Self-Replication of Mesa Patterns in Reaction-Diffusion Systems

T. KOLOKOLNIKOV ⋆, M.J. WARD † J. WEI ‡

⋆ *Department of Mathematics, Dalhousie University, Halifax, N.S., Canada*
tkolokol@gmail.com
† *Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada*
ward@math.ubc.ca
‡ *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*
wei@math.cuhk.edu.hk

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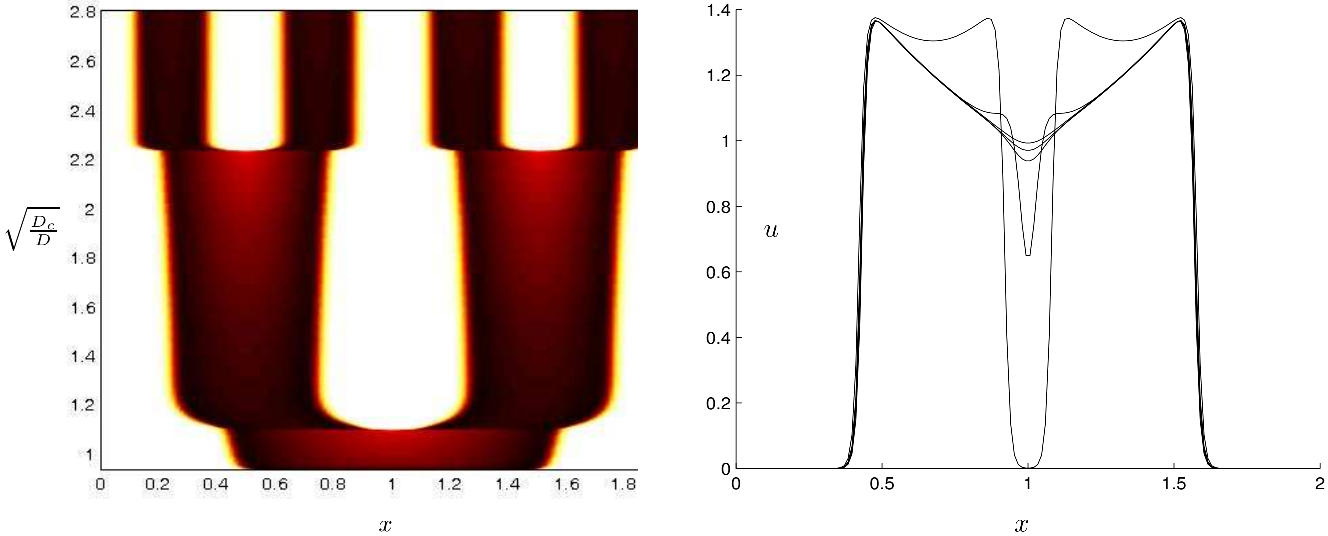
Certain two-component reaction-diffusion systems on a finite interval are known to possess mesa (box-like) steady-state patterns in the singularly perturbed limit of small diffusivity for one of the two solution components. As the diffusivity D of the second component is decreased below some critical value D_c, with $D_c = O(1)$, the existence of a steady-state mesa pattern is lost, triggering the onset of a mesa self-replication event that ultimately leads to the creation of additional mesas. The initiation of this phenomena is studied in detail for a particular scaling limit of the Brusselator model. Near the existence threshold D_c of a single steady-state mesa, it is shown that an internal layer forms in the center of the mesa. The structure of the solution within this internal layer is shown to be governed by a certain *core problem*, comprised of a single non-autonomous second-order ODE. By analyzing this core problem using rigorous and formal asymptotic methods, and by using the Singular Limit Eigenvalue Problem (SLEP) method to asymptotically calculate small eigenvalues, an analytical verification of the conditions of Nishiura and Ueyema [Physica D, **130**, No. 1, (1999), pp. 73–104], believed to be responsible for self-replication, is given. These conditions include: (1) The existence of a saddle-node threshold at which the steady-state mesa pattern disappears; (2) the dimple-shaped eigenfunction at the threshold, believed to be responsible for the initiation of the replication process; and (3) the stability of the mesa pattern above the existence threshold. Finally, we show that the *core problem* is universal in the sense that it pertains to a class of reaction-diffusion systems, including the Gierer-Meinhardt model with saturation, where mesa self-replication also occurs.

1 Introduction

In [**29**] Pearson used numerical simulations to show that the two-component Gray-Scott reaction-diffusion model in the singularly perturbed limit can exhibit many intricate types of spatially localized patterns. Many of these numerically computed patterns for this model have been observed qualitatively in certain chemical experiments (cf. [**12**], [**13**]). An important new phenomenon that was discovered in [**29**], [**12**], and [**13**], is the occurrence of self-replication behavior of pulse and spot patterns. In recent years, many theoretical and numerical studies have been made in both one and two spatial dimensions to analyze self-replication behavior for the Gray-Scott model in different parameter regimes (cf. [**34**], [**33**], [**27**], [**28**], [**35**], [**20**], [**2**], [**1**], [**16**]). In addition to the Gray-Scott model, many other reaction-diffusion systems have been found to exhibit self-replication behavior. These include the ferrocyanide-iodide-sulfite system (cf. [**13**]), the Belousov-Zhabotinsky reaction (cf. [**19**]), the Gierer-Meinhardt model (cf. [**18**], [**4**], [**17**]), and the Bonhoffer van-der-Pol-type system (cf. [**8**], [**9**]).

Despite a large number of studies on the subject, the detailed mechanisms responsible for self-replication are still not clear. In an effort to classify reaction-diffusion systems that can exhibit pulse self-replication, Nishiura and Ueyema [**27**] (see also [**5**]) proposed a set of necessary conditions for this phenomenon to occur. Roughly stated, these conditions are the following:



(a) (b)

FIGURE 1. (a) Numerical simulation showing mesa-splitting in the Brusselator model (1.2). The fixed parameters are $\beta_0 = 1.5$, $\varepsilon = 0.01$, $\tau = 0.7$, $x \in [0,2]$, while D is slowly decreased in time according to $D = (1 + 5 \times 10^{-6} t)^{-2}$. Parameters β and α are determined through (1.3), i.e. $\alpha = \varepsilon^2/D$, $\beta = \alpha\beta_0$. The vertical axis is $K = \sqrt{\frac{D_c}{D}}$, where $D_c = 0.88$. Splitting events occur for $K \approx 1$ and $K \approx 2$. (b) Snapshots of $u(x)$ during a splitting event. The distinctive two-mesa pattern is shown at $t = 35100$. Further snapshots are plotted every 1000 time units.

(1) The disappearance of the K-pulse steady-state solution due to a saddle-node (or fold point) bifurcation that occurs when a control parameter is decreased below a certain threshold value.

(2) The existence of a dimple eigenfunction at the existence threshold, which is believed to be responsible for the initiation of the pulse-splitting process. By definition, a dimple eigenfunction is an even eigenfunction $\Phi(y)$ associated with a zero eigenvalue, that decays as $|y| \to \infty$ and that has precisely one positive zero.

(3) Stability of the steady-state solution above the threshold value for existence.

(4) The alignment of the existence thresholds, so that the disappearance of K pulses, with $K = 1, 2, 3, \ldots$, occurs at asymptotically the same value of the control parameter.

For the Gray-Scott model in the weak interaction parameter regime where the ratio of the diffusivities is $O(1)$, Nishiura and Ueyema in [**27**] verified these conditions numerically for a given fixed diffusivity ratio. Alternatively, for the Gray-Scott model in the semi-strong regime, where the ratio of the diffusivities is asymptotically large, it was shown in [**20**] and in equation (2.9) of [**2**] that the following core problem determines the spatial profile of a pulse in the self-replicating parameter regime:

$$V'' - V + UV^2 = 0\,, \qquad U'' - UV^2 = 0\,; \qquad U'(0) = V'(0) = 0\,, \quad V \to 0\,, \quad U' \to A \ \text{ as } \ y \to \infty\,. \tag{1.1}$$

By using a combination of asymptotic and numerical methods, and by coupling (1.1) to an appropriate outer solution away from a localized pulse, conditions (1)–(4) of Nishiura and Ueyema were verified in [**16**]. In [**3**] a detailed study of the intricate bifurcation structure of (1.1) was given.

In this paper we study self-replication of *mesa patterns*. A single mesa solution is a spatial pattern that consists of two back-to-back transition layers. An example of such a steady-state pattern is shown in Fig. 2 below. Our goal is to analytically verify whether the conditions (1)–(4) of Nishiura and Ueyama [**27**], originally formulated for analyzing pulse self-replication behavior, also hold for mesa self-replication. In addition, we seek to derive and study a certain core problem, analogous to (1.1), that pertains to self-replicating mesa patterns.

For concreteness, we concentrate on the Brusselator model. This model was introduced in [**31**], and is based on the following hypothetical chemical reaction:

$$A \to X\,, \qquad C + X \to Y + F\,, \qquad 2X + Y \to 3X, \qquad X \to E.$$

The autocatalytic step $2X + Y \to 3X$ introduces a cubic non-linearity in the rate equations. Since the 1970's, various weakly-nonlinear Turing patterns in the Brusselator have been studied both numerically and analytically in one, two, and three dimensions. These include spots, stripes, labyrinths and hexagonal patterns (cf. [**6**], [**21**], [**30**], [**36**], [**37**]), and oscillatory instabilities and spatio-temporal chaos (cf. [**14**], [**38**]).

After a suitable rescaling, we write the one-dimensional Brusselator model on a domain of length $2L$ as

$$u_t = \varepsilon^2 u_{xx} - u + \alpha + u^2 v\,, \qquad \tau v_t = \varepsilon^2 v_{xx} + (1-\beta)\,u - u^2 v\,; \qquad u_x(\pm L, t) = v_x(\pm L, t) = 0\,. \tag{1.2}$$

In this paper we make the following assumptions on the parameters:

$$\varepsilon \ll 1\,; \quad \alpha \ll 1\,; \quad \beta \ll 1\,; \quad D = \frac{\varepsilon^2}{\alpha} = O(1)\,; \quad \beta_0 \equiv \frac{\beta}{\alpha} = O(1)\,, \;\text{ with }\; \beta_0 > 1\,; \quad \tau = 0\,. \tag{1.3}$$

The full numerical results in Fig. 1 illustrate the mesa self-replication behavior for (1.2). To trigger mesa self-replication events we started with a single mesa as initial condition and slowly decreased D in time (see the figure caption for the parameter values). At the critical value $D_1 \sim 0.8$, a mesa splits into two mesas, which then repel and move away from each other. The splitting process is repeated when D is decreased below $D_2 \sim 0.2$.

In §2 we calculate a threshold value D_c of D for the existence of a single-mesa steady-state solution for the Brusselator (1.2) in the limit $\varepsilon \to 0$, and under the assumptions (1.3) on the parameter values. The result, summarized in Proposition 1 of §2, shows the existence of a value D_c such that a K-mesa steady-state solution exists if and only if $D > D_c/K^2$. Analytical upper and lower bounds for D_c are also derived. Similar thresholds for the existence of steady-state mesa patterns were derived for other reaction-diffusion systems in [**11**] using more heuristic means. Our analysis is based on a systematic use of the method of matched asymptotic expansions.

For a single-mesa steady-state solution, we show in §3 that an internal layer of width $O(\varepsilon^{2/3})$ forms in the center of the mesa when D is asymptotically close to the threshold value D_c. This internal layer is illustrated below in Fig. 4. By analyzing this internal layer region using matched asymptotic analysis, we show that the solution u is determined locally in terms of the solution $U(y)$ to a single non-homogeneous ODE of the form

$$U'' = U^2 - A - y^2\,; \qquad U'(0) = 0\,, \quad U'(y) \to 1 \;\text{ as }\; y \to \infty\,. \tag{1.4}$$

Here A is related to the parameter values in (1.2). We refer to (1.4) as the *core problem* for the onset of self-replication. Unlike (1.1) for self-replicating pulses in the Gray-Scott model, the problem (1.4) is not coupled and, consequently, is easier to study analytically than (1.1). The proof of conditions (1) and (2) of Nishiura and Ueyama is then reduced to a careful study of (1.4). More specifically, by using rigorous techniques we prove analytically the existence of a saddle-node bifurcation for (1.4) and we analyze the solution behavior on the bifurcation diagram. The result is summarized below in Theorem 2. In §3.1 we use some rigorous properties of the core problem, together with a formal matched asymptotic analysis, to asymptotically construct a dimple eigenfunction corresponding to the zero eigenvalue at the saddle-node bifurcation value. This construction, summarized in Proposition 3, establishes condition (2) of Nishiura and Ueyama.

In §3.2 we show that the core problem (1.4) is universal in the sense that it can be readily derived for other reaction-diffusion systems where mesa self-replication occurs. The universal nature of (1.4) is illustrated for the Gierer-Meinhardt model with saturation (cf. [**18**]). For this specific model, mesa-splitting was computed numerically in Figure 28 of [**17**]. Although the phenomena of mesa self-replication is qualitatively described in Chapter 11 of [**11**], the core problem (1.4) governing the onset of mesa self-replication and its analysis has not, to our knowledge, appeared in the literature.

Since the saddle-node existence value for a K-mesa steady-state solution is $D = D_c/K^2$, the condition (4) of

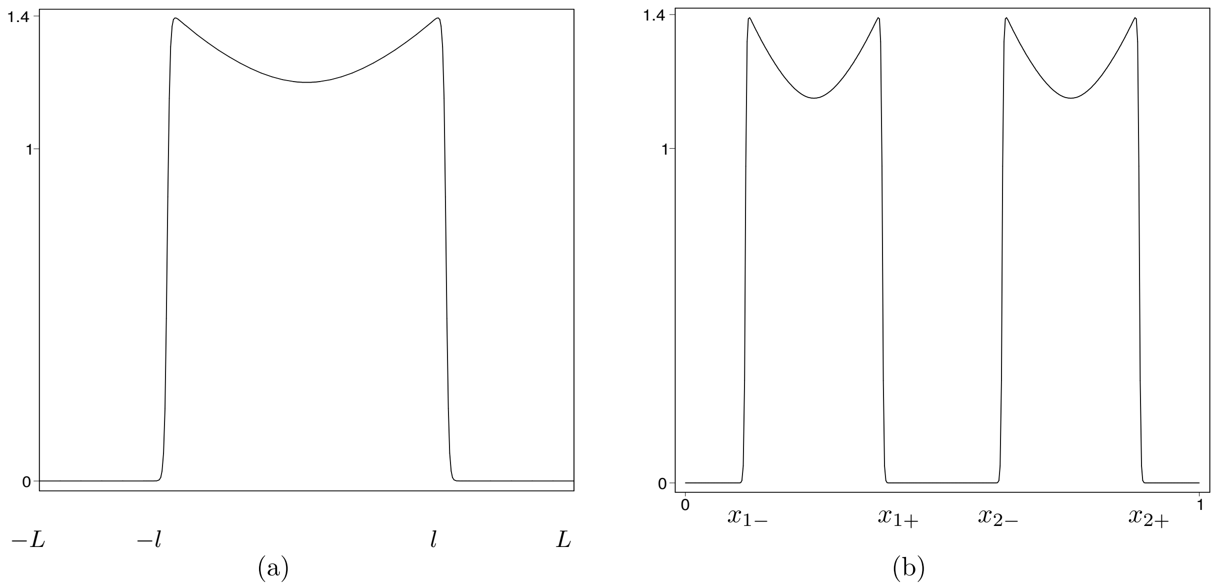


FIGURE 2. (a) A single mesa steady-state on $[-L, L]$. The parameter values are $L = 1,\ \beta_0 = 1.5,\ D = 1.29,\ \varepsilon = 0.005$. (b) A two-mesa steady state ($K = 2, L = 1/4,\ x \in [0, 1]$). The parameter values are $D = 0.068,\ \varepsilon = 0.00125,\ \beta_0 = 1.5$.

