Eigenvalue Optimization, Spikes, and the Neumann Green's Function

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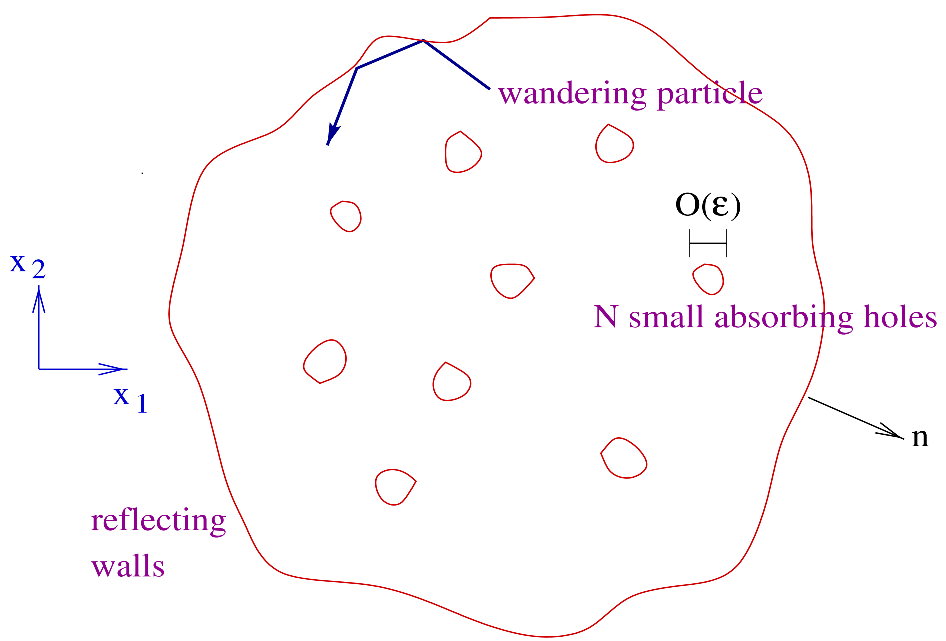
Outline of the Talk

1. An Eigenvalue Optimization Problem
   - Asymptotic expansions and the Neumann Green's function
   - Non-uniqueness of the optimal trap location
   - Ring-shaped configurations of traps
2. Spike Solutions to the Gierer-Meinhardt Model
   - equilibrium location of spikes and the Neumann Green's function
   - the reduced-wave Green's function: a bifurcation result.
3. Other applications related to the Neumann Green's function: (interacting Coulombic particles, locations of vortices in superconductivity, persistence problem in ecology in patchy environments)

The Perturbed Eigenvalue Problem

$$\Delta u + \lambda u = 0 \,, \quad x \in \Omega \backslash \Omega_p \,; \quad \int_{\Omega \backslash \Omega_p} u^2 \, dx = 1 \,,$$
$$\partial_n u = 0 \quad x \in \partial \Omega \,, \quad u = 0 \,, \quad x \in \partial \Omega_p \,.$$

- Here $\Omega_p = \cup_{i=1}^{N} \Omega_{\varepsilon_i}$ is a collection of N small interior non-overlapping holes or traps, each of ‘radius’ $O(\varepsilon) \ll 1$. The holes are assumed to be identical up to a translation and rotation.
- Also $\Omega_{\varepsilon_i} \to x_i$ as $\varepsilon \to 0$, for $i = 1, \ldots, N$. The centers x_i are arbitrary.



The Eigenvalue Optimization Problem

Goal: Let $\lambda_0 > 0$ be the fundamental eigenvalue. In the limit $\varepsilon \to 0$ (small hole radius) we seek to determine the hole locations x_i, for $i = 1, \ldots, N$, that maximize λ_0. In other words, we want to chose the trap locations to minimize the lifetime of a wandering particle in the domain, i.e. where are the best places to fish?

Specific Questions:

- For $N = 1$ (one hole), is there a unique location x_0 that maximizes λ_0? Can one construct domains Ω where there are several values for x_0 that locally maximize λ_0?
- For the unit ball $\Omega = |x| \leq 1$, can we determine ring-type configurations of holes $x_1, ..., x_N$ that maximize λ_0.
- Is this problem related (indirectly) to other concentration problems (i.e. spikes, vortices, persistence problems).

