Asymptotic Methods for PDE Problems in Fluid Mechanics and Related Systems with Strong Localized Perturbations in Two-Dimensional Domains

Michael J. Ward (UBC)

CISM Advanced Course; Asymptotic Methods in Fluid Mechanics: Surveys and Recent Advances

Lecture I: Infinite Logarithmic Expansions and Linear Elliptic Problems

Outline of Lecture I

TWO SPECIFIC PROBLEMS CONSIDERED:

1. A Model Pipe Flow Problem
2. Oxygenation of Muscle Tissue by Capillaries

Key Point: We show how to deal with certain classes of problems yielding infinite logarithmic expansions of the form

$$V \sim a_1 \left(\frac{-1}{\log \varepsilon} \right) + a_2 \left(\frac{-1}{\log \varepsilon} \right)^2 + a_3 \left(\frac{-1}{\log \varepsilon} \right)^3 + \cdots .$$

Rather than computing the coefficients a_j directly, we formulate a hybrid method for a function $A(\nu)$ that embeds all of the infinite logarithmic terms

$$V \sim A(\nu) + \mathcal{O}(\sigma) \, ,$$

where $\nu = -1/\log \varepsilon$ and $\sigma \ll \nu^k$ for any $k > 0$.

Model Pipe Flow Problem I

We consider steady, incompressible, laminar flow in a straight pipe containing a thin core. Both the pipe and the core have a constant cross-section of arbitrary shape, and thus the problem is two-dimensional. With these assumptions, the pipe flow is unidirectional and the velocity component w in the axial direction satisfies

$$\triangle w = -\beta\,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon\,, \tag{1.1a}$$

$$w = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{1.1b}$$

$$w = 0\,, \quad \mathbf{x} \in \partial\Omega_\varepsilon\,. \tag{1.1c}$$

- $\Omega \in \mathbb{R}^2$ is the dimensionless pipe cross-section and Ω_ε is the cross-section of the thin core.
- Let Ω_ε have radius $\mathcal{O}(\varepsilon)$ and $\Omega_\varepsilon \to \mathbf{x}_0 \in \Omega$ as $\varepsilon \to 0$.
- The constant $\beta \equiv \mu^{-1} dp/dz$. Here μ is the dynamic viscosity μ of the fluid and dp/dz the constant pressure gradient.
- The mean flow velocity $\bar{w}$ is defined by

$$\bar{w} \equiv \frac{1}{A_\Omega} \int_{\Omega \backslash \Omega_\varepsilon} w \, d\mathbf{x}\,. \tag{1.2}$$

Model Pipe Flow Problem: Hybrid I

The asymptotic solution to (1.1) is constructed in two different regions: an outer region defined at an $\mathcal{O}(1)$ distance from the perturbing core, and an inner region defined in an $\mathcal{O}(\varepsilon)$ neighborhood of the thin core Ω_ε.

We show how to account for all the logarithmic terms for w in in the limit of small core radius $\varepsilon \to 0$.

In the outer region we expand the solution to (1.1) as

$$w(\mathbf{x};\varepsilon) = W_0(\mathbf{x};\nu) + \sigma(\varepsilon)W_1(\mathbf{x};\nu) + \cdots \,. \tag{1.3}$$

Here $\nu = \mathcal{O}(1/\log\varepsilon)$ is a gauge function to be chosen. We assume that $\sigma \ll \nu^k$ for any $k > 0$ as $\varepsilon \to 0$. Thus, W_0 contains all of the log terms.

Substitute (1.3) into (1.1a,b) and let $\Omega_\varepsilon \to \mathbf{x}_0$ as $\varepsilon \to 0$;

$$\triangle W_0 = -\beta\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\}\,, \tag{1.4a}$$

$$W_0 = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{1.4b}$$

$$W_0 \quad \text{is singular as} \quad \mathbf{x} \to \mathbf{x}_0\,. \tag{1.4c}$$

Matching to an inner expansion will yield a singularity structure for W_0 as $\mathbf{x} \to \mathbf{x}_0$.

Model Pipe Flow Problem: Hybrid II

In the inner region near Ω_ε we introduce

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0)\,, \qquad v(\mathbf{y};\varepsilon) = W(\mathbf{x}_0 + \varepsilon\mathbf{y};\varepsilon)\,, \quad \Omega_1 \equiv \varepsilon^{-1}\Omega_\varepsilon\,. \tag{1.5}$$

Remark: If we assume that $v = \mathcal{O}(1)$ in the inner region, we obtain the leading-order problem $\triangle_{\mathbf{y}} v = 0$ outside Ω_1, with $v = 0$ on $\partial\Omega_1$ and $v \to W_0(\mathbf{x}_0)$ as $|\mathbf{y}| \to \infty$. There is no solution to this problem!

To overcome this difficulty, we require that $v = \mathcal{O}(\nu)$ in the inner region and we allow v to be logarithmically unbounded as $|\mathbf{y}| \to \infty$. Therefore, we expand v as

$$v(\mathbf{y};\varepsilon) = V_0(\mathbf{y};\nu) + \mu_0(\varepsilon)V_1(\mathbf{y}) + \cdots\,, \tag{1.6a}$$

where we write V_0 in the form

$$V_0(\mathbf{y};\nu) = \nu\gamma v_c(\mathbf{y})\,. \tag{1.6b}$$

Here $\gamma = \gamma(\nu)$ is a constant to be determined with $\gamma = \mathcal{O}(1)$ as $\nu \to 0$, and we assume that $\mu_0 \ll \nu^k$ for any $k > 0$ as $\varepsilon \to 0$.