Nishiura and Ueyama, regarding an asymptotically close alignment of saddle-node bifurcation values, does not hold in a strict sense. However, this condition is satisfied in the same approximate sense as in the study of self-replicating pulses for the Gray-Scott model in the semi-strong interaction regime (see Table 3 of [**1**] and equation (1.2) of [**16**]).

In §4 we study the stability of K-mesa steady-state solutions when $D > D_c/K^2$. We show that such a pattern is stable when $\tau = 0$, and moreover all asymptotically small eigenvalues are purely real. This proves condition (3) of Nishiura and Ueyema. Our analysis is similar to the SLEP method, originally developed by Nishiura et. al. in [**22**] and [**23**], and that has been used successfully to prove the stability of mesa-type patterns in reaction-diffusion systems and in related contexts (cf. [**24**], [**26**]). In our analysis a formal matched asymptotic analysis is used to derive a reduced problem that capture the asymptotically small eigenvalues of the linearization. This reduced system is then studied rigorously using several tools, including the maximum principle and matrix theory. In this way we prove that the small eigenvalues are purely real and negative when $\tau = 0$.

Finally, in §5 we relate our results regarding mesa self-replication for the Brusselator model with previous results concerning the coarsening phenomenon of mesa patterns that occurs when D is sufficiently large (cf. [**15**]). In addition, we propose some open problems.

2 The Steady State Mesa and the Universal Core Problem

In this section we study the steady-state problem for (1.2), and we prove analytically the first two conditions of Nishiura and Ueyema. We will analyze an even symmetric solution of the type shown on Fig. 2(a), consisting of a single mesa on a domain $[-L, L]$ with interfaces at $x = \pm l$. The K-mesa solution on a domain of length $2KL$ can then be constructed by reflecting and gluing together K such solutions.

We first reformulate (1.2) to emphasize the slow-fast structure. We define w by

$$w = v + u\,,$$

so that (1.2) becomes

$$u_t = \varepsilon^2 u'' - u + \alpha + u^2(w-u)\,, \qquad \tau\,(w_t - u_t) + u_t = \varepsilon^2 w'' + \alpha - \beta\,. \tag{2.1}$$

Here the primes indicate derivatives with respect to x. We then introduce $\beta_0 = O(1)$ and $D = O(1)$ defined by

$$\beta_0 \equiv \frac{\beta}{\alpha}\,, \qquad D \equiv \frac{\varepsilon^2}{\alpha}\,.$$

Then, (2.1) becomes

$$u_t = \varepsilon^2 u'' - u + \alpha + u^2(w-u)\,, \qquad \frac{\tau}{\alpha} w_t + \frac{(1-\tau)}{\alpha} u_t = D w'' + 1 - \beta_0\,. \tag{2.2}$$

The corresponding steady-state problem, with $\alpha = O(\varepsilon^2)$, is

$$\varepsilon^2 u'' - u + u^2(w-u) + \alpha = 0\,, \qquad D w'' + 1 - \beta_0 u = 0\,. \tag{2.3}$$

Since $\varepsilon^2 \ll D$ from (1.3), w is the slow variable and u is the fast variable. Upon integrating (2.3) for w and using $w'(L) = 0$ and the symmetry condition $w'(0) = 0$, we obtain the integral constraint

$$L = \beta_0 \int_0^L u\,dx\,. \tag{2.4}$$

Near the interface at $x = l$ we introduce the inner expansion

$$u = U_0\,(y) + \varepsilon U_1\,(y) + \ldots\,, \qquad w = W_0(y) + \varepsilon W_1\,(y) + \ldots\,, \qquad y = \varepsilon^{-1}(x-l)\,. \tag{2.5}$$

Upon substituting this expansion into (2.3), we obtain the leading-order problem

$$U_0'' - f(U_0, W_0) = 0\,, \qquad W_0'' = 0\,, \tag{2.6}$$

where $f(u,w)$ is defined by

$$f(u,w) \equiv u - u^2(w-u)\,. \tag{2.7}$$

At next order, we obtain

$$\mathcal{L}U_1 \equiv U_1'' - f_u(U_0, W_0)U_1 = f_w(U_0, W_0)W_1\,, \qquad W_1'' = 0\,. \tag{2.8}$$

From (2.6) we get that W_0 is a constant to be determined. To ensure that there exists a heteroclinic connection for U_0 we require that f satisfy the *Maxwell line condition*, which states that the area between the first two roots of f is the negative of the area between its last two roots of f. Since f is a cubic, this is equivalent to simultaneously solving $f = 0$ and $f'' = 0$ for W_0. In this way, we obtain

$$W_0 = \frac{3}{\sqrt{2}}\,, \qquad U_0 = \frac{1}{\sqrt{2}}\left[1 \pm \tanh\left(\frac{y}{2}\right)\right]\,. \tag{2.9}$$

For the mesa solution as shown in Fig. 2(a), we must take the minus sign in (2.9) above.

To determine the interface location l, we now study the outer problem away from the interface at $x = l$. Since $\alpha = O(\varepsilon^2)$, we obtain to leading order from (2.3) that $u + u^2(w-u) = 0$. This yields either $u = 0$ or

$$w \sim h\,(u) \equiv \frac{1}{u} + u\,. \tag{2.10}$$

Moreover, we have $U_0 \to 0$ as $y \to \infty$ and $U_0 \to \sqrt{2}$ as $y \to -\infty$. Therefore, by matching to U_0 and W_0, and by using the symmetry condition at $x = 0$, we obtain the following outer problem in the mesa region $0 \le x \le l$:

$$w = h(u); \qquad D w'' = g\,(u) \equiv \beta_0 u - 1\,, \qquad 0 < x < l\,. \tag{2.11 a}$$

$$u\,(l) = \sqrt{2}\,, \qquad w\,(l) = \frac{3}{\sqrt{2}}\,, \qquad u'\,(0) = w'\,(0) = 0\,. \tag{2.11 b}$$

In contrast, the leading-order outer problem on $l \le x \le L$ is $u = 0$ and $D w'' = -1$.

The solution to the second-order inner problem (2.8) for W_1 is $W_1 = W_{11}y + W_{12}$, where W_{11} and W_{12} are constants to be determined. Since $\mathcal{L}U_0' = 0$, and $f_w(U_0, W_0) = -U_0^2$, the solvability condition for (2.8) yields

$$0 = \int_{-\infty}^{\infty} U_0' f_w(U_0, W_0) W_1 \, dy = -\int_{-\infty}^{\infty} U_0' U_0^2 \left(W_{11} y + W_{12} \right) \, dy \, .$$

This yields one relation between W_{11} and W_{12}. The second relation is obtained by matching W to the the outer solution w. This yields $W_{11} = w'(l^{\pm})$. In this way, we obtain

$$W_1' \equiv W_{11} = w'(l) \, , \qquad W_{12} = -\frac{W_{11}}{2\sqrt{2}} \int_{-\infty}^{\infty} y \left(U_0^3 \right)' \, dy \, . \tag{2.12}$$

We now solve the outer problem (2.11) in terms of $u_0 \equiv u(0)$. We first define $F(u; u_0)$ by

$$F(u; u_0) \equiv \int_{u_0}^{u} g(s) \, h'(s) \, ds \, . \tag{2.13}$$

By multiplying (2.11 a) for w by w' we get

$$\frac{D w'^2}{2} = F(u; u_0) \, , \qquad w' = \sqrt{\frac{2F(u; u_0)}{D}} \, .$$

In the outer region on $l \leq x \leq L$, we have $u = 0$. Therefore, by integrating w'' from $x = 0$ to $x = L$, we obtain $\int_0^l g(u) \, dx + \int_l^L (-1) \, dx = 0$. This yields,

$$\int_0^l g(u) \, dx = L - l \, . \tag{2.14}$$

The left-hand side of (2.14) is evaluated by integrating w'' from $x = 0$ to $x = l$ to get $\int_0^l g(u) \, dx = D w'(l) = \sqrt{(2D) F(\sqrt{2}; u_0)}$. In addition, by using $w' = h'(u) u'$, we obtain

$$\frac{du}{dx} = \sqrt{\frac{2}{D}} \frac{\sqrt{F(u; u_0)}}{h'(u)} \, . \tag{2.15}$$

We then integrate (2.15) with $u(0) = u_0$ and $u(l) = \sqrt{2}$. In this way, we obtain

$$\sqrt{2F(\sqrt{2}; u_0)} = \frac{L - l}{\sqrt{D}} \, , \qquad \frac{l}{\sqrt{D}} = \int_{u_0}^{\sqrt{2}} \frac{h'(u)}{\sqrt{2F(u; u_0)}} \, du \, . \tag{2.16}$$

Upon integrating the second expression in (2.16) by parts we get

$$\int_{u_0}^{\sqrt{2}} \frac{h'(u)}{\sqrt{2F(u; u_0)}} \, du = \frac{\sqrt{2F\left(\sqrt{2}; u_0\right)}}{g\left(\sqrt{2}\right)} + \int_{u_0}^{\sqrt{2}} \frac{g'(u)}{\left[g(u)\right]^2} \sqrt{2F(u; u_0)} \, du \, .$$

By combining this relation with (2.16), and by calculating $g(\sqrt{2})$, we obtain the following expression relating u_0 to the overall length of the domain:

$$\chi(u_0) \equiv \sqrt{2F(\sqrt{2}; u_0)} \left(\frac{\beta_0 \sqrt{2}}{\beta_0 \sqrt{2} - 1} \right) + \int_{u_0}^{\sqrt{2}} \frac{g'(u)}{\left[g(u)\right]^2} \sqrt{2F(u; u_0)} \, du = \frac{L}{\sqrt{D}} \, . \tag{2.17}$$

We note that the function $h(u)$ has a minimum at $u_0 = 1$, and that $\frac{dF}{du_0} = -g(u_0) \, h'(u_0) < 0$ for $u_0 > 1$ from (2.13). Hence, $\chi(u_0)$ is a decreasing function of u_0 for $u_0 > 1$. Upon defining D_c by $D_c = L^2 / \left[\chi(1)\right]^2$, we conclude that there exists a value of u_0, and consequently an outer solution on $0 < x < l$ exists, if and only if $D \geq D_c$. Therefore, we obtain a threshold value D_c for the existence of a single mesa solution on $[-L, L]$.

Next, we obtain explicit bounds on D_c. The function $w = h(u)$ satisfies $h(1) = 2$ and $h\left(\sqrt{2}\right) = 3/\sqrt{2}$. Therefore, for $x \in [0, l]$ we have that u and w are both increasing functions for $x \in [0, l]$ with $u(l) = \sqrt{2}$, $w(l) = 3/\sqrt{2}$, and

$$1 < u < \sqrt{2} \, , \qquad 2 < w < 3/\sqrt{2} \, . \tag{2.18}$$

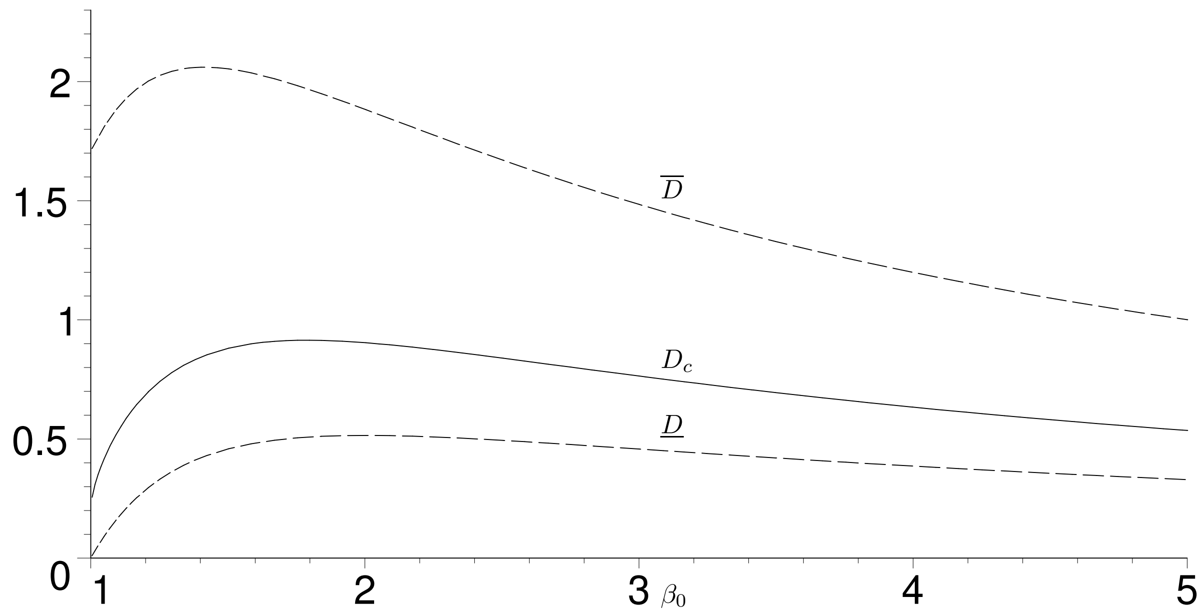


FIGURE 3. The graphs of D_c, $\overline{D}$ and $\underline{D}$ versus β_0 when $L = 1$

Using (2.18) and (2.11 *a*) for w we estimate $w'' = D^{-1}(\beta_0 u - 1) \geq D^{-1}(\beta_0 - 1)$. This yields that

$$w(x) \geq w(0) + \frac{(\beta_0 - 1)}{2D}x^2, \qquad \text{which implies} \qquad \frac{3}{\sqrt{2}} \geq 2 + \frac{(\beta_0 - 1)}{2D}l^2. \tag{2.19}$$

Then, we use (2.4) with $u = 0$ for $l \leq x \leq L$ to get $L = \beta_0 \int_0^l u\, dx \leq \beta_0 l \sqrt{2}$. Combining this inequality with (2.19), we get

$$l^2 \geq \frac{L^2}{2\beta_0^2}, \qquad \frac{3}{\sqrt{2}} \geq 2 + \frac{\beta_0 - 1}{4\beta_0^2}\frac{L^2}{D}.$$

By solving this second relation for D we conclude that a necessary condition for the existence of a solution to (2.11) is that $D \geq \underline{D}$, where

$$\underline{D} = \frac{(\beta_0 - 1)}{4\beta_0^2}\frac{1}{3/\sqrt{2} - 2}L^2. \tag{2.20}$$

This relation provides a lower bound for D_c. To find an upper bound we calculate $w'' \leq \left(\beta_0\sqrt{2} - 1\right)/D$. This inequality can be integrated, and with $w(l) = \sqrt{3}/2$, we get

$$w(0) \geq \frac{3}{\sqrt{2}} - \frac{l^2}{2D}\left(\beta_0\sqrt{2} - 1\right). \tag{2.21}$$

Then, from (2.4), we obtain $l \leq L/\beta_0$. Therefore, using (2.21) we conclude that $w(0) \geq 2$ provided that $\frac{3}{\sqrt{2}} - \frac{L^2}{2D}\frac{1}{\beta_0^2}\left(\beta_0\sqrt{2} - 1\right) > 2$. Hence, a sufficient condition for the existence of a solution to (2.11) is that $D \geq \overline{D}$, where

$$\overline{D} = \frac{\sqrt{2}\beta_0 - 1}{2\beta_0^2}\frac{1}{3/\sqrt{2} - 2}L^2.$$

It follows that $\underline{D} \leq D_c \leq \overline{D}$. We summarize the results of this analysis of a single mesa solution as follows:

