculate \( du/dx = 2x \) and \( v = -e^{-x} \), so that one step of integration by parts gives

\[
I = u v - \int v \frac{du}{dx} \, dx = -x^2 e^{-x} + \int 2xe^{-x} \, dx.
\]

Now we apply integration by parts again to \( J = \int 2xe^{-x} \, dx \) choosing \( u = 2x \) and \( dv/dx = e^{-x} \), obtaining

\[
J = u v - \int v \frac{du}{dx} \, dx = -2xe^{-x} + \int 2e^{-x} \, dx = e^{-x}(-2x - 2) + C.
\]

Plugging this into our first equation for \( I \) we get

\[
I = -x^2 e^{-x} + J = e^{-x}(-x^2 - 2x - 2) + C.
\]
Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.
   
   (a) Calculate \( \int_{0}^{\pi/2} \cos^3(x) \, dx \).

   \[ \text{Answer: } \frac{2}{3} \]

   \[ \text{Solution:} \text{ Since the power of cosine is odd, we hold on to one copy of it and turn the rest into sines. We get} \]
   \[ \int_{0}^{\pi/2} \cos^3(x) \, dx = \int_{0}^{\pi/2} \cos(x)(1 - \sin^2(x)) \, dx. \]

   Now we can use the substitution \( u = \sin(x) \). We have \( du/dx = \cos(x) \), and that \( x = 0 \) and \( x = \pi/2 \) map to \( u = 0 \) and \( u = 1 \). This yields,
   \[ \int_{0}^{\pi/2} \cos(x)(1 - \sin^2(x)) \, dx = \int_{0}^{1} (1 - u^2) \, du = \left[ u - \frac{u^3}{3} \right]_{0}^{1} = \frac{2}{3}. \]

   (b) Calculate \( \int_{3}^{0} \frac{9x^2}{x^2 + 9} \, dx \).

   \[ \text{Answer: } 27 - \frac{27\pi}{4} \]

   \[ \text{Solution:} \text{ We first note that} \]
   \[ \frac{x^2}{x^2 + 9} = 1 - \frac{9}{x^2 + 9}. \text{ Then} \]
   \[ \int_{3}^{0} \frac{9x^2}{x^2 + 9} \, dx = 9 \int_{0}^{3} \left( 1 - \frac{9}{x^2 + 9} \right) \, dx = 9 \cdot 3 - 81 \int_{0}^{3} \frac{1}{x^2 + 9} \, dx. \]

   To solve the second integral we use the substitution \( x = 3u \), so that
   \[ \int_{0}^{3} \frac{1}{x^2 + 9} = \int_{0}^{1} \frac{3}{9(u^2 + 1)} \, du = \left[ \frac{\arctan(u)}{3} \right]_{0}^{1} = \frac{\pi}{12} - 0. \]

   Plugging this back into the first equation we get
   \[ \int_{0}^{3} \frac{9x^2}{x^2 + 9} \, dx = 27 - \frac{27\pi}{4}. \]
(c) (A Little Harder): Calculate \( \int_1^{e^2} \frac{\ln x}{x^2} \, dx \).

| Answer: \( 1 - \frac{3}{e^2} \) |

**Solution:** We use integration by parts, picking \( dv/dx = \frac{1}{x^2} \) and \( u = \ln x \). We compute \( v = -\frac{1}{x} \) and \( du/dx = \frac{1}{x} \). Thus applying the IBP formula we get

\[
I = uv - \int v \, du = \left[ \frac{-\ln x}{x} \right]_1^{e^2} + \int \frac{1}{x^2} \, dx = -\frac{2}{e^2} + \left[ -\frac{1}{x} \right]_1^{e^2} = -\frac{2}{e^2} + \left[ -\frac{1}{e^2} \right] = 1 - \frac{3}{e^2}.
\]
Riemann Sum, FTC, and Volumes

3. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.
   
   (a) Calculate the infinite sum
   \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2}{n^3} \sqrt{1 + \frac{i^3}{n^3}} \]
   by first writing it as a definite integral. Then, evaluate this integral.
   
   Answer: \( \frac{2}{3} \left( 2\sqrt{2} - 1 \right) \).

   Solution: We try to pick \( \Delta x = \frac{1}{n}, a = 0, b = 1, x_i = \frac{i}{n} \), so we have to write the summand in the form \( \Delta x f \left( \frac{i}{n} \right) \). By collecting a \( \frac{1}{n} \) in the expression we get
   \[ 3 \left( \frac{i}{n} \right)^2 \sqrt{1 + \left( \frac{i}{n} \right)^3} \]
   so we have \( f(x) = 3x^2 \sqrt{1 + x^3} \), and thus
   \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2}{n^3} \sqrt{1 + \frac{i^3}{n^3}} = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x f(x_i) = \int_{0}^{1} 3x^2 \sqrt{1 + x^3} \, dx. \]
   
   Using the substitution \( u = 1 + x^3 \), we get
   \[ \int_{0}^{1} 3x^2 \sqrt{1 + x^3} \, dx = \int_{1}^{2} \sqrt{u} \, du = \frac{2}{3} \left[ u^{3/2} \right]_{1}^{2} = \frac{2}{3} \left( 2\sqrt{2} - 1 \right). \]

   (b) For \( x \geq 0 \) define \( F(x) \) and \( g(x) \) by \( F(x) = \int_{0}^{x} \cos^2(t) \, dt \) and \( g(x) =xF(x^2) \). Calculate \( g'(\sqrt{\pi}) \).
   