Model Pipe Flow Problem: Bybrid III

This yields the canonical inner problem for $v_c(\mathbf{y})$:

$$\triangle_{\mathbf{y}} v_c = 0 \,, \quad \mathbf{y} \notin \Omega_1 \,; \qquad v_c = 0 \,, \quad \mathbf{y} \in \partial\Omega_1 \,, \tag{1.7a}$$

$$v_c \sim \log|\mathbf{y}| \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{1.7b}$$

The unique solution for v_c has the far-field asymptotic behavior

$$v_c(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + \frac{\mathbf{p}\cdot\mathbf{y}}{|\mathbf{y}|^2} + \cdots \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{1.7c}$$

- The constant $d > 0$, called the logarithmic capacitance of Ω_1, depends on the shape of Ω_1 but not on its orientation.
- The vector $\mathbf{p}$ is called the dipole vector (needed to account for transcendentally small terms beyond the infinite logarithmic expansion)
- Numerical values for d can be calculated by conformal mapping for different shapes of Ω_1. A boundary integral method to compute d for arbitrarily-shaped domains Ω_1 can be formulated.

Model Pipe Flow Problem: Bybrid IV

Shape of $\Omega_1 \equiv \varepsilon^{-1}\Omega_\varepsilon$	Logarithmic Capacitance d
circle, radius a	$d = a$
ellipse, semi-axes a, b	$d = \frac{a+b}{2}$
equilateral triangle, side h	$d = \frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)^3 h}{8\pi^2} \approx 0.422h$
isosceles right triangle, short side h	$d = \frac{3^{3/4}\Gamma\left(\frac{1}{4}\right)^2 h}{2^{7/2}\pi^{3/2}} \approx 0.476h$
square, side h	$d = \frac{\Gamma\left(\frac{1}{4}\right)^2 h}{4\pi^{3/2}} \approx 0.5902h$

The logarithmic capacitance, or shape-dependent parameter, d, for some cross-sectional shapes of $\Omega_1 = \varepsilon^{-1}\Omega_\varepsilon$.

Model Pipe Flow Problem: Hybrid V

Match the inner and outer solutions to determine the constant γ. Upon using the far-field behavior of v_c in (1.7c) in (1.6b), and writing the resulting expression in outer variables, we get the far-field behavior

$$v(\mathbf{y};\varepsilon) \sim \gamma\nu \left[\log|\mathbf{x}-\mathbf{x}_0| - \log(\varepsilon d)\right] + \cdots \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{1.8}$$

Therefore, we should choose ν as

$$\nu(\varepsilon) = -1/\log(\varepsilon d) \,. \tag{1.9}$$

Matching v to W_0 gives the singularity structure for W_0,

$$W_0 = \gamma + \gamma\nu \log|\mathbf{x}-\mathbf{x}_0| + o(1) \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.10}$$

Remark: The singularity structure in (1.10) specifies both the regular and singular parts of a Coulomb singularity. As such, it must provide one constraint for the determination of γ. More specifically, for a linear elliptic equation we can freely impose $W_0 \sim S \log|\mathbf{x}-\mathbf{x}_0|$ as $\mathbf{x} \to \mathbf{x}_0$ for any S. However, we cannot impose a condition on the regular part without introducing a constraint.

To sum all logarithmic terms we must solve (1.4) for W_0 subject to (1.10).

Model Pipe Flow Problem: Hybrid VI

The solution for W_0 is decomposed as

$$W_0(\mathbf{x};\nu) = W_{0H}(\mathbf{x}) - 2\pi\gamma\nu G_d(\mathbf{x};\mathbf{x}_0)\,, \qquad (1.11)$$

where $W_{0H}(\mathbf{x})$ satisfies the unperturbed problem

$$\triangle W_{0H} = -\beta\,, \quad \mathbf{x}\in\Omega\,; \qquad W_{0H} = 0\,, \quad \mathbf{x}\in\partial\Omega\,, \qquad (1.12)$$

and $G_d(\mathbf{x};\mathbf{x}_0)$ is the Dirichlet Green's function satisfying

$$\triangle G_d = -\delta(\mathbf{x}-\mathbf{x}_0)\,, \quad \mathbf{x}\in\Omega\,; \qquad G_d = 0\,, \quad \mathbf{x}\in\partial\Omega\,, \qquad (1.13a)$$

$$G_d(\mathbf{x};\mathbf{x}_0) = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| + R_d(\mathbf{x}_0;\mathbf{x}_0) + o(1)\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_0\,. \qquad (1.13b)$$

Here $R_{d00} \equiv R_d(\mathbf{x}_0;\mathbf{x}_0)$ is the regular part of the Dirichlet Green's function $G_d(\mathbf{x};\mathbf{x}_0)$ at $\mathbf{x}=\mathbf{x}_0$. This regular part is also known as either the self-interaction term or the Robin constant.

Remark: G_d can be found by the method of images for a circle, and for other domains it is easily computed numerically.

Model Pipe Flow Problem: Hybrid VII

Expand the outer solution (1.11) as $\mathbf{x} \to \mathbf{x}_0$ and compare it with the required singularity structure (1.10):

$$W_{0H}(\mathbf{x}_0) - 2\pi\gamma\nu \left[-\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_0| + R_{d00} \right] \sim \gamma + \gamma\nu \log|\mathbf{x} - \mathbf{x}_0| \,. \quad (1.14)$$

This determines γ as (a geometric series)

$$\gamma = \frac{W_{0H}(\mathbf{x}_0)}{1 + 2\pi\nu R_{d00}} \,, \quad (1.15)$$

provided that

$$0 < \varepsilon < \varepsilon_c \,, \qquad \varepsilon_c \equiv \frac{1}{d} \exp\left[2\pi R_{d00}\right] \,. \quad (1.16)$$