Proposition 1 (Nishiura-Ueyema's Condition 1: The Steady State and its Disappearance) *Consider the steady* *state solution to the Brusselator model (2.3) with $\beta_0 > 1$ in the limit $\varepsilon \to 0$. We define $F(u;u_0)$ and $\chi(u_0)$ by*

$$F\left(u;u_0\right) \equiv \int_{u_0}^{u} \left(\beta_0 s - 1\right)\left(1 - \frac{1}{s^2}\right) ds\,, \qquad \chi\left(u_0\right) \equiv \sqrt{2F(\sqrt{2};u_0)}\left(\frac{\beta_0\sqrt{2}}{\beta_0\sqrt{2}-1}\right) + \int_{u_0}^{\sqrt{2}} \frac{\beta_0\sqrt{2F\left(u;u_0\right)}}{\left(\beta_0 u - 1\right)^2}\, du\,.$$

In terms of $\chi(1)$, we define the threshold D_c by

$$D_c = L^2/\chi\left(1\right)^2\,.$$

Suppose that $D > D_c$. Then, there exists unique $u_0 \in \left(1,\sqrt{2}\right)$ and $l \in (0,L)$, given implicitly by

$$\chi\left(u_0\right) = \frac{L}{\sqrt{D}}, \qquad l = L - \sqrt{D}\sqrt{2F(\sqrt{2};u_0)}\,,$$

such that there exists a symmetric mesa solution on the interval $[-L,L]$ with interfaces at $\pm l$ and with $u\left(0\right) = u_0$. In the region $x \in (0,l)$, w and u are given implicitly by

$$w = \frac{1}{u} + u\,, \qquad Dw'' = g\left(u\right) \equiv \beta_0 u - 1\,, \quad 0 < x < l\,; \qquad u\left(0\right) = u_0\,, \quad u\left(l\right) = \sqrt{2}\,. \tag{2.22}$$

In the region $x \in (l,L)$, the leading-order outer solutions for u and w are

$$u = 0\,, \qquad w = -\frac{1}{2D}\left(x-L\right)^2 + \frac{1}{2D}\left(2/\sqrt{3} - L\right)^2 + 2/\sqrt{3}\,.$$

Moreover, we have $\underline{D} \le D_c \le \overline{D}$, where

$$\underline{D} = \frac{\left(\beta_0 - 1\right)}{4\beta_0^2}\frac{1}{3/\sqrt{2}-2}L^2\,, \qquad \overline{D} = \frac{\sqrt{2}\beta_0 - 1}{2\beta_0^2}\frac{1}{3/\sqrt{2}-2}L^2\,.$$

In Fig. 3 we plot D_c, $\overline{D}$, and $\underline{D}$, versus β_0. A single-mesa steady-state does not exist if $D < D_c$. By reflections and translations, a single-mesa solution can be extended to a K-mesa solution on the domain $[-KL,KL]$.

3 Core problem

In this section we verify Nishiura and Ueyema's second condition by first deriving and then analyzing a limiting differential equation that is valid in the vicinity of the critical threshold $D = D_c$. When D is decreased slightly below D_c (at which point $u(0) \sim 1, w(0) \sim 2$) the single-mesa solution ceases to exist. To study the solution near this critical value, we fix $u(0) = u_0$ and consider $D = D(u_0)$. From numerical computations of the steady state solution as shown in Fig. 4, an internal layer forms near the origin when u_0 is decreased below $u(0) = 1$.

To study the initial formation of this internal layer near the origin, we expand (2.3) near $D = D_c$ as

$$u = 1 + \delta u_1 + \cdots\,, \qquad w = 2 + \delta^2 w_1 + \cdots\,, \qquad D = D_c + \cdots\,.$$

The nonlinear term in (2.3) becomes $f(u,w) = \delta^2(u_1^2 - w_1) + \cdots$. From (2.3) we then obtain

$$\frac{\varepsilon^2}{\delta}u_1'' = u_1^2 - w_1\,, \qquad D_c\delta^2 w_1'' = \beta_0 - 1\,. \tag{3.1}$$

We introduce the inner-layer variable z by $z = x/\delta$ with $\delta \ll 1$. Then, (3.1) for w_1 becomes

$$w_{1zz} = \frac{\beta_0 - 1}{D_c}\,.$$

The solution is

$$w_1 = \mathcal{A} + \mathcal{B}z^2\,, \qquad \mathcal{B} = \frac{\beta_0 - 1}{2D_c} > 0\,, \qquad \mathcal{A} = w_1\left(0\right)\,. \tag{3.2}$$

Then, (3.1) for u_1 is

$$\frac{\varepsilon^2}{\delta^3}u_{1zz} = u_1^2 - \left(\mathcal{A} + \mathcal{B}z^2\right)\,. \tag{3.3}$$

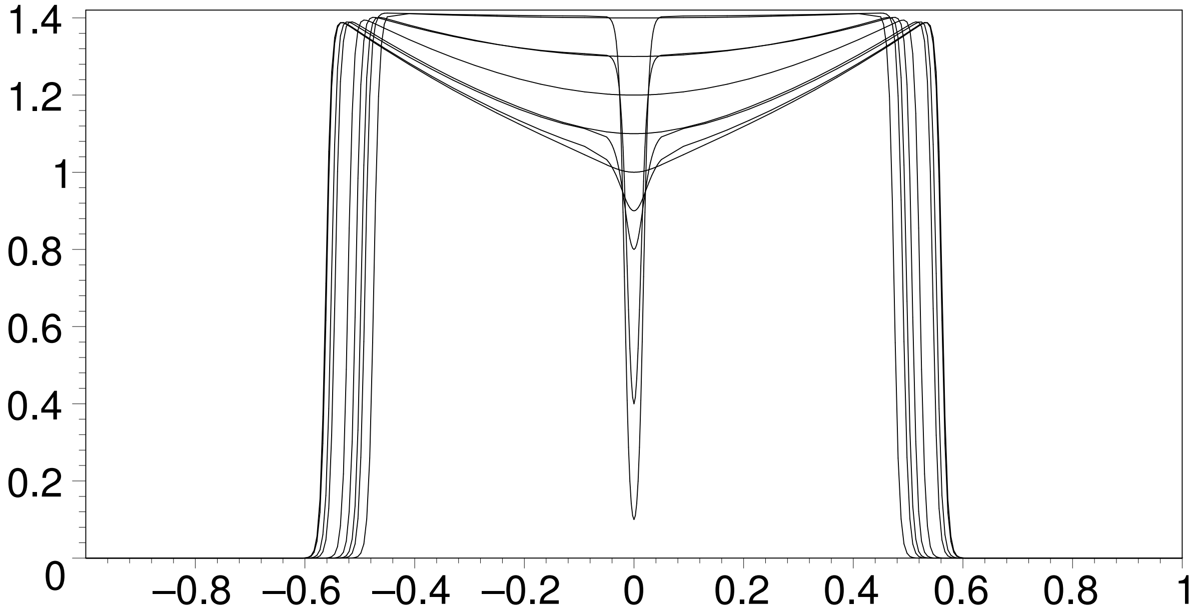


FIGURE 4. Formation of a boundary layer near the center of a mesa. The steady state and D are solved simultaneously, while $u(0)$ is fixed at one of the values $1.4, 1.3, 1.2, 1.0, 0.9, 0.4, 0.1$. The corresponding values for D are as follows: $u(0) = 1.4, D = 17.109$; $u(0) = 1.3, D = 2.257$; $u(0) = 1.2, D = 1.290$; $u(0) = 1.1, D = 0.96$; $u(0) = 1.0, D = 0.846$; $u(0) = .9, D = 0.863$; $u(0) = .8, D = 0.938$; $u(0) = .7, D = 1.068$; $u(0) = .6, D = 1.27$; $u(0) = .5, D = 1.624$; $u(0) = .4, D = 2.244$; $u(0) = .3, D = 3.525$; $u(0) = .2, D = 6.982$; $u(0) = .1, D = 24.34$.

This suggests the internal-layer scaling $\delta = \varepsilon^{2/3}$ so that $u_{1zz} = u_1^2 - \left(\mathcal{A} + \mathcal{B}z^2\right)$. The boundary conditions for u_1 are $u_{1z}(0) = 0$ and $u_1 \to z\sqrt{\mathcal{B}}$ as $z \to \infty$. Finally, we introduce U and y as

$$u_1 = \mathcal{B}^{1/3} U \,, \qquad z = \mathcal{B}^{-1/6} y \,.$$

This yields the following *core problem* for $U(y)$ on $0 < y < \infty$:

$$U'' = U^2 - A - y^2 \,; \qquad U'(0) = 0 \,, \qquad U' \sim 1 \quad \text{as} \quad y \to \infty \,. \tag{3.4}$$

Here A is related to $\mathcal{A}$ and $\mathcal{B}$ by

$$A = \mathcal{A}\mathcal{B}^{-2/3} \,.$$

In terms of the original variables, we have that

$$u(x) - 1 \sim \varepsilon^{2/3}\mathcal{B}^{1/3} U(y) \,, \qquad w(x) - 2 \sim \varepsilon^{4/3}\mathcal{B}^{2/3}\left(A + y^2\right) \qquad y = x/(\varepsilon^{2/3}\mathcal{B}^{-1/6}) \,, \tag{3.5 a}$$

$$u_1(0) = \mathcal{B}^{1/3} U(0) = \varepsilon^{-2/3}\left[u(0) - 1\right] \,, \qquad w_1(0) = \mathcal{B}^{2/3} A = \varepsilon^{-4/3}\left[w(0) - 2\right] \,. \tag{3.5 b}$$

Here $\mathcal{B}$ is defined in (3.2). The following main result pertains to the solution behavior of (3.4).

Theorem 2 *Suppose that $A \gg 1$. Then, the core problem (3.4) admits exactly two solutions $U^{\pm}(y)$ with $U' > 0$ for $y > 0$. They have the following uniform expansions:*

$$U^+ \sim \sqrt{A + y^2} \,, \qquad U^+(0) \sim \sqrt{A} \,, \tag{3.6 a}$$

$$U^- \sim \sqrt{A + y^2}\left(1 - 3\ \operatorname{sech}^2\left(\frac{\sqrt{A}y}{\sqrt{2}}\right)\right) \,, \qquad U^-(0) \sim -2\sqrt{A} \,. \tag{3.6 b}$$

These two solutions are connected. For any such solution, let $s = U(0) \equiv U_0$ and consider the solution branch

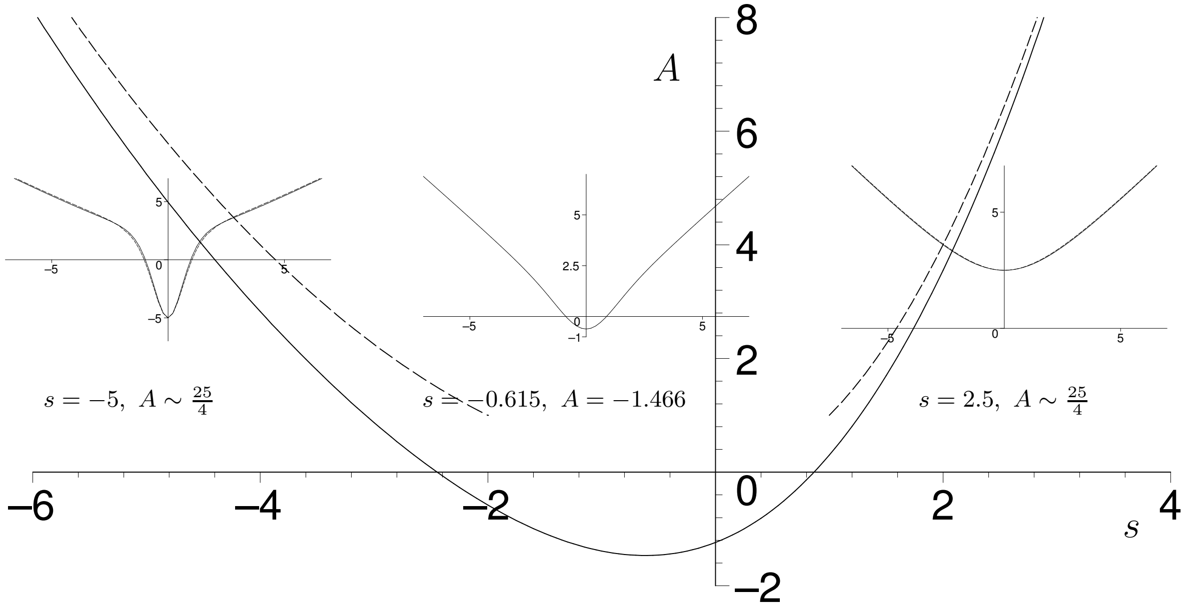


FIGURE 5. Plot of A versus $s = U(0)$ showing the fold point for the core problem (3.4). The inserts show the solution U versus y at the parameter values as indicated. The middle insert shows U at the fold point. The dotted lines are the limiting approximations of A versus $U^{\pm}(0)$ in (3.6). See Theorem 2 for more details.

$A = A(s)$. Then, $A(s)$ has a unique (minimum) critical point at $s = s_c,\ \ A = A_c$. Moreover, define $\Phi(y)$ by

$$\Phi = \frac{\partial U}{\partial s}\Big|_{s=s_c}. \tag{3.7}$$

Then, $\Phi > 0$ for all $y \geq 0$ and $\Phi \to 0$ as $y \to 0$. Numerically, we calculate that $A_c \approx -1.46638$ and $s_c \approx -0.61512$. The graph of $A(s)$ is shown in Fig. 5.

Proof. The proof consists of four steps. In Step 1 we use formal asymptotics to show that when $A \gg 1$, there are exactly two possible solutions with $U' > 0$ for $y > 0$, as given by (3.6). In Step 2 we rigorously show that there are no solutions when $-A$ is large enough. In Step 3 we show that the solution branch with $U' > 0,\ y > 0$ cannot connect with any branch for which $U' \leq 0$ at some $y > 0$. It then follows that the two branches U^+ and U^- must connect to each other. In Step 4 we show that Φ is positive and that the fold point s_c is unique.

Step 1:. We first consider the case $A \gg 1$. After rescaling $U = \sqrt{A}v,\ \ y = \alpha t$ for some α to be determined, (3.4) becomes

$$\frac{1}{\alpha^2\sqrt{A}}v_{tt} = v^2 - 1 - \frac{\alpha^2}{A}y^2.$$

If we choose $\alpha = \sqrt{A}$ then the leading-order equation for v becomes $v^2 - 1 - t^2 \sim 0$. This yields $v \sim \sqrt{1+t^2}$. The only other possible choice is $\alpha = A^{-1/4}$, which yields the leading-order equation

$$v_{tt} = v^2 - 1\,,$$

with $v'(0) = 0$, $v(t) \sim 1$ for large t. This ODE admits exactly two monotone solutions satisfying $v(t) \to 1$ as

$t \to \infty$. These solutions are given by

$$v = 1 \qquad \text{and} \qquad v = 1 - 3\ \operatorname{sech}^2\left(\frac{t}{\sqrt{2}}\right),$$

which correspond to the inner expansion of U^+ and U^-, respectively. Matching the inner and outer expansion into a uniform solution yields (3.6 a) and (3.6 b).