Reference: T. Kolokolnikov, M. Titcombe, MJW, Optimizing the Fundamental Neumann Eigenvalue for the Laplacian in a Domain with Small Traps, EJAM Vol. 16, No. 2, (2005), pp. 161-200.

Previous Studies I

Perturbed Eigenvalue Problems (Dirichlet, N-Dimensions): Swanson (1963); Ozawa (1980-85); Ward, Keller (1993); Flucher (1995).

For the Neumann problem, with N circular holes each of radius $\varepsilon \ll 1$, Ozawa (Duke Math. J. 1981) proved that

$$\lambda_0 \sim \frac{2\pi N \nu}{|\Omega|} + O(\nu^2), \quad \nu \equiv -1/\log \varepsilon \ll 1 .$$

Since this is independent of x_i, $i = 1, \ldots, N$, we need the further $O(\nu^2)$ term to optimize λ_0. For the Dirichlet problem, Ozawa (1981) proved

$$\lambda_0 \sim \lambda_{0d} + 2\pi \sum_{i}^{N} \left[u_0(x_i)\right]^2 \nu + O(\nu^2) , \quad \nu \equiv -1/\log \varepsilon \ll 1 .$$

To optimize λ_0 with one hole, put it at local maxima of u_0. For the Neumann or Dirichlet case, MJW, Henshaw, Keller (SIAP, 1993) showed

$$\lambda_0 \sim \lambda_*(\nu; x_1, ..x_N) + O(\varepsilon/\nu) .$$

For given $x_1, .., x_N$, λ_* (which "sums" all the log terms) satisfies a simple PDE that must be solved numerically. Highly accurate results for λ_0, but no analytical insight into how to optimize λ_0 wrt hole locations.

Eigenvalue Expansion: A Two-Term Result

A formal analysis (KTW) using matched asymptotic expansions leads to:

Principal Result: Suppose that there are N small (non-overlapping) circular holes of a common radius ε. Then,

$$\lambda_0(\varepsilon) \sim \lambda^* \,, \qquad \lambda^* = \frac{2\pi N \nu}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|} p(x_1, \ldots, x_N) + O(\nu^3) \,,$$

where $\nu \equiv -1/\log \varepsilon$ and $p(x_1, \ldots, x_N)$ is defined by

$$p(x_1, \ldots, x_N) = \sum_{j=1}^{N} \sum_{k=1}^{N} (\mathcal{G})_{jk} \,.$$

Here $(\mathcal{G})_{jk}$ are the entries of a certain Green's function matrix $\mathcal{G}$.

- Therefore, for $\nu \ll 1$, λ_0 has a local maximum at a local minimum point of the "Energy-like" function $p(x_1, \ldots, x_N)$.
- For identical (up to a rotation) non-circular holes, replace ε by εd, where d is the logarithmic capacitance of the hole.
- Similar results can be obtained when each hole has a different shape.

The Neumann Green's Function

The Neumann Green's function $G_m(x; x_0)$, with regular part $R_m(x; x_0)$, satisfies:

$$\Delta G_m = \frac{1}{|\Omega|} - \delta(x - x_0)\,, \quad x \in \Omega\,,$$

$$\partial_n G_m = 0\,, \quad x \in \partial\Omega\,; \quad \int_\Omega G_m(x; x_0)\, dx = 0\,,$$

$$G_m(x, x_0) = -\frac{1}{2\pi} \log|x - x_0| + R_m(x, x_0)\,;$$

The Green's matrix $\mathcal{G}$ is determined in terms of the interaction term $G_m(x_i; x_j) \equiv G_{mij}$, and the self-interaction $R_m(x_i; x_i) \equiv R_{mii}$ by

$$\mathcal{G} \equiv \begin{pmatrix} R_{m11} & G_{m12} & \cdots & \cdots & G_{m1N} \\ G_{m21} & R_{m22} & G_{m23} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G_{mN1} & \cdots & \cdots & G_{mNN-1} & R_{mNN} \end{pmatrix}.$$

One Hole: Uniqueness of Maximizer?