Summary: The outer expansion is

$$w \sim W_0(\mathbf{x}; \nu) = W_{0H}(\mathbf{x}) - \frac{2\pi\nu W_{0H}(\mathbf{x}_0)}{1 + 2\pi\nu R_{d00}} G_d(\mathbf{x}; \mathbf{x}_0) \,, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_0| = \mathcal{O}(1) \,. \quad (1.17)$$

The inner expansion with $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0)$ is

$$w \sim V_0(\mathbf{y}; \nu) = \frac{\nu W_{0H}(\mathbf{x}_0)}{1 + 2\pi\nu R_{d00}} v_c(\mathbf{y}) \,, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_0| = \mathcal{O}(\varepsilon) \,. \quad (1.18)$$

Model Pipe Flow Problem: Hybrid VIII

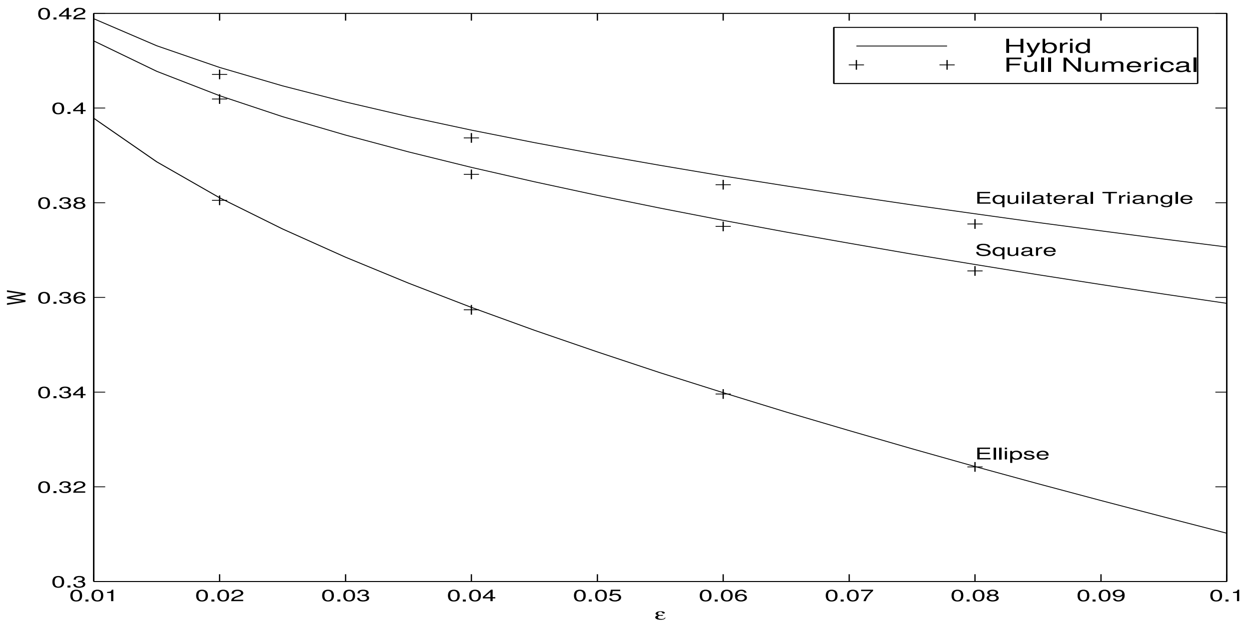
- formulation is referred to as a hybrid asymptotic-numerical method since it uses the asymptotic analysis as a means of reducing the original problem with a hole to the simpler asymptotically related problem for W_0 with singularity structure.
- The numerics required for the hybrid problem involve the computation of the unperturbed solution W_{0H} and the Dirichlet Green's function $G_d(\mathbf{x};\mathbf{x}_0)$. In terms of G_d we then identify its regular part $R_d(\mathbf{x}_0;\mathbf{x}_0)$ at the singular point.
- From the canonical inner problem we must compute the logarithmic capacitance d.
- The asymptotics depends on the product of εd and not on ε itself (Kaplun's equivalence pricnipe). Thus, a change of the shape of Ω_1 requires us only to re-calculate the constant d
- In contrast to solving the full problem numerically, we do not have any stiff $\varepsilon-$dependent problems to solve.

Model Pipe Flow Problem: Validation I

Compare results of the hybrid method with results obtained either analytically or numerically from the full perturbed problem (1.1).

Example 1: Let Ω be a circular pipe of cross-sectional radius $r_0 = 2$ that contains a concentric core Ω_ε of various cross-sectional shapes centered at the origin. We use the Table for the logarithmic capacitance d. The hybrid solution is simply

$$w(\mathbf{x};\varepsilon) \sim \frac{\beta}{4}\left[r_0^2 - r^2 - r_0^2 \frac{\log(r_0/r)}{\log(r_0/[\varepsilon d])}\right], \qquad r = |\mathbf{x}|. \tag{1.19}$$