Step 2: Next we show the non-existence of a solution to the core problem when $-A$ is positive and sufficiently large. To show this, we rescale

$$u = \sqrt{-A}v, \quad y = (-A)^{-1/4}\, t\,.$$

From (3.4), we obtain

$$v'' = v^2 + 1 - \varepsilon t^2\,, \qquad \varepsilon \equiv (-A)^{-3/2}\,. \tag{3.8}$$

We will show that no solution to (3.8) exists when $\varepsilon > 0$ is small enough. First we choose any $a \in (0,1)$ and define T by

$$T \equiv \sqrt{\frac{1-a}{\varepsilon}}\,.$$

Then, for $0 < t < T$ we have

$$v'' > v^2 + a\,; \qquad v'(0) = 0\,, \quad v(0) = v_0\,.$$

In particular, $v' > 0$ for all $t \in (0,T)$. First, we suppose that $v(0) = v_0 \leq 0$. Then, under this assumption, we derive

$$\frac{v'^2}{2} \geq \frac{1}{3}v^3 + av - \left(\frac{1}{3}v_0^3 + av_0\right) \geq \frac{1}{3}v^3 + av\,.$$

The first step is to show that when ε is sufficiently large, $v(t)$ crosses zero at some value $t = t_1$. There two subcases to consider. For the first subcase, suppose that $v_0 < -1$. Then

$$\frac{v'^2}{2} \geq \frac{1}{3}v^3 - \frac{1}{3}v_0^3\,,$$

so that

$$t_1 \leq \int_{v_0}^{0} \frac{dv}{\sqrt{\frac{1}{3}v^3 - \frac{1}{3}v_0^3}} \leq C\,|v_0|^{-1/2} \leq C\,.$$

Therefore, by choosing ε small enough so that $T \geq C$, we have $t_1 \in [0,T]$. For the subcase $v_0 > -1$, we have $v'' \geq a,\ \ v \geq \frac{a}{2}t^2 + v_0$ so that $t_1 \leq \sqrt{\frac{2}{a}}$. Therefore, $t_1 \in [0,T]$ by choosing ε small enough so that $T \geq \sqrt{\frac{2}{a}}$.

The second step is to show that v blows up for some $T_b \in (0,T)$, provided that T is large enough. This would yield a contradiction. Indeed we have $\frac{v'^2}{2} \geq \frac{1}{3}v^3 + av$ for $v \geq 0$ so that

$$T_b \leq I_2 + t_1, \quad I_2 = \int_0^\infty \frac{dv}{\sqrt{2}\sqrt{\frac{1}{3}v^3 + av}} < \infty.$$

Therefore a contradiction is attained by choosing ε so small that $T > I_2 + t_1$.

Finally, if $v_0 > 0$ let $\eta = v_0 + Bt^2$. Then, for large enough B we have $\eta'' - \eta^2 - 1 + \varepsilon t^2 \leq 0$. Hence, by a comparison principle, $v \geq v_0 + Bt^2$ for all $t > 0$. But this is impossible since we must have $v \to \sqrt{\varepsilon}t$ for large values of t. This shows that no solution to the core problem (3.4) can exist if $-A$ is sufficiently large.

Step 3: We now show that the solution branch with $U' > 0$ for $y > 0$ can never connect to a non-monotone solution branch. We argue by contradiction. Suppose not. Then consider the first parameter value A for which a connection occurs. For such a value of A, there must be a point $y_0 \in [0,\infty)$ such that $U'(y_0) = 0$ with $U' \geq 0$ for any other y. Suppose first that $y_0 > 0$. Then we have $U''' = 2UU' - 2y$ so that $U'''(y_0) = -2y_0 < 0$. But this contradicts the assumption that $U'(y_0) = 0$ is a minimum of U'. If on the other hand $y_0 = 0$, then we consider three cases. First if $U''(0) = 0$, then from a Taylor expansion we obtain $U(0) = \sqrt{A}$; $U'(0) = U''(0) = U'''(0) = 0$,

$U^{(4)}(0) = -2$. This expansion shows that U is decreasing to the right of the origin, which contradicts the assumption that $U' \geq 0$ for all $y \neq y_0$. Similarly, if $U''(0) < 0$ then again U is decreasing to the right of the origin, which yields a similar contradiction. Finally, $U''(0)$ cannot be positive when $y_0 = 0$, since we assumed that A is the connection point.

Step 4: Define $\Phi(y)$ by

$$\Phi = \frac{\partial U}{\partial s}, \qquad U(0) = s \,.$$

At the fold point $s = s_c$ where $A'(s) = 0$, we obtain upon differentiating (3.4) that

$$\Phi'' = 2U\Phi, \qquad \Phi(0) = 1 \,. \tag{3.9}$$

To show that Φ is positive at the fold point, we define $\chi(y)$ by

$$\chi = \frac{\Phi}{U'} \,.$$

We readily derive that

$$\chi'' U' - 2y\chi + 2U''\chi' = 0 \,. \tag{3.10}$$

Since $\Phi(0) = 1$, and $U' > 0$ for $y > 0$, we obtain that χ is positive near the origin. In addition, for large y, (3.10) reduces to $\chi'' \sim 2y\chi$, which implies $\chi \to 0$ as $y \to \infty$. It follows by the maximum principle that $\chi > 0$. This shows the positivity of Φ. Finally, we establish that the fold point $A'(s) = 0$ is unique. Assuming that $A'(s) = 0.$, we differentiate (3.4) twice with respect to s to obtain

$$\Phi_s'' = 2U\Phi_s + 2\Phi^2 - A''(s) \,.$$

By multiplying both sides of this expression by Φ, and integrating the resulting expression by parts, we obtain

$$A''(s) = 2\frac{\int_0^\infty \Phi^3 \, dy}{\int_0^\infty \Phi \, dy} \,.$$

However, since Φ is positive then $A''(s) > 0$ whenever $A'(s) = 0$. This implies that the fold point is unique. ■

3.1 The Dimple Eigenfunction

Next, we study the qualitative properties of the eigenfunction pair associated with linearizing (2.2) around the steady-state solution at the fold point where $D = D_c$. We label the steady-state solution at the fold point $D = D_c$ by $u_c(x)$ and $w_c(x)$. From (3.5 b) and Theorem 2, we obtain at $D = D_c$ that

$$u_{c0} \equiv u_c(0) \sim 1 + \varepsilon^{2/3}\mathcal{B}^{1/3}U(0), \qquad U(0) = s_c = -0.61512 \,.$$

We linearize (2.2) around u_c and w_c by setting

$$u(x,t) = u_c(x) + e^{\lambda t}\phi(x), \qquad w(x,t) = w_c(x) + e^{\lambda t}\psi(x) \,.$$

This leads to the eigenvalue problem

$$\lambda\phi = \varepsilon^2\phi_{xx} - f_u(u_c, w_c)\,\phi - f_w(u_c, w_c)\,\psi, \qquad \frac{\lambda}{\alpha}\left[\phi + \tau(\psi - \phi)\right] = D\psi_{xx} - \beta_0\phi, \tag{3.11}$$

with $\psi_x = \phi_x = 0$ at $x = \pm L$ and $D = D_c$. Here $f(u, w)$ is defined in (2.7).

Let $u_0 = u(0)$ and $D = D(u_0)$. Then, if we define ϕ and ψ by

$$\phi = \left.\frac{\partial}{\partial u_0} u(x)\right|_{u_0 = u_{c0}}, \qquad \psi = \left.\frac{\partial}{\partial u_0} w(x)\right|_{u_0 = u_{c0}}, \tag{3.12}$$

it follows from (3.11) and $D'(u_{c0}) = 0$ that (3.12) is an eigenfunction pair corresponding to $\lambda = 0$.

We now construct an asymptotic approximation to this eigenpair ϕ, ψ of (3.11) corresponding to $\lambda = 0$. In particular, we show that ϕ is an even function that has a dimple shape when $D = D_c$. This is shown below to

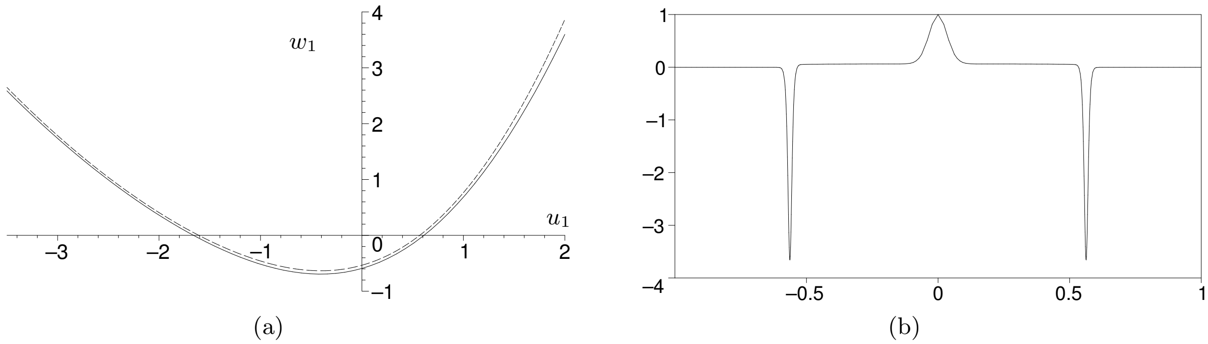


FIGURE 6. (a) The plot of $u_1(0)$ versus $w_1(0)$ computed from (3.5). The solid and dashed lines correspond to numerical computations of the full steady-state system (2.3) and the core problem (3.4), respectively. (b) The dimple eigenfunction ϕ at the fold point. The parameter values are $\varepsilon = 0.005$, $\beta_0 = 1.5$, with D determined numerically as a function of $u_1(0)$

be a consequence of the positivity of the function Φ in (3.9), together with the integral constraint $\int_0^L \phi\, dx = 0$, which is readily obtained from (3.11). We normalize this eigenfunction by imposing that $\phi(0) = 1$. For $\varepsilon \ll 1$ and $D = D_c$, our analysis below shows that the asymptotic structure of ϕ has four distinct regions: an inner region of width $O(\varepsilon^{2/3})$ near $x = 0$ where $\phi = O(1)$; an outer region on $x \in (0, l)$ where $\phi = O(\varepsilon^{2/3})$; an inner region of width $O(\varepsilon)$ near $x = l$ where $\phi = O(\varepsilon^{-1/3})$; and an outer region on $x \in (l, L]$ where $\phi = 0(\varepsilon)$. The first three regions give asymptotically comparable contributions of order $O(\varepsilon^{2/3})$ to the integral constraint $\int_0^L \phi\, dx = 0$, whereas the contribution from the fourth region can be neglected. For a particular set of parameter values the resulting dimple-shape of the eigenfunction ϕ at $D = D_c$ is shown in Fig. 6(b).

We now give the details of the asymptotic construction of ϕ. We begin with the internal layer region of width $O(\varepsilon^{2/3})$ near $x = 0$. In this region, we use (2.7) for $f(u, w)$ and (3.5 a) to calculate

$$f_u = 1 + 3u_c^2 - 2w_c u_c \sim 2\varepsilon^{2/3}\mathcal{B}^{1/3}U_c\,; \qquad f_w = -u_c^2 \sim -1\,. \tag{3.13}$$

Here $U_c(y)$ is the solution to the core problem (3.4) at the fold point location $A'(s_c) = 0$ where $D \sim D_c$. Using (3.13) in (3.11) with $\lambda = 0$, we obtain the following leading-order system on $0 \le y < \infty$:

$$\varepsilon^2\phi_{xx} - 2\varepsilon^{2/3}\mathcal{B}^{1/3}U_c\phi + \psi = 0\,; \qquad D_c\psi_{xx} - \beta_0\phi = 0\,, \tag{3.14}$$

with normalization condition $\phi(0) = 1$ and with $\phi_x = \psi_x = 0$ at $x = 0$. We then introduce the inner variables $y = x/(\varepsilon^{2/3}\mathcal{B}^{-1/6})$, $\phi = \Phi(y)$, and $\psi = \Psi(y)$. Then, (3.14) becomes

$$\Phi'' - 2U_c\Phi + \varepsilon^{-2/3}\mathcal{B}^{-1/3}\Psi = 0\,; \qquad D_c\Psi'' = \beta_0\varepsilon^{4/3}\mathcal{B}^{-1/3}\Phi\,. \tag{3.15}$$

This shows that $\Psi = O(\varepsilon^{4/3})$, and hence $\varepsilon^{-2/3}\Psi = O(\varepsilon^{2/3}) \ll 1$. Therefore, to leading-order, we obtain that Φ satisfies (3.9) of Theorem 2 at $s = s_c$, and that

$$\Psi \sim \frac{\beta_0}{D_c}\varepsilon^{4/3}\mathcal{B}^{-1/3}\Psi_0\,; \qquad \Psi_0'' = \Phi\,, \quad \Psi_0'(0) = 0\,. \tag{3.16}$$

Therefore, as $y \to \infty$, we have $\Psi_0' \sim \left(\int_0^\infty \Phi(y)\, dy\right)$. Writing the far-field expansion of Ψ in terms of the outer x-variable, we obtain the far-field matching condition

$$\Psi \sim \frac{\beta_0}{D_c}\varepsilon^{2/3}\mathcal{B}^{-1/6}\left(\int_0^\infty \Phi(y)\, dy\right) x\,. \tag{3.17}$$

Next, we analyze the inner region of width $O(\varepsilon)$ near the transition layer at $x = l$. We introduce the inner variables $y_l = (x - l)/\varepsilon$, $\phi = \Phi_l(y)$ and $\psi = \Psi_l(y)$. From (3.11) with $\lambda = 0$, we obtain on $-\infty < y_l < \infty$ that

$$\Phi_l'' - f_u\,(U_0, W_0)\,\Phi_l - f_w\,(U_0, W_0)\,\Psi_l = 0\,, \qquad \Psi_l'' = 0\,. \tag{3.18}$$

Here U_0 and W_0, given in (2.9), satisfy the leading-order steady-state inner problem (2.6). Upon comparing (2.6) and (3.18), we conclude that Φ_l is proportional to U_0' and that $\Psi_l = 0$. By using (2.9) to get U_0', we obtain

$$\phi \sim \Phi_l = c\,\mathrm{sech}^2\left(\frac{x-l}{2\varepsilon}\right), \qquad \psi \sim \Psi_l = 0\,. \tag{3.19}$$

Here c is an unknown constant, possibly depending on ε, that is found below by the global constraint $\int_0^L \phi\, dx = 0$.

In the outer region $x \in (l, L]$, where $u_c = 0$, we obtain the leading-order solution $\phi = \psi = 0$. A higher-order construction, which we omit, shows that $\psi = \phi = O(\varepsilon)$ in this near-boundary region. In contrast, in the outer mesa plateau region $x \in (0, l)$, we set $\lambda = 0$ in (3.11) to obtain $\phi = -\left[f_w(u_c, w_c)/f_u(u_c, w_c)\right]\psi$ and $D\psi_{xx} - \beta_0\phi = 0$. Then, by using the solution u_c and w_c to (2.11) at $D = D_c$, we obtain $f_w/f_u = -u_c^2/(u_c^2-1)$. The boundary conditions for ψ as $x \to 0$ and at $x = l$ are obtained by matching to (3.17) and (3.19), respectively. In this way, we obtain the following formulation of the leading-order outer problem for ψ and ϕ on $x \in (0, l)$:

$$D_c\psi_{0xx} - \frac{\beta_0 u_c^2}{u_c^2-1}\psi_0 = 0\,, \quad x \in (0,l)\,; \qquad D_c\psi_{0x} \to \beta_0 \quad \text{as} \quad x \to 0^+\,, \quad \psi_0(l) = 0\,. \tag{3.20 a}$$

In terms of ψ_0, we have

$$\psi \sim \varepsilon^{2/3}\mathcal{B}^{-1/6}\left(\int_0^\infty \Phi\, dy\right)\psi_0\,, \qquad \phi \sim \varepsilon^{2/3}\frac{u_c^2}{u_c^2-1}\mathcal{B}^{-1/6}\left(\int_0^\infty \Phi\, dy\right)\psi_0\,. \tag{3.20 b}$$

By the maximum principle the solution ψ_0 to (3.20 a), which depends only on β_0, satisfies $\psi_0 > 0$ on $x \in (0, l)$. Therefore, since $\Phi > 0$ from Theorem 2, we conclude that $\phi = O(\varepsilon^{2/3})$ with $\phi > 0$ on $x \in (0, l)$ and $\phi(l) = 0$.