   Answer: \( \frac{5}{2\pi} \)

   Solution: By the product rule, chain rule, and FTC I we have
   \[ g'(x) = F(x^2) + 2x^2 F'(x^2) = F(x^2) + 2x^2 \cos^2(x^2). \]
   
   Setting \( x = \sqrt{\pi} \), we get \( g'(\sqrt{\pi}) = F(\pi) + 2\pi \cos^2(\pi) = F(\pi) + 2\pi \). Now,
   \[ F(\pi) = \int_{0}^{\pi} \cos^2(t) \, dt = \int_{0}^{\pi} \frac{1 + \cos(2t)}{2} \, dt = \frac{1}{2} \left[ t + \frac{\sin(2t)}{2} \right]_{0}^{\pi} = \frac{\pi}{2}. \]
   
   So we conclude that \( g'(\sqrt{\pi}) = \frac{\pi}{2} + 2\pi = \frac{5}{2}\pi \).
(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between \( x = -y^2 \) and \( x = -4+y^2 \) about the vertical line \( x = 2 \). **Do not evaluate the integral.**

\[
\text{Answer: } V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} ((6 - y^2)^2 - (2 + y^2)^2) \, dy.
\]

**Solution:** The plot is as shown:

The red curve is \( x = -y^2 \), the blue curve is \( x = -4+y^2 \) and the green line is the axis of rotation \( x = 2 \). The red and blue curves meet where \(-y^2 = -4+y^2\), which gives us \( y = \pm \sqrt{2} \) and \( x = -2 \).

A slice of the rotational solid at height \( y \) will be a circular crown with inner radius \( r_y = 2 + y^2 \) (the distance between \( x = 2 \) and \( x = -y^2 \)) and outer radius \( R_y = 6 - y^2 \) (the distance between \( x = 2 \) and \( x = -4+y^2 \)).

Thus the area of the slice will be

\[
A_y = \pi (R_y^2 - r_y^2) = \pi ((6 - y^2)^2 - (2 + y^2)^2).
\]

This gives the volume \( V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} ((6 - y^2)^2 - (2 + y^2)^2) \, dy. \)
4. (a) \(2\) marks  Plot the finite area enclosed by \(y^2 = x\) and \(x = 8 - 2y\).

**Solution:** The area is the region enclosed between the blue and red curves:

![Graph of the area enclosed by \(y^2 = x\) and \(x = 8 - 2y\).]

(b) \(4\) marks  Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral.**

**Answer:** \(A = \int_{-4}^{2} (8 - 2y - y^2) \, dy\)

**Solution:** The two curves meet when \(y^2 = 8-2y\), which has solutions \(y = 2, -4\). Seeing as how for both curves \(x\) is expressed as a function of \(y\), we choose to write the area as an integral in the variable \(y\), with \(x_B(y) = y^2\) and \(x_T(y) = 8 - 2y\). By evaluating at \(y = 0\) we see that \(x_T(y) \geq x_B(y)\) on the interval \([-4, 2]\). Then we get the integral

\[
\int_{-4}^{2} [x_T(y) - x_B(y)] \, dy = \int_{-4}^{2} (8 - 2y - y^2) \, dy .
\]

Alternatively, as an integral in \(x\), the area is

\[
A = 2 \int_{0}^{4} \sqrt{x} \, dx + \int_{4}^{16} \left(4 - \frac{x}{2} + \sqrt{x}\right) \, dx .
\]
5. A solid has as its base the region in the $xy$-plane between $y = 1 - x^2/9$ and the $x$-axis. The cross-sections of the solid perpendicular to the $x$-axis are semi-circles with the diameter of the semi-circle in the base.

(a) 4 marks Write a definite integral that determines the volume of the solid.

$$\text{Answer: } V = \frac{\pi}{8} \int_{-3}^{3} (1 - \frac{x^2}{9})^2 \, dx$$

**Solution:** The points of intersection with the $x$-axis are given by $x^2/9 = 1$, so we get $x = \pm 3$.

A vertical slice of the solid at $x$ will be a half circle whose diameter is $1 - \frac{x^2}{9}$, and thus will have area $A_x = \frac{\pi}{2} \left( \frac{1}{2}(1 - \frac{x^2}{9}) \right)^2$. Then the volume is given by

$$V = \int_{-3}^{3} A_x \, dx = \int_{-3}^{3} \frac{\pi}{8} (1 - \frac{x^2}{9})^2 \, dx.$$ 

(b) 2 marks Evaluate the integral to find the volume of the solid.

$$\text{Answer: } \frac{25}{5}\pi$$

**Solution:** We simplify by using the substitution $u = x/3$ so that

$$V = \int_{-3}^{3} \frac{\pi}{8} (1 - \frac{x^2}{9})^2 \, dx = \frac{3\pi}{8} \int_{-1}^{1} (1 - u^2)^2 \, du.$$ 

Now we note that the function is even, so that

$$V = \frac{3\pi}{8} \int_{-1}^{1} (1 - u^2)^2 \, du = \frac{3\pi}{4} \int_{0}^{1} (1 - u^2)^2 \, du.$$ 

Finally we expand the formula and compute the integral:

$$V = \frac{3\pi}{4} \int_{0}^{1} (1 - u^2)^2 \, du = \frac{3\pi}{4} \int_{0}^{1} (u^4 - 2u^2 + 1) \, du = \frac{3\pi}{4} \left[ \frac{u^5}{5} - \frac{2u^3}{3} + u \right]_0^1.$$ 

We calculate that

$$V = \frac{3\pi}{4} \left( \frac{1}{5} - \frac{2}{3} + 1 \right) = \frac{2}{5}\pi.$$