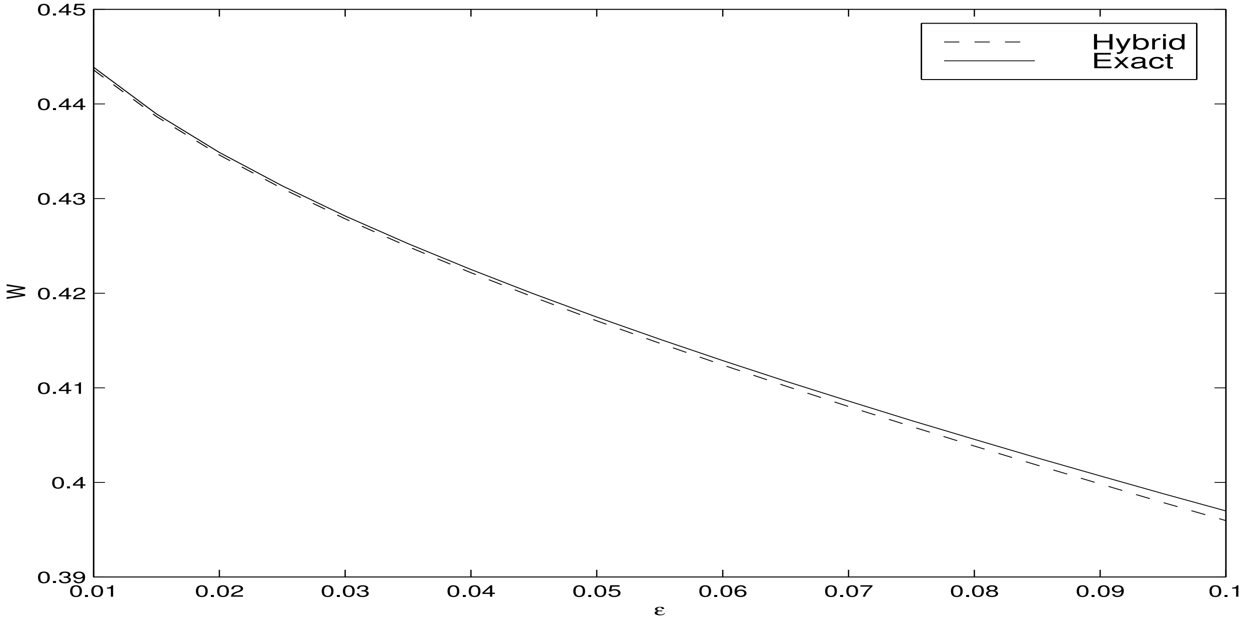


Model Pipe Flow Problem: Validation II

Example 2: Let Ω be a circular pipe of cross-sectional radius $r_0 = 2$ that contains a circular core Ω_ε of radius ε centered at $\mathbf{x}_0 = (-1, 0)$. There is a complicated exact solution to this problem. For the hybrid method we use $d = 1$, so that $\nu = -1/\log\varepsilon$, and

$$G_d(\mathbf{x};\mathbf{x}_0) = -\frac{1}{2\pi}\log\left(\frac{|\mathbf{x}-\mathbf{x}_0| r_0}{|\mathbf{x}-\mathbf{x}_0'||\mathbf{x}_0|}\right), \qquad R_{d00} = -\frac{1}{2\pi}\log\left[\frac{r_0}{|\mathbf{x}_0-\mathbf{x}_0'||\mathbf{x}_0|}\right],$$

where $\mathbf{x}_0'$ is the image of $\mathbf{x}_0$ in the circle $|\mathbf{x}| = r_0$. Also, $W_{0H}(r) = \frac{\beta}{4}(r_0^2 - r^2)$. **Remark:** For a non-circular core there is no exact solution; for the hybrid method we simply ε by εd.



Pipe Flow Problem: Direct Approach I

Problem 1: Consider a conventional infinite-order logarithmic expansion for the outer solution in the form

$$W \sim \sum_{j=0}^{\infty} \left(\frac{-1}{\log(\varepsilon d)} \right)^j W_{0j}(\mathbf{x}) + \sigma(\varepsilon) W_1 + \cdots \,, \tag{1.20}$$

with $\sigma(\varepsilon) \ll \nu^k$ for any $k > 0$. By formulating a similar series for the inner solution, derive a recursive set of problems for the W_{0j} for $j \geq 0$ from the asymptotic matching of the inner and outer solutions. *Show that this series can be summed and leads to the result of the hybrid method.*

Recall that the model pipe flow problem is

$$\triangle w = -\beta \,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon \,, \tag{2.1a}$$

$$w = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{2.1b}$$

$$w = 0 \,, \quad \mathbf{x} \in \partial\Omega_\varepsilon \,. \tag{2.1c}$$

Pipe Flow Problem: Direct Approach II

Solution:

In the outer region we pose an explicit infinite-order logarithmic expansion:

$$w(\mathbf{x};\varepsilon) = W_{0H}(\mathbf{x}) + \sum_{j=1}^{\infty} \nu^j W_{0j}(\mathbf{x}) + \cdots \,. \tag{2.2}$$

Here $\nu = \mathcal{O}(1/\log\varepsilon)$ is to be chosen. The smooth function W_{0H} satisfies the unperturbed problem in the unperturbed domain, given by

$$\triangle W_{0H} = -\beta \,, \quad \mathbf{x} \in \Omega \,; \quad W_{0H} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,. \tag{2.3}$$

Letting $\Omega_\varepsilon \to \mathbf{x}_0$ as $\varepsilon \to 0$, we get that W_{0j} for $j \geq 1$ satisfies the infinite sequence of problems

$$\triangle W_{0j} = 0 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\} \,, \tag{2.4a}$$

$$W_{0j} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{2.4b}$$

$$W_{0j} \quad \text{is singular as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{2.4c}$$

The matching of the outer and inner expansions will determine a singularity behavior for W_{0j} as $\mathbf{x} \to \mathbf{x}_0$ for each $j \geq 1$.