Finally, we use the global condition $\int_0^L \phi\, dx = 0$ to calculate the constant c in (3.19). Upon using $\phi \sim \Phi$ for $x = O(\varepsilon^{2/3})$, together with (3.19) and (3.20 b) for ϕ in the plateau and transition regions, we estimate

$$\int_0^L \phi\, dx = \varepsilon^{2/3}\mathcal{B}^{-1/3}\left(\int_0^\infty \Phi\, dy\right) + \varepsilon^{2/3}\mathcal{B}^{-1/6}\left(\int_0^\infty \Phi\, dy\right)\left(\int_0^l \frac{u_c^2}{u_c^2-1}\psi_0\, dx\right) + \varepsilon c\int_{-\infty}^\infty \mathrm{sech}^2\left(y/2\right)\, dy\,,$$

$$0 = \varepsilon^{2/3}\mathcal{B}^{-1/6}\int_0^\infty \Phi\, dy\left(1 + \int_0^l \frac{u_c^2}{u_c^2-1}\psi_0\, dx\right) + 4\varepsilon c\,.$$

Therefore, the constant c in (3.19) satisfies

$$c \sim c_0\varepsilon^{-1/3}\,, \qquad c_0 \equiv -\frac{1}{4}\mathcal{B}^{-1/6}\left(\int_0^\infty \Phi\, dy\right)\left[1 + I(\beta_0)\right]\,, \qquad I(\beta_0) \equiv \int_0^l \frac{u_c^2}{u_c^2-1}\psi_0\, dx\,. \tag{3.21}$$

Here $\mathcal{B} > 0$ is defined in (3.2). Since $\int_0^\infty \Phi\, dy > 0$ by Theorem 2, and $\psi_0 > 0$ on $x \in (0, l)$, we get that $c_0 < 0$. Numerically, we compute from (3.9) that $\int_0^\infty \Phi\, dy \approx 1.1857$. Alternatively, $I(\beta_0)$ must be calculated numerically from the solution to (3.20 a). We remark that the integrand in $I(\beta_0)$ is well-defined as $x \to 0$, since although $u_c \to 1$ as $x \to 0^+$, we have $\psi_0 \sim \beta_0 x/D$ as $x \to 0^+$ to cancel the apparent singularity in the integrand.

We summarize the asymptotic construction of the dimple eigenfunction as follows:

Proposition 3 (Nishiura-Ueyama's Condition 2: Dimple eigenfunction) *Consider a single-mesa steady-state solution at the fold point $D = D_c$. Let ϕ be the corresponding eigenfunction. For $x = O(\varepsilon^{2/3})$, we have*

$$\phi \sim \Phi\left(\mathcal{B}^{1/6}\varepsilon^{-2/3}x\right)\,, \qquad \Phi(0) = 1\,.$$

Here $\mathcal{B} = (\beta_0 - 1)/(2D_c) > 0$ and $\Phi(y)$, defined in (3.9) of Theorem 2 at $s = s_c$, is a strictly positive function that decays at infinity. Alternatively, in an $O(\varepsilon)$ region near $x = l$, we have

$$\phi \sim c_0\,\varepsilon^{-1/3}\,\mathrm{sech}^2\left(\frac{x-l}{2\varepsilon}\right)\,,$$

where c_0 is the negative constant, independent of ε, given in (3.21). In the outer plateau region $0 < x < l$, then $\phi = O(\varepsilon^{2/3})$ is determined from (3.20), and this outer approximation for ϕ has a unique zero crossing at $x = l$. This establishes the dimple-shape of ϕ when $\varepsilon \ll 1$.

3.2 Universality of the Core Problem

In this section we show that the core problem can be derived for a class of reaction-diffusion systems that have steady-state mesa solutions. On $x \in [-L.L]$, we begin by constructing a single mesa steady-state solution for

$$u_t = \varepsilon^2 u_{xx} + a(u,v) \,, \qquad \sigma v_t = D v_{xx} - v + b(u,v) \,; \qquad u_x(\pm L, t) = v_x(\pm L, t) = 0 \,. \tag{3.22}$$

We assume that there exists three roots to $a(u,v) = 0$ on the interval $0 < v < v_m$ at $u = 0$, $u = u_-(v)$, and $u = u_+(v)$, with $0 < u_-(v) < u_+(v)$. Furthermore, we assume that

$$a_u(0,v) < 0 \,, \qquad a_u(u_-, v) > 0 \,, \qquad a_u(u_+, v) < 0 \,, \quad \text{for } \ 0 < v < v_m \,. \tag{3.23 a}$$

We write the two roots $u = u_\pm(v)$ on $0 < v < v_m$ as $v = h(u)$. When $v = v_m$ the two roots are assumed to coalesce so that $u_m \equiv u_-(v_m) = u_+(v_m)$ and $v_m = h(u_m)$. Furthermore, we assume that there exists a unique value v_c with $0 < v_c < v_m$ such that the Maxwell line condition

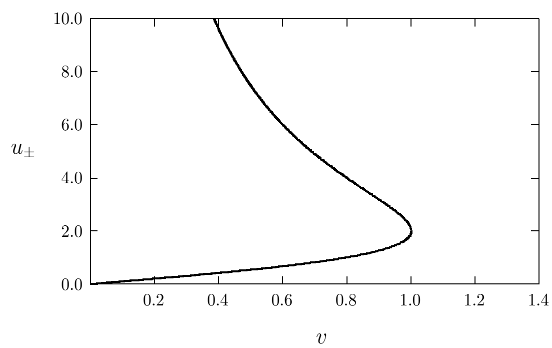
$$\int_0^{u_c} a(u, v_c)\, du = 0 \,, \qquad u_c \equiv u_+(v_c) \tag{3.23 b}$$

is satisfied. We also assume that $h'(u) < 0$ for $u > u_m$ and $h'(u) > 0$ for $u < u_m$. With these assumptions on $a(u,v)$, we conclude at the coalescence point that

$$a_{uu}^0 \equiv a_{uu}(u_m, v_m) < 0 \,, \qquad a_v^0 \equiv a_v(u_m, v_m) < 0 \,. \tag{3.23 c}$$

For the function $b(u,v)$ in (3.22) we will assume that

$$b(0,v) = 0 \,; \qquad g(u) \equiv h(u) - b\left[u, h(u)\right] < 0 \quad \text{for } \ u > u_m \,. \tag{3.23 d}$$



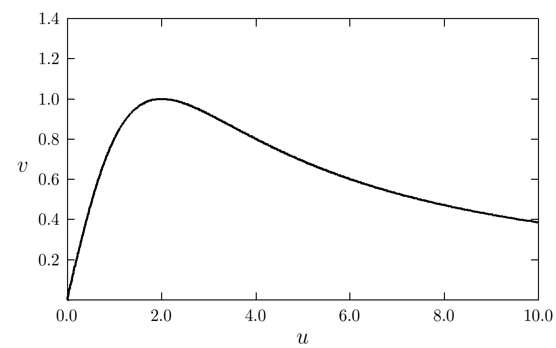


FIGURE 7. Left figure: Plot of $u_\pm(v)$ from (3.24) for the Gierer-Meinhardt model with saturation parameter $k = 0.25$. Right figure: corresponding plot of the inverse function $v = h(u)$ from (3.24).

A specific example of (3.22) is the Gierer-Meinhardt model with saturation where $a(u,v) = -u + u^2/[v(1+ku^2)]$ with $k > 0$, and $b(u,v) = u^2$. For this system we calculate

$$u_\pm(v) = \frac{1}{2kv}\left[1 \pm \sqrt{1 - 4kv^2}\right] \,, \qquad v = h(u) = \frac{u}{1+ku^2} \,, \qquad h'(u) = \frac{1-ku^2}{(1+ku^2)^2} \,. \tag{3.24}$$

Hence, $v_m = 1/[2\sqrt{k}]$, $u_m = 1/\sqrt{k}$, and $h'(u) < 0$ for $u > u_m$. In Fig. 7(a) we plot $u = u_\pm(v)$, and in Fig. 7(b) we plot $v = h(u)$. The Maxwell-line condition (3.23 b) is satisfied when (cf. [**17**])

$$v_c = \frac{0.4597}{\sqrt{k}}\,, \qquad u_c \equiv u_+(v_c) = \frac{1.515}{\sqrt{k}}\,. \tag{3.25}$$

In addition, we calculate from (3.23 d) that

$$g(u) = \frac{u}{1+ku^2}\left[1 - u(1+ku^2)\right]\,. \tag{3.26}$$

Since $g'(u) < 0$ for $u > u_m = k^{-1/2}$, and $g\left(k^{-1/2}\right) < 0$ when $0 < k < 4$, we get $g(u) < 0$ for $u > u_m$ when $0 < k < 4$.

We now return to the general case under the assumptions (3.23) and we construct a single-mesa steady-state solution of the type shown in Fig. 4. We first derive an expression for the critical value D_c of D for which no single-mesa steady-state solution exists when $D < D_c$.

Near the interface at $x = l$ we introduce the inner expansion

$$u = U_0\,(y) + \varepsilon U_1\,(y) + \ldots\,, \qquad v = V_0(y) + \varepsilon V_1\,(y) + \ldots\,, \qquad y = \varepsilon^{-1}(x-l)\,. \tag{3.27}$$

From the steady-state problem for (3.22), we obtain

$$U_0'' + a\,(U_0, V_0) = 0\,, \qquad V_0'' = 0\,, \tag{3.28 a}$$

$$U_1'' + a_u(U_0,V_0)U_1 = -a_v(U_0,V_0)V_1\,, \qquad V_1'' = 0\,. \tag{3.28 b}$$

The solution to the leading-order problem is $V_0 = v_c$, where v_c satisfies (3.23 b), and $U_0(y)$ is the unique heteroclinic connection satisfying

$$U_0(-\infty) = u_+(v_c) = u_c\,, \qquad U_0(\infty) = 0\,, \qquad U_0(0) = u_c/2\,. \tag{3.29}$$

At next order we obtain that $V_1 = V_{11}y + V_{12}$, for some constants V_{11} and V_{12}. The solvability condition for (3.28 b) determines V_{12} in terms of V_{11} as

$$V_{12}\int_{-\infty}^{\infty} a_v(U_0,v_c)\,U_0'\,dy = -V_{11}\int_{-\infty}^{\infty} a_v(U_0,v_c)\,y\,U_0'\,dy\,.$$

Then, by matching to the outer solution for v we obtain $V_{11} = v'(l^{\pm})$.

The outer problems for v determine $v'(l^{\pm})$. In the mesa region $0 \le x \le l$, where $v = h(u)$, we readily derive the outer problem

$$Dv'' = g\,(u)\;, \qquad 0 < x < l\,; \qquad v\,(l) = v_c\,, \qquad v'\,(0) = 0\,. \tag{3.30}$$

Here $g(u)$ is defined in (3.23 d). The corresponding u is given by $u(x) = u^+[v(x)]$ with $u(l) = u^+(v_c) \equiv u_c$. We require that $0 < v < v_m$ at each $x \in (0,l)$ so that $u > u_m$ on $x \in (0,l)$. In contrast, since $b(0,v) = 0$ by (3.23 d), we obtain in the outer region $l \le x \le L$ that $u = 0$ and that

$$Dv'' = v\,, \qquad l < x < L\,; \qquad v\,(l) = v_c\,, \qquad v'\,(L) = 0\,. \tag{3.31}$$

Since V_{11} is a constant, the solutions to (3.30) and (3.31) are joined by the condition that $v'(l^-) = v'(l^+)$.

The reduction of (3.30) and (3.31) to a quadrature relating $u(0) \equiv u_0$ to the length of the domain L is very similar to that done for the Brusselator. We first multiply (3.30) by $v' = h'(u)u'$ and integrate to get

$$\frac{Dv'^2}{2} = F\,(u;u_0)\;, \qquad F\,(u;u_0) \equiv \int_{u_0}^{u} g\,(s)\,h'(s)\,ds\,.$$

Since $h'(u) < 0$ and $g(u) < 0$ for $u > u_m$ (see (3.23 d)), we obtain that $F(u;u_0) > 0$ for $u > u_0$. By taking the negative square root, we calculate

$$\frac{dv}{dx} = -\sqrt{\frac{2F(u;u_0)}{D}} < 0\,, \qquad \frac{du}{dx} = -\sqrt{\frac{2}{D}}\,\frac{\sqrt{F\,(u;u_0)}}{h'(u)} > 0\,. \tag{3.32}$$

By integrating (3.32) with $u(0) = u_0$ and $u(l) = u_c = u_+(v_c)$, we obtain a relation between l and u_0

$$-\frac{l}{\sqrt{D}} = \int_{u_0}^{u_+(v_c)} \frac{h'(u)}{\sqrt{2F(u;u_0)}}\,du = \frac{\sqrt{2F(u_+(v_c);u_0)}}{g\left[u_+(v_c)\right]} + \int_{u_0}^{u_+(v_c)} \frac{g'(u)}{\left[g(u)\right]^2}\sqrt{2F(u;u_0)}\,du < 0\,. \tag{3.33}$$

In the outer region $l < x < L$, we solve (3.31) to obtain

$$v(x) = v_c \left(\frac{\cosh\left[(L-x)/\sqrt{D}\right]}{\cosh\left[(L-l)/\sqrt{D}\right]} \right), \qquad v'(l^+) = -\frac{v_c}{\sqrt{D}}\tanh\left[(L-l)/\sqrt{D}\right]. \tag{3.34}$$

By setting $v'(l^-) = v'(l^+)$, we obtain that

$$\frac{(L-l)}{\sqrt{D}} = \tanh^{-1}\left(\frac{\sqrt{2F(u_+(v_c);u_0)}}{v_c} \right), \qquad \text{when} \quad \frac{\sqrt{2F(u_+(v_c);u_0)}}{v_c} < 1\,. \tag{3.35}$$

Finally, upon combining (3.33) and (3.35), we obtain the following expression relating $u_0 \equiv u(0)$ to L:

$$\frac{L}{\sqrt{D}} = \chi(u_0) \equiv \tanh^{-1}\left(\frac{\sqrt{2F(u_+(v_c);u_0)}}{v_c} \right) - \frac{\sqrt{2F(u_+(v_c);u_0)}}{g\left[u_+(v_c)\right]} - \int_{u_0}^{u_+(v_c)} \frac{g'(u)}{\left[g(u)\right]^2}\sqrt{2F(u;u_0)}\,du\,. \tag{3.36}$$

Noting that $\frac{dF(u;u_0)}{du_0} < 0$ for $u > u_0$, it follows that $\chi(u_0)$ is a decreasing function of u_0 when $u_0 > u_m$. Therefore, for the existence of a single mesa steady-state solution, we require that $D > D_c$, where $D_c \equiv L^2/[\chi(u_m)]^2$.