Pipe Flow Problem: Direct Approach III

In the inner region near Ω_ε we introduce

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x}-\mathbf{x}_0)\,, \qquad v(\mathbf{y};\varepsilon) = W(\mathbf{x}_0+\varepsilon\mathbf{y};\varepsilon)\,, \qquad \Omega_1 \equiv \varepsilon^{-1}\Omega_\varepsilon\,. \tag{2.5}$$

We then pose the explicit infinite-order logarithmic inner expansion

$$v(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty} \gamma_j \nu^{j+1} v_c(\mathbf{y})\,. \tag{2.6}$$

Here γ_j are ε-independent coefficients to be determined. The function $v_c(\mathbf{y})$ satisfies the canonical inner problem

$$\triangle_{\mathbf{y}} v_c = 0\,, \quad \mathbf{y} \notin \Omega_1\,; \qquad v_c = 0\,, \quad \mathbf{y} \in \partial\Omega_1\,, \tag{2.7a}$$

$$v_c \sim \log|\mathbf{y}| - \log d + o(1)\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{2.7b}$$

Upon using the far-field behavior (2.7b) in (2.6), and writing the resulting expression in terms of the outer variable $\mathbf{x}-\mathbf{x}_0 = \varepsilon\mathbf{y}$, we obtain that

$$v \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1}\log|\mathbf{x}-\mathbf{x}_0| + \gamma_j\right]\,. \tag{2.8}$$

Pipe Flow Problem: Direct Approach IV

Matching the infinite-order outer expansion (2.2) as $\mathbf{x} \to \mathbf{x}_0$ and the far-field behavior (2.8) of the inner expansion gives

$$W_{0H}(\mathbf{x}_0) + \sum_{j=1}^{\infty} \nu^j W_{0j}(\mathbf{x}) \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1} \log |\mathbf{x} - \mathbf{x}_0| + \gamma_j \right] . \tag{2.9}$$

The leading-order match gives $\gamma_0 = W_{0H}(\mathbf{x}_0)$. At higher order, the solution W_{0j} to (2.4) must have the singularity behavior

$$W_{0j} \sim \gamma_{j-1} \log |\mathbf{x} - \mathbf{x}_0| + \gamma_j \, , \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \, . \tag{2.10}$$

The solution for W_{0j} with $W_{0j} \sim \gamma_{j-1} \log |\mathbf{x} - \mathbf{x}_0|$ as $\mathbf{x} \to \mathbf{x}_0$ is

$$W_{0j}(\mathbf{x}) = -2\pi \gamma_{j-1} G_d(\mathbf{x}; \mathbf{x}_0) \, , \tag{2.11}$$

Expand (2.12) as $\mathbf{x} \to \mathbf{x}_0$ and compare it with (2.11):

$$-2\pi\gamma_{j-1} \left[-\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0| + R_{d00} \right] \sim \gamma_{j-1} \log |\mathbf{x} - \mathbf{x}_0| + \gamma_j \, , \tag{2.12}$$

where $R_{d00} \equiv R_d(\mathbf{x}_0; \mathbf{x}_0)$.

Pipe Flow Problem: Direct Approach V

By comparing the non-singular parts, we get a recursion relation for γ_j:

$$\gamma_j = -2\pi R_{d00} \gamma_{j-1} \,, \qquad \gamma_0 = W_{0H}(\mathbf{x}_0) \,, \tag{2.13}$$

which has the explicit solution

$$\gamma_j = \left[-2\pi R_{d00}\right]^j W_{0H}(\mathbf{x}_0) \,, \quad j \geq 0 \,. \tag{2.14}$$

Finally, the outer solution is given by

$$\begin{aligned} w &\sim W_{0H}(\mathbf{x}) + \sum_{j=1}^{\infty} \nu^j \left(-2\pi \gamma_{j-1}\right) G_d(\mathbf{x}; \mathbf{x}_0) \,, \\ &\sim W_{0H}(\mathbf{x}) - 2\pi\nu G_d(\mathbf{x}; \mathbf{x}_0) \sum_{j=0}^{\infty} \nu^j \gamma_j \\ &\sim W_{0H}(\mathbf{x}) - 2\pi\nu W_{0H}(\mathbf{x}_0) G_d(\mathbf{x}; \mathbf{x}_0) \sum_{j=0}^{\infty} \left[-2\pi\nu R_{d00}\right]^j \\ &\sim W_{0H}(\mathbf{x}_0) - \frac{2\pi\nu W_{0H}(\mathbf{x}_0)}{1 + 2\pi\nu R_{d00}} G_d(\mathbf{x}_0; \mathbf{x}_0) \,. \end{aligned} \tag{2.15}$$

Pipe Flow Problem: Direct Approach VI

Correspondingly, the inner solution is given by

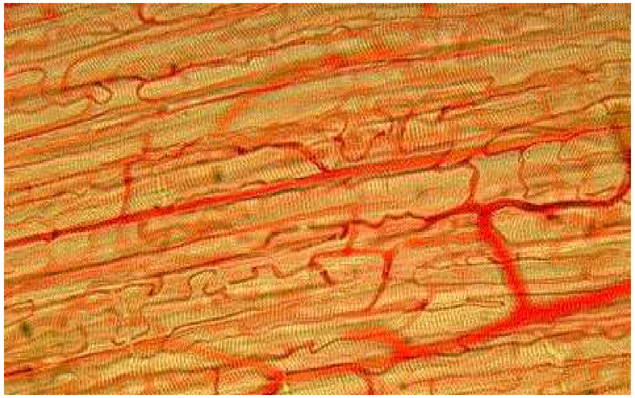
$$v(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty} \gamma_j \nu^{j+1} v_c(\mathbf{y}) = \nu W_{0H}(\mathbf{x}_0) v_c(\mathbf{y}) \sum_{j=0}^{\infty} [-2\pi R_{d00}\nu]^j \tag{2.16}$$

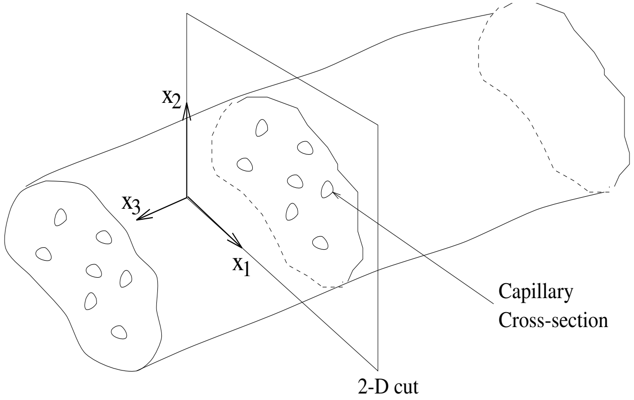
$$= \frac{\nu W_{0H}(\mathbf{x}_0)}{1 + 2\pi\nu R_{d00}} v_c(\mathbf{y})\,. \tag{2.17}$$

This reproduces the result from the hybrid formulation.

Remark: The direct formulation involving the infinite sequence of outer problems determines the coefficients γ_j recursively. The hybrid method avoids computing the γ_j directly.

Oxygen Transport via Capillaries





The steady-state model for the oxygen partial pressure is

$$\triangle p = \mathcal{M}\,, \quad \mathbf{x} \in \Omega \backslash \Omega_p \qquad \Omega_p \equiv \bigcup_{j=1}^{N} \Omega_{\varepsilon_j}\,, \tag{4.1a}$$

$$\partial_n p = 0, \quad \mathbf{x} \in \partial\Omega\,. \tag{4.1b}$$

$$\varepsilon \partial_n p + \kappa_j (p - p_{cj}) = 0\,, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1, \ldots, N, \tag{4.1c}$$

- $\kappa_i > 0$ is the permeability coefficient of the i^{th} capillary and p_{ci} is the oxygen partial pressure within the i^{th} capillary (assumed constant).
- the oxygen consumption rate $\mathcal{M}$, modeling the effect of mitochondria, is spatially-dependent.

Oxygen Transport: Hybrid I

In the outer region we expand the solution as

$$p(\mathbf{x};\varepsilon) = P_0(\mathbf{x};\nu_1,\ldots,\nu_N) + \sigma(\varepsilon) P_1(\mathbf{x};\nu_1,\ldots,\nu_N) + \cdots \,. \qquad (4.2)$$

Here $\nu_j = \mathcal{O}(1/\log\varepsilon)$ for $j = 1,\ldots,N$ are gauge functions to be chosen, and we assume that $\sigma \ll \nu_j^k$ for any $k > 0$ as $\varepsilon \to 0$. Thus, P_0 contains all of the logarithmic terms in the expansion.

Substituting (4.2) into (4.1a,b) and letting $\Omega_{\varepsilon_j} \to \mathbf{x}_j$ as $\varepsilon \to 0$, so that

$$\triangle P_0 = \mathcal{M}\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1,\ldots,\mathbf{x}_N\}\,, \qquad (4.3a)$$

$$\partial_n P_0 = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \qquad (4.3b)$$

$$P_0 \quad \text{is singular as } \mathbf{x} \to \mathbf{x}_j\,, \quad j = 1,\ldots,N\,. \qquad (4.3c)$$

The matching of the outer and inner expansions will determine singularity structures for P_0 as $\mathbf{x} \to \mathbf{x}_j$ for $j = 1,\ldots,N$.

In the inner region near the j^{th} capillary Ω_{ε_j} we introduce

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)\,, \quad p(\mathbf{y};\varepsilon) = q_j(\mathbf{x}_j + \varepsilon\mathbf{y};\varepsilon)\,, \quad \Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}\,. \qquad (4.4)$$

Oxygen Transport: Hybrid II

We then introduce the local expansion

$$q_j = p_{cj} + q_{0j}(\mathbf{y}; \nu_1, \ldots, \nu_N) + \mu q_{1j}(\mathbf{y}; \nu_1, \ldots, \nu_N) + \cdots \,, \tag{4.5}$$

where we assume that $\mu \ll \nu_j^k$ for any $k > 0$. We then write q_{0j} in the form

$$q_{0j} = A_j q_{cj}(\mathbf{y}) \,, \tag{4.6}$$

where $A_j = A_j(\nu_1, \ldots, \nu_N)$ is an unknown constant to be determined, and $q_{cj}(\mathbf{y}) \sim \log|\mathbf{y}|$ as $\mathbf{y} \to \infty$. The canonical inner solution satisfies

$$\triangle_{\mathbf{y}} q_{cj} = 0 \,, \quad \mathbf{y} \notin \Omega_j \,; \qquad \partial_n q_{cj} + \kappa_j q_c = 0 \,, \quad \mathbf{y} \in \partial\Omega_j \,, \tag{4.7a}$$

$$q_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d_j + o(1) \,, \quad |\mathbf{y}| \to \infty \,. \tag{4.7b}$$

- For a particular cross-sectional shape of the capillary and for a given value of κ_j, one must compute $d_j = d_j(\kappa_j)$ numerically.
- For a circular capillary of radius ε, for which q_{cj} can be found analytically, then

$$d_j = \exp\left(-1/\kappa_j\right) \,. \tag{4.8}$$

Oxygen Transport: Hybrid III

By using (4.7b) in (4.5) and (4.6), we re-write the far-field form for $|\mathbf{y}| \gg 1$ of the inner solution in terms of the outer variables as

$$q_j \sim p_{cj} + A_j \log|\mathbf{x} - \mathbf{x}_j| + \frac{A_j}{\nu_j} \, . \tag{4.9a}$$

Here we have defined ν_j by

$$\nu_j \equiv -\frac{1}{\log(\varepsilon d_j)} \, . \tag{4.9b}$$

The matching condition is that the far-field form (4.9a) of the inner solution must agree with the near-field behavior of the outer solution for p.

Therefore, P_0 satisfies (4.3) subject to the following singularity structure as $\mathbf{x} \to \mathbf{x}_j$ for $j = 1, \ldots, N$:

$$P_0 \sim p_{cj} + \frac{A_j}{\nu_j} + A_j \log|\mathbf{x} - \mathbf{x}_j| + o(1) \, , \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \, . \tag{4.10}$$

The regular part of the singularity structure is prescribed at each $\mathbf{x}_j$, which yields N equations for the determination of the unknown constants A_j for $j = 1, \ldots, N$.