Next, we show that the core problem (3.4) is universal in the sense that it determines the local internal layer solution behavior near the origin when $D = D_c$ for the class of systems (3.22). In the this layer near $y = 0$ we expand

$$u = u_m + \delta u_1 + \cdots\,, \qquad v = v_m + \delta^2 v_1 + \cdots\,, \qquad z = x/\delta\,, \qquad D = D_c + \cdots\,, \tag{3.37}$$

where $\delta \ll 1$. The nonlinear terms in (3.22) are calculated as

$$a(u,v) \sim a^0 + a_u^0(u-u_m) + \frac{a_{uu}^0}{2}(u-u_m)^2 + a_v^0(v-v_m) + \cdots \sim \delta^2\left(\frac{u_1^2}{2}a_{uu}^0 + a_v^0 v_1\right). \tag{3.38}$$

Here the superscript 0 denotes the evaluation of partial derivatives of a at $u = u_m$ and $v = v_m$. In obtaining (3.38) we used $a^0 = a_u^0 = 0$. By substituting (3.37) and (3.38) into the steady-state problem for (3.22), and by choosing $\delta = \varepsilon^{2/3}$, we obtain

$$u_{1zz} + \frac{a_{uu}^0}{2}u_1^2 + a_v^0 v_1 = 0\,, \qquad D_c v_{1zz} = g(u_m)\,. \tag{3.39}$$

Here $g(u)$, with $g(u_m) < 0$, is defined in (3.23 *d*). The solution for v_1 is written as

$$v_1 = -\mathcal{A} - \mathcal{B}z^2\,, \qquad \mathcal{B} = -\frac{g(u_m)}{2D_c} > 0\,, \qquad \mathcal{A} = -v_1(0)\,. \tag{3.40}$$

Then, (3.39) for u_1 becomes

$$u_{1zz} + \frac{a_{uu}^0}{2}u_1^2 - a_v^0\left(\mathcal{A} + \mathcal{B}z^2\right) = 0\,. \tag{3.41}$$

From (3.23 *c*) we recall that $a_{uu}^0 < 0$ and $a_v^0 < 0$. Finally, we rescale (3.41) by introducing C, μ, and A, by $u_1 = CU$, $z = \mu y$, and $\mathcal{A} = \mathcal{B}\mu^2 A$. Then, (3.41) is transformed precisely to the core problem (3.4) for $U(y)$ and A when

$$\mu = \left(\frac{2}{a_{uu}^0 a_v^0}\right)^{1/6}\mathcal{B}^{-1/6}\,, \qquad C = -\frac{2}{a_{uu}^0}\left(\frac{2}{a_{uu}^0 a_v^0}\right)^{-1/3}\mathcal{B}^{1/3}\,, \qquad \mathcal{A} = \mathcal{B}\mu^2 A\,. \tag{3.42}$$

By combining these transformations, we obtain the following characterization of the internal layer near the origin:

$$u - u_m \sim \varepsilon^{2/3} C U(y) \sim -\frac{2\varepsilon^{2/3}}{a^0_{uu}} \left(\frac{2}{a^0_{uu} a^0_v} \right)^{-1/3} \mathcal{B}^{1/3}\, U(y)\,, \tag{3.43 a}$$

$$v - v_m \sim \varepsilon^{4/3} v_1 \sim -\varepsilon^{4/3} \mathcal{B}^{2/3} \left(\frac{2}{a^0_{uu} a^0_v} \right)^{1/3} (A + y^2)\,, \tag{3.43 b}$$

$$y = \frac{x}{\mu \varepsilon^{2/3}} = \frac{x}{\varepsilon^{2/3}} \left(\frac{2}{a^0_{uu} a^0_v} \right)^{-1/6} \mathcal{B}^{1/6}\,. \tag{3.43 c}$$

Here $\mathcal{B}$ is defined in (3.40).

Using the result from Theorem 2 for the core problem (3.4), we conclude that the bifurcation diagram near the existence threshold of D for a single mesa steady-state solution of (3.22) has a saddle-node structure. Recall that at the saddle-node point $U(0) \approx -0.61512 < 0$ and $A = A_c \approx -1.46638 < 0$ (see Theorem 2). Therefore, from (3.43), we have $u(0) < u_m$ and $v(0) > v_m$ at this point, as expected.

The constants in (3.43) must be calculated for each specific reaction-diffusion system. In particular, for the Gierer-Meinhardt model with saturation where $a(u,v) = -u + u^2/[v(1+ku^2)]$, $b(u,v) = u^2$, $u_m = 1/\sqrt{k}$, and $v_m = 1/[2\sqrt{k}]$, we readily calculate that

$$a^0_v = -2\,, \qquad a^0_{uu} = -\sqrt{k}\,, \qquad g(u_m) = \frac{1}{2\sqrt{k}} \left[1 - \frac{2}{\sqrt{k}} \right] \quad \text{with } 0 < k < 4\,. \tag{3.44}$$

The existence threshold D_c can be computed numerically from (3.36) for a given domain half-length L and saturation parameter k.

Finally, we remark on the local behavior of the time-dependent solution to (3.22) in the internal layer region. If we substitute (3.37) with $u_1 = u_1(z,t)$ and $v_1 = v_1(z,t)$, we readily obtain that

$$\varepsilon^{-2/3} u_{1t} = u_{1zz} + \frac{a^0_{uu}}{2} u_1^2 + a^0_v v_1\,, \qquad \sigma \varepsilon^{4/3} v_{1t} = D_c v_{1zz} - g(u_m)\,. \tag{3.45}$$

We then introduce C and τ defined by $t = \varepsilon^{-2/3} \mu^2 \tau$ and $u_1 = CU$. In this way we obtain, $\sigma \varepsilon^2 \mu^{-2} v_{1t} = D_c v_{1zz} - g(u_m)$. Thus, v_1 is quasi-steady, and $D_c v_{1zz} = g(u_m)$. The corresponding equation for $U(y,\tau)$ is

$$U_\tau = U_{yy} - U^2 + A + y^2\,. \tag{3.46}$$

If we take $A < A_c$ and even initial data $U(y,0)$ with $U(0,0) < s_c$, which is below the existence threshold for the steady-state core problem, then (3.46) should exhibit the finite-time blowup $U \to -\infty$ as $\tau \to T^-$. The local structure of the solution near the blowup point $y = 0$ and $\tau = T$ is independent of the lower-order terms $A + y^2$ in (3.46), and is given from [**7**] as

$$U(y,\tau) \sim -(T-\tau)^{-1/2} \left[1 + \frac{1}{4|\log(T-\tau)|} - \frac{y^2}{8(T-\tau)|\log(T-\tau)|} \right]. \tag{3.47}$$

The analysis leading to (3.46) is, of course, not a valid description of the solution to (3.22) when $\tau \to T^-$ since full nonlinear effects in (3.22) must be accounted for near the singularity time. However, this analysis does suggest the formation of a *large amplitude* finger, such as shown in Fig. 1(b), when D is reduced significantly below D_c.

4 The Stability of the Mesa Pattern

In this section we show that the steady-state K-mesa pattern is stable when $D > D_c$ and $\tau = 0$. We linearize (2.2) around this steady-state solution by letting

$$u\,(x,t) = u\,(x) + e^{\lambda t} \phi(x)\,, \qquad w\,(x,t) = w\,(x) + e^{\lambda t} \psi(x)\,,$$

to obtain (3.11). Upon setting $\tau = 0$ in (3.11), we obtain the eigenvalue problem

$$\lambda \phi = \varepsilon^2 \phi_{xx} - f_u\,(u,w)\,\phi - f_w\,(u,w)\,\psi\,, \qquad \frac{\lambda}{\alpha} \phi = D \psi_{xx} - \beta_0 \phi\,, \tag{4.1}$$

where $f(u,w)$ is defined in (2.7). The main result of this section is as follows:

Theorem 4 (Nishiura-Ueyama's Condition 3) *Consider a symmetric K-mesa steady-state solution as constructed in Proposition 1 on a domain of length $2KL$, with $D > D_c$. When $\tau = 0$, the spectrum of (4.1) admits only real eigenvalues with $\lambda < 0$.*

To show Theorem 4, we first reformulate (4.1) as a singular limit eigenvalue problem (SLEP) in the limit $\varepsilon \to 0$ with $\alpha = O\left(\varepsilon^2\right)$, in order to derive a reduced set of equations, independent of ε, for the eigenvalues. The following Lemma characterizes this reduced system and its eigenvalues:

Lemma 5 (SLEP reduction) *Let λ be an eigenvalue associated with the K-mesa steady-state solution w, u on an interval of length $2KL$, with an interface at $x = l$, as described in Proposition 1. The leading $2K$ eigenvalues of the eigenvalue problem (4.1) are of order $O\left(\alpha\right)$ with $\alpha = O(\varepsilon^2)$. These eigenvalues are characterized as follows: Define λ_1 by*

$$\lambda = \alpha\lambda_1 \,,$$

where $\alpha = O(\varepsilon^2)$, and let u_e, u_o be the solutions of the differential equation

$$D\psi'' - \left(\beta_0 + \lambda_1\right)\frac{u'}{w'}\psi = 0 \,,$$

satisfying the boundary conditions

$$u_o\left(0\right) = 0 \,, \quad u_o'\left(l\right) = 1 \,; \qquad u_e'\left(0\right) = 0 \,, \quad u_e'\left(l\right) = 1 \,.$$

Then λ_1 satisfies

$$-\frac{1}{\sigma}\left(\beta_0 + \lambda_1\right) = \frac{L-l}{\sqrt{2}} \,, \tag{4.2}$$

where σ is one of the eigenvalues of the $2K \times 2K$ matrix

$$M = \begin{bmatrix} -\frac{b}{\delta} & \frac{a}{\delta} & & & & & & \\ \frac{a}{\delta} & -\frac{b}{\delta} - \frac{1}{2d} & \frac{1}{2d} & & & & & \\ & \frac{1}{2d} & -\frac{b}{\delta} - \frac{1}{2d} & \frac{a}{\delta} & & & & \\ & & \frac{a}{\delta} & -\frac{b}{\delta} - \frac{1}{2d} & \ddots & & & \\ & & & \ddots & \ddots & \frac{1}{2d} & & \\ & & & & \frac{1}{2d} & -\frac{b}{\delta} - \frac{1}{2d} & \frac{a}{\delta} & \\ & & & & & \frac{a}{\delta} & -\frac{b}{\delta} & \end{bmatrix} . \tag{4.3}$$

The entries of the matrix M are

$$a = \frac{u_e\left(l\right) - u_o\left(l\right)}{2} \,, \qquad b = \frac{u_e\left(l\right) + u_o\left(l\right)}{2} \,, \qquad \delta = b^2 - a^2 \,, \qquad d = L - l \,. \tag{4.4}$$

The eigenvalues of M are given explicitly by

$$\sigma_{j\pm} = -\frac{1}{2d} - \frac{b}{\delta} \pm \sqrt{\left(\frac{a}{\delta}\right)^2 + \left(\frac{1}{2d}\right)^2 + \frac{a}{d\delta}\cos\left(\frac{\pi j}{K}\right)} \,, \quad j = 1, \ldots, K-1 \,; \qquad \sigma_{0\pm} = -\left(\frac{1}{b \pm a}\right) . \tag{4.5}$$

For the special case of one mesa, where $K = 1$, there are two small eigenvalues corresponding to either an even or an odd eigenfunction. These eigenvalues satisfy

$$\left(\beta_0 + \lambda_1\right) u_e\left(l\right) = \frac{L-l}{\sqrt{2}} \qquad \textit{and} \qquad \left(\beta_0 + \lambda_1\right) u_o\left(l\right) = \frac{L-l}{\sqrt{2}} \,. \tag{4.6}$$

The proof of this lemma is given in Appendix A. Our SLEP analysis in Appendix A, although somewhat similar to that in [**22**], [**23**], and [**25**], for the generalized Fitzhugh-Nagumo model, shows that $\lambda = O(\varepsilon^2)$ for the Brusselator model. In contrast, the analysis in [**22**], [**23**], and [**25**], showed that $\lambda = O(\varepsilon)$ for the generalized Fitzhugh-Nagumo model. This difference in the asymptotic order of the small eigenvalues results from the presence of the cross-term $\frac{\lambda}{\alpha}\phi$ in the Brusselator linearization (4.1). We now use Lemma 5 to prove Theorem 4.

Proof of Theorem 4. Define μ by

$$\mu = \lambda_1 + \beta_0 \,.$$

From (2.10) we obtain on the interval $x \in (0,l)$ that $\frac{u'}{w'} = \frac{u^2}{u^2-1} > 0$ since $u \in \left(1, \sqrt{2}\right]$ on this interval. With this preliminary result, the proof of Theorem 4 consists of four steps.

Step 1: Let $\mu = \lambda_1 + \beta_0$ and define $f(\mu) = \mu u_o(l)$. In this step we will show that the function $\mu \to u_0(l)$ is decreasing whereas $f(\mu)$ is increasing for all $\mu > 0$.

The former claim is easy to show. Indeed, let u_i be a solution of $u'' - h_i u = 0$ with $u_i(0) = 0,\ u_i'(l) = 1$, for $i = 1, 2$, and with $0 < h_1(x) < h_2(x)$. Then, from the comparison principle, we find that $u_1 > u_2$. Now take $0 < \mu_1 < \mu_2$. Applying this comparison principle with $h_1 = \frac{\mu_1}{D}\frac{u'}{w'}$ and $h_2 = \frac{\mu_2}{D}\frac{u'}{w'}$, we immediately find that $u_o(l;\mu_1) > u_o(l;\mu_2)$.

Next we show the more difficult result that $f(\mu)$ is increasing. Define $h(x) = \frac{1}{D}\frac{u'}{w'}$ so that u_0 satisfies

$$u_o'' - \mu h(x) u_o = 0\,, \quad h > 0\,; \qquad u_o(0) = 0\,, \quad u_o'(l) = 1\,. \tag{4.7}$$

Define v and v_μ by $v = \frac{\partial}{\partial\mu}(\mu u_0)$ and $v_\mu = \frac{\partial}{\partial\mu} v$. Then, we readily obtain

$$v'' - \mu h v = \mu h u_o\,; \qquad v(0) = 0\,, \quad v'(l) = 1\,,$$
$$v_\mu'' - \mu h v_\mu = 2hv\,; \qquad v_\mu(0) = 0\,, \quad v_\mu'(l) = 0\,.$$

First note that by the maximum principle, $u_o > 0$ for all $x \in (0,l)$ so that $v'' - \mu h v > 0$ in $(0,l)$. Now suppose that $v(l) \leq 0$. Then, by the maximum principle, $v < 0$ for all $x \in (0,l)$. But this implies that $v_\mu - \mu h v_\mu < 0$ inside $(0,l)$. It then follows by the maximum principle that $v_\mu > 0$ for all $x \in (0,l)$. Since $f'(\mu) = v(l)$ and $f''(\mu) = v_\mu(l)$, we conclude that $f''(\mu) > 0$ whenever $f'(\mu) < 0$. It follows that f has no local maximum. Therefore, to complete the proof of Step 1, it suffices to show that $f'(0) > 0$.

For $\mu \ll 1$, the leading-order solution to (4.7) is $u_o(x) \sim x$. It follows that $f(\mu) \sim \mu l$ as $\mu \to 0$ so that $f'(0) = l > 0$. This completes the proof of Step 1.

Step 2: We show that $\sigma_{\min} < \sigma < 0$ where

$$\sigma_{\min} \equiv -\left(\frac{1}{u_o(l)} + \frac{1}{d}\right)\,, \qquad d = L - l\,. \tag{4.8}$$

To show this result, we must establish that $0 < u_o(l) < u_e(l)$, where $u_o(l) = b - a$, and $u_e(l) = b + a$ from (4.4). This result follows from a comparison principle, which yields that $0 < u_o(x) < u_e(x)$ for all $x \in (0, l]$. A simple calculation shows that the result $\sigma_{\min} < \sigma < 0$ readily follows from (4.5) upon using $0 < u_o(l) < u_e(l)$. This completes the proof of Step 2.