Oxygen Transport: Hybrid IV

By using the divergence theorem on the P_0 problem:

$$\sum_{j=1}^{N} A_j = -\frac{1}{2\pi}\int_{\Omega} \mathcal{M}(\mathbf{x})\, d\mathbf{x}\,. \qquad (4.11)$$

Next, we decompose the solution for P_0 in the form

$$P_0 = P_R(\mathbf{x}) - 2\pi \sum_{i=1}^{N} A_i G_N(\mathbf{x};\mathbf{x}_i) + \chi\,. \qquad (4.12)$$

Here χ is an unknown constant, and $P_R(\mathbf{x})$ is the unique solution of

$$\triangle P_R = \mathcal{M} - \frac{1}{|\Omega|}\int_{\Omega} \mathcal{M}(\mathbf{x})\, d\mathbf{x}\,, \quad \mathbf{x}\in\Omega\,; \quad \partial_n P_R = 0\,, \quad \mathbf{x}\in\partial\Omega\,, \qquad (4.13)$$

with $\int_\Omega P_R(\mathbf{x})\, d\mathbf{x} = 0$. Also, $G_N(\mathbf{x};\boldsymbol{\xi})$ is the Neumann Green's function;

$$\Delta G_N = \frac{1}{|\Omega|} - \delta(\mathbf{x}-\boldsymbol{\xi})\,, \quad \mathbf{x}\in\Omega\,; \quad \partial_n G_N = 0\,, \quad \mathbf{x}\in\partial\Omega\,,$$

$$G_N(\mathbf{x};\boldsymbol{\xi}) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}| + R_N(\boldsymbol{\xi};\boldsymbol{\xi}) + o(1)\,, \quad \text{as} \quad \mathbf{x}\to\boldsymbol{\xi}\,,$$

with $\int_\Omega G_N(\mathbf{x};\boldsymbol{\xi})\, d\mathbf{x} = 0$ and regular part $R_N(\boldsymbol{\xi};\boldsymbol{\xi})$.

Oxygen Transport: Hybrid V

Finally, we expand P_0 as $\mathbf{x} \to \mathbf{x}_j$ and we compare the regular part of the resulting expression with the regular part of the required singularity structure in (4.10). This gives,

$$P_R(\mathbf{x}_j) - 2\pi \left[A_j R_{Njj} + \sum_{\substack{i=1 \\ i \neq j}}^{N} A_i G_{Nji} \right] + \chi = \frac{A_j}{\nu_j} + p_{cj}\,, \quad j = 1, \ldots, N\,.$$

Here we have defined $R_{Njj} \equiv R_N(\mathbf{x}_j;\mathbf{x}_j)$ and $G_{Nji} \equiv G_N(\mathbf{x}_j;\mathbf{x}_i)$. The remaining equation relating these unknowns is obtained from the divergence theorem on the P_0 equation

$$\sum_{j=1}^{N} A_j = -\frac{1}{2\pi} \int_{\Omega} \mathcal{M}(\mathbf{x})\, d\mathbf{x}\,.$$

- In summary, we have $N+1$ algebraic equations for the $N+1$ unknown constants χ and $A_1, \ldots, A_N$
- The constant χ can be interpreted as the average oxygen partial pressure $\chi = |\Omega|^{-1} \int_{\Omega} P_0\, d\mathbf{x}$.

Oxygen Transport: Hybrid VI

We summarize our asymptotic construction as follows:

Principal Result: For $\varepsilon \to 0$, the inner solution near the j^{th} capillary, is

$$p \sim p_{cj} + A_j q_{cj}(\mathbf{y}) \,, \qquad \mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j) = \mathcal{O}(1) \,. \tag{4.15a}$$

In the outer region, defined at $\mathcal{O}(1)$ distances from the centers of the capillaries, we have

$$p \sim P_R(\mathbf{x}) - 2\pi \sum_{i=1}^{N} A_i G_N(\mathbf{x}; \mathbf{x}_i) + \chi \,. \tag{4.15b}$$

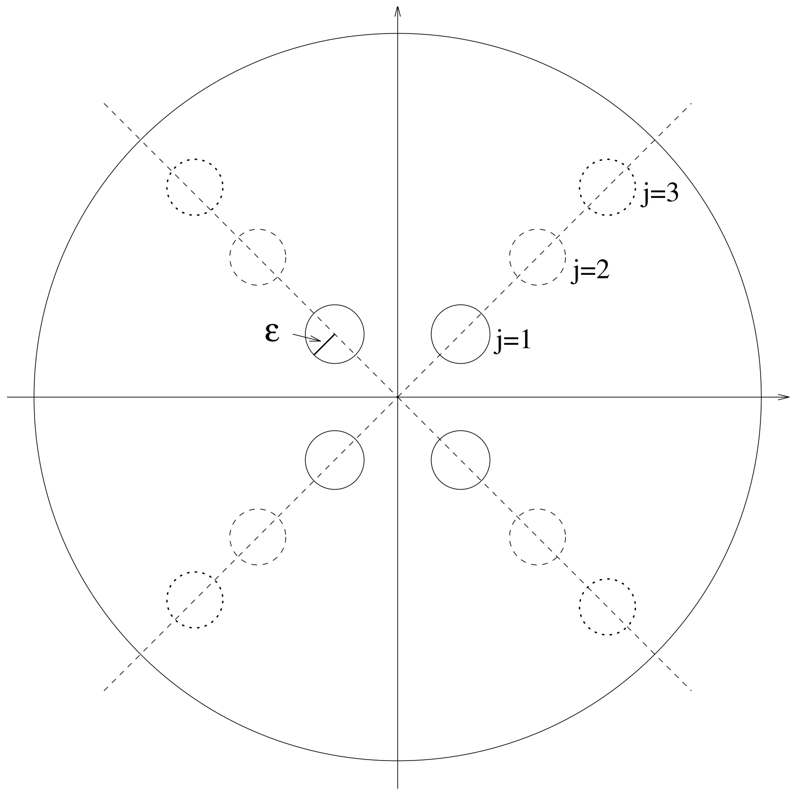
- The hybrid method requires us to determine P_R, G_N, R_N and the shape parameters $d_j(\kappa_j)$ for $j = 1, \ldots, N$. Then, solve a linear algebraic system
- For the unit disk, G_N and R_N are given explicitly by