Step 3: Since $\sigma > \sigma_{\min}$, we derive that

$$-\frac{1}{\sigma}(\beta_0 + \lambda_1) > \mathcal{G}(\mu) \equiv \frac{\mu u_o(l)}{\frac{1}{L-l}u_o(l) + 1}\,. \tag{4.9}$$

We now calculate $\mathcal{G}(\beta_0)$ corresponding to $\lambda_1 = 0$. When $\lambda_1 = 0$, (4.7) becomes

$$u_o'' - \frac{u'}{w'}\beta_0 u_o = 0\,; \qquad u_o(0) = 0\,, \quad u_o'(l) = 1\,. \tag{4.10}$$

By differentiating (2.22), we note that w' satisfies (4.10) on $[0,l]$ with $w'(0) = 0$. Therefore, $u_o(x) = w'(x)/w''(l)$. We then calculate using (2.22) that

$$w''(l) = \frac{1}{D}\left(\beta_0\sqrt{2} - 1\right) \qquad \text{and} \qquad w'(l) = \frac{L-l}{D}\,.$$

Therefore, for $\mu = \beta_0,$ we obtain $u_o\,(l) = (L-l)/(\beta_0\sqrt{2}-1)$ and consequently

$$\mathcal{G}(\beta_0) \equiv \frac{\beta_0 u_o\,(l)}{\frac{1}{L-l}u_o\,(l)+1} = \frac{L-l}{\sqrt{2}} \quad \text{when} \quad \mu = \beta_0\,.$$

Next, by Step 1, we readily find that $\mathcal{G}(\mu)$, defined in (4.9), is an increasing function of μ. Therefore,

$$-\frac{1}{\sigma}\,(\beta_0+\lambda_1) > \mathcal{G}(\beta_0) \equiv \frac{L-l}{\sqrt{2}} \quad \text{for all} \quad \mu > \beta_0\,.$$

This contradicts (4.2). We conclude that no solution to (4.2) exists if λ_1 is real and positive.

Step 4: To complete the proof of Theorem 4, it suffices to show that all roots λ_1 to (4.2) are purely real. To do so, we decompose M into the two block-diagonal matrices

$$M = M_1 + M_2\,,$$

where

$$M_1 = \begin{bmatrix} \boxed{\begin{matrix} -\frac{b}{\delta} & \frac{a}{\delta} \\ \frac{a}{\delta} & -\frac{b}{\delta} \end{matrix}} & & & \\ & \boxed{\begin{matrix} -\frac{b}{\delta} & \frac{a}{\delta} \\ \frac{a}{\delta} & -\frac{b}{\delta} \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} -\frac{b}{\delta} & \frac{a}{\delta} \\ \frac{a}{\delta} & -\frac{b}{\delta} \end{matrix}} \end{bmatrix}; \quad M_2 = \begin{bmatrix} 0 & & & & \\ & \boxed{\begin{matrix} -\frac{1}{2d} & \frac{1}{2d} \\ \frac{1}{2d} & -\frac{1}{2d} \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} -\frac{1}{2d} & \frac{1}{2d} \\ \frac{1}{2d} & -\frac{1}{2d} \end{matrix}} & \\ & & & & 0 \end{bmatrix}.$$

We first note that (4.2) is equivalent to

$$Mv = -\mu\left(\frac{\sqrt{2}}{L-l}\right)v\,, \tag{4.11}$$

where v is an eigenvector corresponding to σ and $\mu = \lambda_1 + \beta_0$. Let v_k be the k^{th} component of v. Then, upon using $\delta = b^2 - a^2$, we calculate the inner product as

$$\bar{v}^t M_1 v = -\frac{b}{\delta}\left(|v_1|^2 + |v_2|^2 + \cdots + |v_{2K-1}|^2 + |v_{2K}|^2\right) + \frac{a}{\delta}\left(v_1\overline{v_2} + v_2\overline{v_1} + \cdots + v_{2K-1}\overline{v_{2K}} + v_{2K}\overline{v_{2K-1}}\right)\,, \tag{4.12 a}$$

$$= -\frac{1}{2(b-a)}\left(|v_1 - v_2|^2 + \cdots + |v_{2K-1} - v_{2K}|^2\right) - \frac{1}{2(b+a)}\left(|v_1 + v_2|^2 + \cdots + |v_{2K-1} + v_{2K}|^2\right)\,, \tag{4.12 b}$$

$$= -\frac{C_2}{u_o\,(l)} - \frac{C_1}{u_e\,(l)}\,. \tag{4.12 c}$$

Here and below C_i denotes a non-negative constant that may change from line to line. Similarly, we obtain

$$\bar{v}^t M_2 v = -\frac{1}{2d}\left(|v_2 - v_3|^2 + \cdots + |v_{2K-2} - v_{2K-1}|^2\right) = -C_3\,. \tag{4.13}$$

We premultiply (4.11) by $\bar{v}^t$, we use (4.12) and (4.13), and we then divide the resulting expression by μ to obtain

$$\frac{C_1}{\mu u_e\,(l)} + \frac{C_2}{\mu u_o\,(l)} + \frac{C_3}{\mu} = C_4\,.$$

This equation can be rewritten as

$$C_1\overline{\mu u_e\,(l)} + C_2\overline{\mu u_o\,(l)} + C_3\bar{\mu} = C_4\,. \tag{4.14}$$

From the expressions for C_i in (4.12) and (4.13) it follows that at least one of the C_1, C_2, or C_3 is strictly positive for any $v \neq 0$. Next, we return to the equation for u_o,

$$Du_o'' - \mu \frac{u'}{w'} u_o = 0 \,; \qquad u_o\,(0) = 0\,, \quad u_o'\,(l) = 1\,.$$

We multiply this equation by $\overline{u_o}$ and then integrate the resulting expression by parts to get

$$\overline{u_o}\,(l) = \int_0^l |u_o'|^2 \; dx + \frac{\mu}{D} \int_0^l \frac{u'}{w'} \, |u_o|^2 \; dx\,.$$

Multiplying this expression by $\overline{\mu}$ we obtain

$$\overline{\mu u_o\,(l)} = \bar{\mu} B_5 + B_6\,.$$

In a similar way, we derive

$$\overline{\mu u_e\,(l)} = \bar{\mu} B_7 + B_8\,.$$

Here B_5, B_6, B_7 and B_8 are strictly positive constants. Substituting these expressions into (4.14) we conclude that

$$\bar{\mu}\,(C_1 B_5 + C_2 B_7 + C_3) = B\,,$$

Here B is a real constant, and we note that $C_1 B_5 + C_2 B_7 + C_3$ is strictly positive for any $v \neq 0$. Finally, by taking the imaginary part of this expression, we get $\mathrm{Im}\,(\mu) = \mathrm{Im}\,\lambda_1 = 0$. This concludes the proof of Theorem 4.

5 Discussion

In [**15**] the one-dimensional Brusselator model was analyzed with the following scaling,

$$\tau_k u_t = \varepsilon_k D_k u_{xx} + \varepsilon_k A_k + u^2 v - (B_k + \varepsilon_k)\, u\,, \qquad v_t = \varepsilon_k D_k v_{xx} + B_k u - u^2 v\,,$$

on the interval $x \in [0,1]$. The assumptions on the parameters were that $\varepsilon_k D_k \ll 1$, $D_k \gg 1$, $A_k = O(1)$, and $B_k = O(1)$. This model is equivalent to (1.2) after the change of variables $u = a\hat{u}$, $v = a\hat{v}$, $t = \frac{\tau_k}{a^2}\hat{t}$, with $a = \sqrt{B_k + \varepsilon_k}$, and after dropping the hat notation. The parameters are mapped to

$$\varepsilon = \sqrt{\frac{\varepsilon_k D_k}{B_k + \varepsilon_k}}\,; \qquad \beta_0 = \frac{\sqrt{B_k + \varepsilon_k}}{A_k}\,; \qquad D = D_k \beta_0\,; \qquad \tau = \frac{1}{\tau_k}\,.$$

One of the main results in [**15**] was that a K-mesa configuration with $K \geq 2$ is unstable only when

$$D_k > \frac{1}{K^2} D_{kc} \quad \text{where} \quad D_{kc} \sim \begin{cases} \dfrac{A_k^2}{2\mathcal{E}_k \ln^2\left(\frac{12\sqrt{2}A_k B_k^{3/2}}{\mathcal{E}_k(\sqrt{2B_k}-A_k)^2}\right)}, & 2A_k^2 < B_k\,, \\[2ex] \dfrac{\left(\sqrt{2B_k}-A_k\right)^2}{2\mathcal{E}_k \ln^2\left(\frac{12\sqrt{2}}{\mathcal{E}_k A_k} B_k^{3/2}\right)}, & 2A_k^2 > B_k\,. \end{cases} \tag{5.1}$$

For D_k above this stability threshold a coarsening phenomenon was observed in [**15**]. This process resulted in the annihilation of some mesas over an exponentially long time scale, until eventually the number of mesas was decreased sufficiently so that (5.1) no longer holds.

The results in this paper together with [**15**] provide analytical bounds on D for the existence and stability of a steady-state K-mesa pattern. When $\beta_0 > \sqrt{2}$, (5.1) reduces to

$$D > \frac{\left(\sqrt{2}\beta_0 - 1\right)^2}{12\sqrt{2}\beta_0} \varepsilon^2 \exp\left(\frac{1}{\sqrt{2}K\beta_0\varepsilon}\right).$$

This provides an exponentially large upper bound on D for the stability of K mesas. Roughly speaking, the stability of K-mesas when τ is sufficiently small is guaranteed on the range

$$O\left(\frac{1}{K^2}\right) \ll D \ll O\left(\varepsilon^2 \exp\left(\frac{1}{\sqrt{2}K\beta_0\varepsilon}\right)\right).$$

If D exceeds an exponentially large upper bound, then the number of mesas is diminished through a coarsening process. Alternatively, if D is too small, then self-replication is observed until such time that DK^2 is large enough.

There are several open problems. The first problem is to study the stability and dynamics of equilibrium and quasi-equilibrium mesa patterns when $\tau > 0$. In [**15**], it was shown that when $D \gg 1$, there is a Hopf bifurcation that occurs for $\tau \sim 1$. However, the analysis there relied on explicit analytical calculations of the small eigenvalues, which is not possible when $D = O(1)$. Moreover, it was shown in [**15**], under some additional conditions, that a Hopf bifurcation can lead to a breather-type instability whereby the center of the mesa remains stationary, but its width slowly oscillates in time. For D above the self-replication threshold of a single-mesa steady-state solution, we suggest that such a breather-type instability for τ sufficiently large can trigger a dynamic mesa self-replication event if the time-oscillating mesa plateau width exceeds its maximum allowable steady-state value. Such a triggering mechanism for mesa self-replication is explored for a reaction-diffusion system with piecewise-linear kinetics in [**10**]. In addition, when $D = O(1)$, some numerical simulations (not shown) suggest that an oscillatory traveling-wave instability is also possible, whereby the position of the center of the mesa oscillates in time, while its width remains constant.

The second area of open problems is to extend the study of the existence and stabilty of mesa patterns to two or three dimensions. Numerical simulations suggest a slew of possible patterns. One possibility is to study radially symmetric patterns in a ball. One can then obtain a blob-like pattern. Self-replication of such a pattern can occur as D is decreased sufficiently, and we expect the core problem in two dimensions to be (3.4) with y replaced by $|y|$. The study of mesa blob-type patterns in an arbitrary two-dimensional domain is also open. Another possibility is to trivially extend the one-dimensional mesa pattern into the second dimension. The resulting mesa-stripe pattern can exhibit transverse instabilities. This stability problem was examined in [**17**] for the Gierer-Meinhardt model with saturation in the near-shadow limit where mesa self-replication does not occur. It is an open problem to perform a similar stability analysis in the mesa self-replication regime. Such an analysis would have implications for establishing stable parameter regimes where mesa-stripe replication can occur on a growing domain. The analysis of these and related problems in two dimensions is the subject of future work.

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Appendix A The SLEP Reduction

Proof of Lemma 5. We label the interface locations by

$$x_{1-} < x_{1+} < x_{2-} < x_{2+} < \cdots < x_{K-} < x_{K+}\,,$$

as illustrated in Fig. 2(b). We then define l and d by

$$l = (x_{i+} - x_{i-})/2\,, \qquad d = (L - l) = (x_{(i+1)-} - x_{i+})/2\,.$$

By symmetry l and d are independent of i.

In the inner region near $x_{i\pm}$, we introduce the inner variables

$$\phi = \Phi\,(y) = \Phi_0 + \varepsilon\Phi_1 + \cdots\,, \qquad \psi = \Psi\,(y) = \Psi_0 + \varepsilon\Psi_1 + \cdots\,, \qquad \lambda = \alpha\lambda_1 + \cdots\,, \qquad y = \varepsilon^{-1}(x - x_{i\pm})\,, \quad (A.1)$$

where $\alpha = \varepsilon^2\alpha_0$. Upon substituting (A.1) into (4.1), we obtain the leading-order system

$$\Phi_0'' - f_u\,(U_0, W_0)\,\Phi_0 - f_w\,(U_0, W_0)\,\Psi_0 = 0\,, \qquad \Psi_0'' = 0\,.$$

Here U_0 and W_0, satisfying (2.6), are given in (2.9). We take the + sign for U_0 in (2.9) for the inner region near

$x = x_{i-}$, and the $-$ sign in (2.9) for the region near $x = x_{i+}$. By differentiating (2.6) with respect to y, we get

$$\Phi_0 = c_{i\pm} U_0'(y) \,, \qquad \Psi_0 = 0 \,, \tag{A.2}$$

where $c_{i\pm}$, for $i = 1, \ldots, K$, are constants to be determined.

Since $\Psi_0 \equiv 0$, we obtain the following problem for Φ_1 and Ψ_1 at next order:

$$\Phi_1'' - f_u(U_0, W_0)\,\Phi_1 - f_w(U_0, W_0)\,\Psi_1 = \Phi_0 \left[f_{uu}(U_0, W_0)\, U_1 + f_{uw}(U_0, W_0)\, W_1 \right] , \qquad \Psi_1'' = 0 \,.$$

In order to match to the outer solution constructed below we require that Ψ_1 is a constant. We then multiply the equation for Φ_1 by U_0', and integrate the resulting expression by parts, to get

$$\int_{-\infty}^{\infty} U_0' f_w(U_0, W_0)\, \Psi_1 \, dy = -\int_{-\infty}^{\infty} \Phi_0 U_0' \left[f_{uu}(U_0, W_0)\, U_1 + f_{uw}(U_0, W_0)\, W_1 \right] \, dy \,. \tag{A.3}$$

To simplify the right-hand side of (A.3), we differentiate the equation for U_1 in (2.8) to obtain

$$U_1''' - f_u(U_0, W_0)\, U_1' = f_w(U_0, W_0)\, W_1' + U_0' \left[f_{uu}(U_0, W_0)\, U_1 + f_{uw}(U_0, W_0)\, W_1 \right] = 0 \,.$$

Upon multiplying this equation by Φ_0, and integrating the resulting expression by parts, we derive the identity

$$\int_{-\infty}^{\infty} \Phi_0 U_0' \left[f_{uu}(U_0, W_0)\, U_1 + f_{uw}(U_0, W_0)\, W_1 \right] \, dy = -\int_{-\infty}^{\infty} f_w(U_0, W_0) W_1' \Phi_0 \, dy \,. \tag{A.4}$$

In (A.3), we use (A.4), $\Phi_0 = c_{i\pm} U_0'$, and the facts that Ψ_1 and W_1' are constants (see (2.12)), to get

$$\Psi_1 \int_{-\infty}^{\infty} U_0' f_w(U_0, W_0) \, dy = c_{i\pm} W_1' \int_{-\infty}^{\infty} f_w(U_0, W_0) U_0' \, dy \,. \tag{A.5}$$

Since $f_w = -U_0^2$ and $\int_{-\infty}^{\infty} U_0^2 U_0' \, dy \neq 0$, (A.5) yields $\Psi_1 \equiv c_{i\pm} W_1'$. However, $W_1' = w'(x_{i\pm})$ from (2.12), and $\Psi_1 = \varepsilon \psi(x_{i\pm})$, where $\psi(x)$ is the outer solution for (4.1). Therefore, we have the following key relationship:

$$\psi(x_{i\pm}) = \varepsilon c_{i\pm} w'(x_{i\pm}) \,. \tag{A.6}$$

Next, we derive an outer equation for ψ. In the outer region, defined on the union of the subintervals $-KL < x < x_{1-}$, $x_{i-} < x < x_{i+}$ for $i = 1, \ldots, K$, and $x_{k+} < x < KL$, we obtain from (4.1) the leading-order system

$$\phi = -\frac{f_w}{f_u} \psi = \frac{u'}{w'} \psi \,, \qquad \lambda_1 \phi = D\psi_{xx} - \beta_0 \phi \,.$$

These equations can be combined to give

$$D\psi_{xx} - (\lambda_1 + \beta_0) \frac{u'}{w'} \psi = 0 \,.$$

To determine the jump condition for ψ across $x = x_{i\pm}$, we use use the inner result $\phi \sim c_{i\pm} U_0'$ to derive

$$[D\psi'] \,|_{i\pm} \equiv D\psi'(x_{i\pm}^+) - D\psi'(x_{i\pm}^-) = c_{i\pm}(\lambda_1 + \beta_0) \int_{-\infty}^{\infty} U_0' \, dy = \mp\sqrt{2}\varepsilon c_{i\pm} (\lambda_1 + \beta_0) \,. \tag{A.7}$$

Therefore, the outer problem for ψ on $-KL < x < KL$ is

$$D\psi'' - (\beta_0 + \lambda_1) \frac{u'}{w'} \psi = -\sqrt{2}\varepsilon (\beta_0 + \lambda_1) \left(\sum_{i=1}^{K} \left[c_{i+} \delta(x - x_{i+}) - c_{i-} \delta(x - x_{i-}) \right] \right) , \tag{A.8}$$

with $\psi'(\pm KL) = 0$. Note that in those outer regions where $u = 0$ to leading order, we have $\frac{u'}{w'} = 0$.