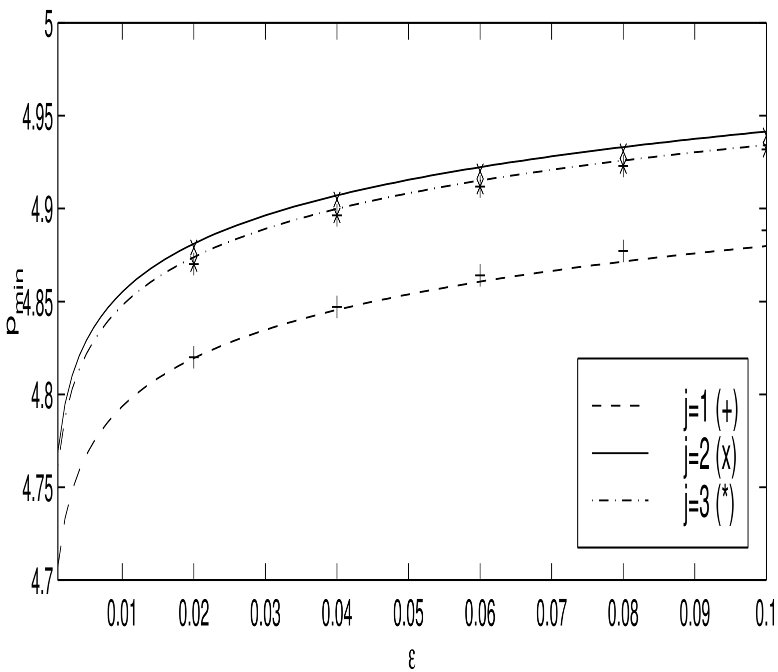
$$G_N(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi}\left(-\log|\mathbf{x} - \boldsymbol{\xi}| - \log\left| \mathbf{x}|\boldsymbol{\xi}| - \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right| + \frac{1}{2}(|\mathbf{x}|^2 + |\boldsymbol{\xi}|^2) - \frac{3}{4} \right) \,,$$

$$R_N(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi}\left(-\log\left| \boldsymbol{\xi}|\boldsymbol{\xi}| - \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right| + |\boldsymbol{\xi}|^2 - \frac{3}{4} \right) \,.$$

Oxygen Transport: Hybrid VII

Example Consider $N = 4$ capillaries of circular cross-section, each of radius ε, located inside a circular tissue domain Ω of unit radius. For each fixed j, with $j = 1, 2, 3$, the capillaries are centered at the locations $\mathbf{x}_i^j = j/4\left(\cos\left((2i-1)\pi/4\right), \sin\left((2i-1)\pi/4\right)\right)$ for $i = 1, \ldots, 4$. For simplicity take $\mathcal{M} = 0.3$, $\kappa_i = \infty$, and $p_{ci} = 5$, for $i = 1, \ldots, 4$. Thus, $d_i = 1$.





Two Linear Problems

Problem 2: Consider the following problem in an arbitrary two-dimensional domain with N small inclusions:

$$\triangle u - m(\mathbf{x}) u = 0 \,, \qquad \mathbf{x} \in \Omega \backslash \cup_{j=1}^{N} \Omega_{\varepsilon_j} \,, \tag{5.1a}$$

$$u = \alpha_j \,, \qquad \mathbf{x} \in \partial \Omega_{\varepsilon_j} \,, \quad j = 1, \ldots, N \,, \tag{5.1b}$$

$$u = f \,, \qquad \mathbf{x} \in \partial \Omega \,. \tag{5.1c}$$

Here $m(\mathbf{x}) > 0$ and f are arbitrary smooth functions, and α_j are constants. Formulate a linear system in terms of a certain Green's function, that effectively sums any infinite-order logarithmic series in the expansion of the solution.

Problem 10: Consider the following problem problem modeling the deflection of a two-dimensional plate with a small hole subject to loading:

$$\triangle^2 u = f(\mathbf{x}) \,, \qquad \mathbf{x} \in \Omega \backslash \Omega_{\varepsilon} \,, \tag{5.2a}$$

$$u = \partial_n u = 0 \,, \quad \mathbf{x} \in \partial \Omega \tag{5.2b}$$

$$u = \partial_n u = 0 \,, \quad \mathbf{x} \in \partial \Omega_{\varepsilon} \tag{5.2c}$$

Determine the asymptotic expansion in the outer and inner regions and show how to sum any infinite-logarithmic series that arise.

References

My paper available at: http://www.math.ubc.ca/ ward/prepr.html

- M. Titcombe, M. J. Ward, *An Asymptotic Study of Oxygen Transport from Multiple Capillaries to Skeletal Muscle Tissue*, SIAM J. Appl. Math., **60**(5), (2000), pp. 1767–1788.
- M. Titcombe, M. J. Ward, *Summing Logarithmic Expansions for Elliptic Equations in Multiply-Connected Domains with Small Holes*, Canad. Appl. Math. Quart., **7**(3), (1999), pp. 313–343.