Single Mesa: We first analyze (A.8), together with (A.6), for the special case of a single mesa where $K = 1$. For this case $x_{1-} = -l$, $x_{1+} = +l$, and and $x \in [-L, L]$. Then (A.8) is equivalent to

$$D\psi'' - (\beta_0 + \lambda_1) \frac{u'}{w'} \psi = 0 \,, \quad x \in (-l, l) \,; \qquad \psi'' = 0 \,, \quad x \in (l, L) \cup (-L, -l) \,;$$

$$D\psi'\left(l^+\right) - D\psi'\left(l^-\right) = -\sqrt{2}\varepsilon (\beta_0 + \lambda_1)\, c_{1+} \,, \qquad D\psi'\left(-l^+\right) - D\psi'\left(-l^-\right) = \sqrt{2}\varepsilon (\beta_0 + \lambda_1)\, c_{1-} \,,$$

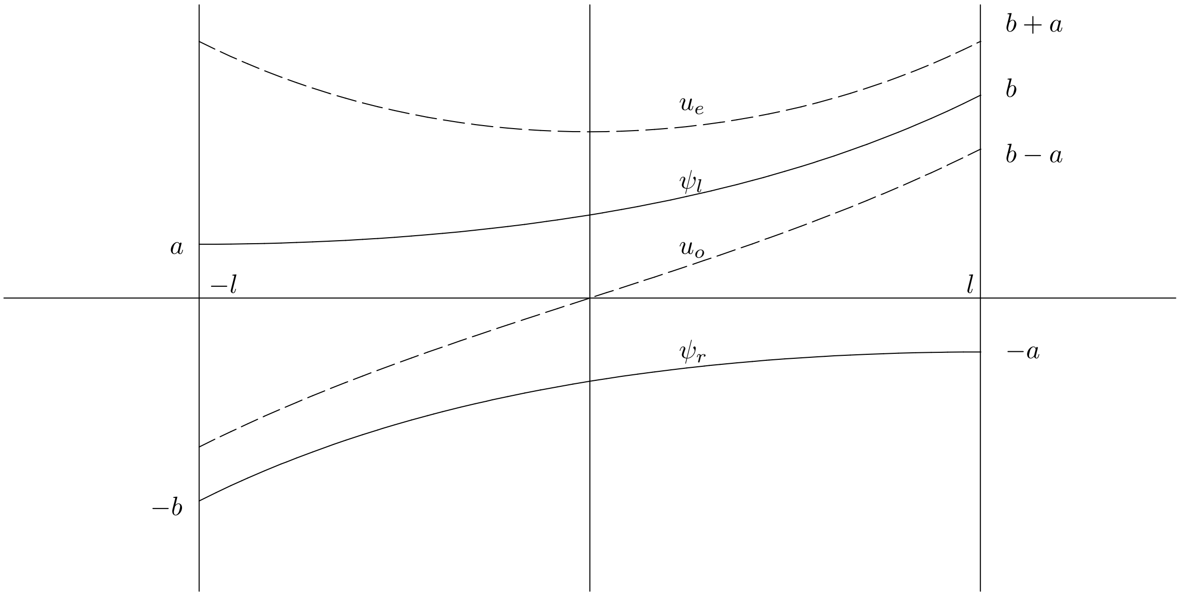

FIGURE A 1. Symmetry of ψ_r, ψ_l, u_o and u_e.

with $\psi'(\pm L) = 0$. By solving for ψ on $x \in (l, L) \cup (-L, -l)$, this system reduces to

$$D\psi'' - (\beta_0 + \lambda_1)\frac{u'}{w'}\psi = 0\,, \quad x \in (-l, l)\,, \tag{A.9 a}$$

$$D\psi'(l) = \sqrt{2}\varepsilon(\beta_0 + \lambda_1)c_{1+}\,; \qquad D\psi'(-l) = \sqrt{2}\varepsilon(\beta_0 + \lambda_1)c_{1-}\,. \tag{A.9 b}$$

We represent ψ in terms of the solutions ψ_l and ψ_r to

$$D\psi'' - (\beta_0 + \lambda_1)\frac{u'}{w'}\psi = 0\,, \quad x \in (-l, l)\,, \tag{A.10 a}$$

with either

$$\psi_l'(-l) = 0\,, \quad \psi_l'(l) = 1 \qquad \text{or} \qquad \psi_r'(-l) = 1\,, \quad \psi_r'(l) = 0\,. \tag{A.10 b}$$

We then define a and b by

$$a \equiv \psi_l(-l)\,, \qquad b \equiv \psi_l(l)\,. \tag{A.11 a}$$

Since $\frac{u'}{w'}$ is an even function, we also have

$$\psi_r(-l) = -b\,, \qquad \psi_r(l) = -a\,. \tag{A.11 b}$$

In terms of ψ_l and ψ_r, the solution for ψ is $\psi = A_l\psi_l + A_r\psi_r$, where $\psi'(-l) = A_r$ and $\psi'(+l) = A_l$. By satisfying the boundary conditions for ψ in (A.9), we get

$$A_l = \frac{\sqrt{2}\varepsilon}{D}(\beta_0 + \lambda_1)c_{1+}\,, \qquad A_r = \frac{\sqrt{2}\varepsilon}{D}(\beta_0 + \lambda_1)c_{1-}\,.$$

In terms of this solution, we write the matrix system

$$\begin{bmatrix} \psi(l) \\ \psi(-l) \end{bmatrix} = \begin{bmatrix} \psi_l(l) & \psi_r(l) \\ \psi_l(-l) & \psi_r(-l) \end{bmatrix}\begin{bmatrix} A_l \\ A_r \end{bmatrix} = \frac{\sqrt{2}\varepsilon}{D}(\beta_0 + \lambda_1)\begin{bmatrix} b & a \\ a & b \end{bmatrix}\begin{bmatrix} c_{1+} \\ -c_{1-} \end{bmatrix}. \tag{A.12}$$

To calculate an independent expression for $\psi(\pm l)$ we use the identity (A.6), which states $\psi(\pm l) = c_{1\pm}\varepsilon w'(\pm l)$.

To calculate $w'(\pm l)$, we recall that $Dw'' = -1$ for $x \in (-L,-l)$ and for $x \in (l,L)$. With $w'(\pm L)=0$, this gives $w'(\pm l) = \pm(L-l)/D$. Therefore,

$$\begin{bmatrix} \psi(l) \\ \psi(-l) \end{bmatrix} = \frac{(L-l)}{D}\varepsilon \begin{bmatrix} c_{1+} \\ -c_{1-} \end{bmatrix}. \tag{A.13}$$

Combining (A.12) and (A.13), we get

$$\begin{bmatrix} b & a \\ a & b \end{bmatrix} \begin{bmatrix} c_{1+} \\ -c_{1-} \end{bmatrix} = \frac{(L-l)}{\sqrt{2}} \frac{1}{(\beta_0+\lambda_1)} \begin{bmatrix} c_{1+} \\ -c_{1-} \end{bmatrix}.$$

The eigenvalues of the matrix on the left-hand side of this expression are $b \pm a$. Therefore, λ_1 must satisfy

$$\frac{L-l}{\sqrt{2}} \frac{1}{(\beta_0+\lambda_1)} = b \pm a. \tag{A.14}$$

Finally, we rewrite (A.14) in terms of new functions $u_o(x)$ and $u_e(x)$ defined by

$$u_o \equiv \psi_l + \psi_r\,, \qquad u_e = \psi_l - \psi_r\,. \tag{A.15}$$

Then, u_o is odd and u_e is even, and both satisfy (A.10 a) with the side conditions

$$u_o(0) = 0, \quad u_o'(l) = 1\,; \qquad u_e'(0) = 0, \quad u_e'(l) = 1\,.$$

Moreover, by using (A.11) and (A.15), we calculate

$$u_o(l) = b - a\,, \qquad u_e(l) = b + a\,.$$

Therefore, from (A.14), the eigenvalues must satisfy

$$\frac{(L-l)}{\sqrt{2}} \frac{1}{(\beta_0+\lambda_1)} = u_e(l)\,, \qquad \text{or} \qquad \frac{(L-l)}{\sqrt{2}} \frac{1}{(\beta_0+\lambda_1)} = u_o(l)\,.$$

The corresponding eigenfunctions are either even or odd. This proves (4.6) of Lemma 5.

General case: We now consider the case of K mesas with $K > 1$. On each subinterval we solve for ψ to obtain

$$\psi = A_{il}\psi_{li} + A_{ir}\psi_{ri}\,, \qquad x \in [x_{i-}, x_{i+}]\,,$$
$$\psi = C_i + D_i(x - x_{i+})\,, \qquad x \in [x_{i+}, x_{(i+1)-}] \cup [-KL, x_{1-}] \cup [x_{K+}, KL]\,,$$

where the coefficients A_{il}, A_{ir}, C_i, and D_i are to be found. The functions ψ_l, ψ_r solve $D\psi'' - (\beta_0+\lambda_1)\frac{u'}{w'}\psi = 0$ with

$$\psi_{li}'(x_{i-}) = 0, \quad \psi_{li}'(x_{i+}) = 1\,; \qquad \psi_{ri}'(x_{i-}) = 1, \quad \psi_{ri}'(x_{i+}) = 0\,.$$

Similar to the case of a single mesa, we have

$$\psi_{li}(x_{i-}) = a\,, \qquad \psi_{li}(x_{i+}) = b\,, \qquad \psi_{ri}(x_{i-}) = -b\,, \qquad \psi_{ri}(x_{i+}) = -a\,.$$

By satisfying the jump condition for $D\psi'$ across $x = x_{i\pm}$ from (A.8) we obtain

$$D\left(\psi'\left(x_{i+}^{+}\right) - \psi'\left(x_{i+}^{-}\right)\right) = D_i - A_{il} = -\sqrt{2}\varepsilon(\beta_0+\lambda_1)c_{i+}\,, \tag{A.16 a}$$
$$D\left(\psi'\left(x_{i-}^{+}\right) - \psi'\left(x_{i-}^{-}\right)\right) = A_{ir} - D_{i-1} = \sqrt{2}\varepsilon(\beta_0+\lambda_1)c_{i-}\,. \tag{A.16 b}$$

Furthermore, since ψ is continuous across $x_{i\pm}$ we get

$$aA_{il} - bA_{ir} = C_{i-1} + 2D_{i-1}d\,, \qquad bA_{il} - aA_{ir} = C_i\,,$$

where $2d = x_{i-} - x_{(i-1)+}$. Then, by using $\psi'(\pm KL) = 0$, we solve for C_i and D_i to obtain

$$D_0 = D_K = 0\,, \qquad C_i = bA_{il} - aA_{ir}\,, \qquad D_i = \frac{1}{2d}\left(aA_{(i+1)l} - bA_{(i+1)r} - bA_{il} + aA_{ir}\right)\,. \tag{A.17}$$

Moreover, we calculate

$$\psi(x_{i-}) = A_{il}a - A_{ir}b\,, \qquad \psi(x_{i+}) = A_{il}b - A_{ir}a\,,$$

so that

$$A_{il} = \frac{1}{\delta}\left(b\psi\left(x_{i+}\right) - a\psi\left(x_{i-}\right)\right), \qquad A_{ir} = \frac{1}{\delta}\left(a\psi\left(x_{i+}\right) - b\psi\left(x_{i-}\right)\right), \tag{A.18}$$

where $\delta \equiv b^2 - a^2$. Therefore, substituting (A.18) and (A.17) into (A.16), we obtain

$$\begin{aligned}
\frac{\sqrt{2}\varepsilon}{D}\left(\beta_0+\lambda_1\right)c_{i-} &= \frac{a}{\delta}\psi\left(x_{i+}\right) - \frac{b}{\delta}\psi\left(x_{i-}\right) - \frac{1}{2d}\psi\left(x_{i-}\right) + \frac{1}{2d}\psi\left(x_{(i-1)+}\right), \quad 1 < i \le K,\\
-\frac{\sqrt{2}\varepsilon}{D}\left(\beta_0+\lambda_1\right)c_{i+} &= \frac{a}{\delta}\psi\left(x_{i-}\right) - \frac{b}{\delta}\psi\left(x_{i+}\right) - \frac{1}{2d}\psi\left(x_{i+}\right) + \frac{1}{2d}\psi\left(x_{(i+1)-}\right), \quad 1 \le i < K,\\
\frac{\sqrt{2}\varepsilon}{D}\left(\beta_0+\lambda_1\right)c_{1-} = \frac{a}{\delta}\psi\left(x_{1+}\right) - \frac{b}{\delta}\psi\left(x_{1-}\right), &\qquad -\frac{\sqrt{2}\varepsilon}{D}\left(\beta_0+\lambda_1\right)c_{K+} = \frac{a}{\delta}\psi\left(x_{K-}\right) - \frac{b}{\delta}\psi\left(x_{K+}\right).
\end{aligned}$$

This system can be written in matrix form as

$$\sqrt{2}\varepsilon\left(\beta_0+\lambda_1\right)v = Mz, \qquad v \equiv \begin{bmatrix} c_{1-} \\ -c_{1+} \\ \vdots \\ c_{K-} \\ -c_{K+} \end{bmatrix} \qquad z \equiv \begin{bmatrix} \psi\left(x_{1-}\right) \\ \psi\left(x_{1+}\right) \\ \vdots \\ \psi\left(x_{K-}\right) \\ \psi\left(x_{K+}\right) \end{bmatrix}. \tag{A.19}$$

Here the matrix M is given in (4.3). Finally, we use (A.6) to calculate $\psi(x_{i\pm}) = \pm c_{i\pm}(L-l)/D$. This yields that $z = \varepsilon(l-L)v/D$. Therefore, (A.19) becomes

$$Mv = \frac{\sqrt{2}}{l-L}(\beta_0+\lambda_1)v. \tag{A.20}$$

The eigenvalue problem (A.20) is equivalent to that in (4.2) of Lemma 5. The eigenvalues of M were calculated explicitly in [**32**], and are as given in (4.5). This completes the proof of Lemma 5.

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