Corollary: Suppose that there is one circular hole of radius ε, centered at x_0. Then,

$$\lambda_0(\varepsilon) \sim \frac{2\pi\nu}{|\Omega|} - \frac{4\pi^2\nu^2}{|\Omega|} R_m(x_0; x_0) + O(\nu^3)\,, \quad \nu \equiv -1/\log\varepsilon\,.$$

Therefore λ_0 is maximized when the hole is centered at a point that minimizes $R_m(x_0; x_0)$. Is there a unique point x_0 in Ω that minimizes $R_m(x_0; x_0)$, and consequently maximizes λ_0?

- We need some properties of $R_m(x; x_0)$ and $\nabla R_{m0} \equiv \nabla R_m(x : x_0)|_{x=x_0}$ (complex analysis).
- Contrast those properties with those of the regular part R_d of the usual Dirichlet Green's function.

Dirichlet Green's Function: Regular Part

Consider the Dirichlet Green's function G_d, with regular part R_d:

$$\triangle G_d = -\delta(x - x_0) \quad x \in \Omega\,, \quad G_d = 0\,, \quad x \in \partial\Omega\,,$$

$$R_d(x, x_0) = G_d(x, x_0) + \frac{1}{2\pi} \log|x - x_0|\,, \qquad \nabla R_{d0} \equiv \nabla R_d(x, x_0)|_{x=x_0}\,.$$

- For a strictly convex domain Ω, $-R_{d0}$ is strictly convex, and thus there is a unique root to $\nabla R_{d0} = 0$. (B. Gustafsson, Duke J. Math (1990), Caffarelli and Friedman, Duke Math J. (1985)).
- ∇R_{d0} can be found for certain mappings $f(z)$ of the unit disk as

$$f^{'}(z_0)\nabla R_{d0} = -\frac{1}{2\pi}\left(\frac{z_0}{1-|z_0|^2} + \frac{f^{''}(\overline{z}_0)}{2f^{'}(\overline{z}_0)}\right)\,.$$

- Let B be the unit disk, and $f(z; a) = \frac{(1-a^2)z}{z^2-a^2}$. Then $f(B)$ is a symmetric but nonconvex dumbbell-shaped domain for $1 < a < 1 + \sqrt{2}$. Using the formula above, Gustafson (1990) proved that $\nabla R_{d0} = 0$ has three roots when $1 < a < \sqrt{3}$.
- Can one derive a similar result for the Neumann Green's function?

An Explicit Formula for ∇R_{m0}

Theorem: (KW) Let $f(z)$ map the unit disk B onto Ω satisfying:

(i) f is analytic and is invertible on $\overline{B}$.

(ii) f has only simple poles at $z_1, .., z_k$, and f is bounded at ∞.

(iii) $f = g/h$, with $g(z_i) \neq 0$, where g and h are analytic everywhere.

(iv) $\overline{f(z)} = f(\overline{z})$.

On the image $\Omega = f(B)$, let R_m be the regular part of G_m. Then, at x_0, with $z_0 \in B$ satisfying $x_0 = f(z_0)$,

$$\nabla R_{m0} = \frac{\nabla s(z_0)}{f'(z_0)}\,, \qquad \nabla s(z_0) = \frac{1}{2\pi}\left(\frac{z_0}{1-|z_0|^2} + \frac{f''(\overline{z}_0)}{2f'(\overline{z}_0)}\right)$$

$$- \frac{f'(\overline{z}_0)\left(f(z_0) - f(\frac{1}{\overline{z}_0})\right) + \sum_j \frac{g(z_j) f'(\frac{1}{z_j})}{z_j^2 h'(z_j)}\chi}{2\pi \sum_j \frac{g(z_j) f'(\frac{1}{z_j})}{z_j^2 h'(z_j)}}\,, \qquad \chi = \left(\frac{1}{z_j - \overline{z}_0} + \frac{z_j}{1 - z_j \overline{z}_0}\right)$$

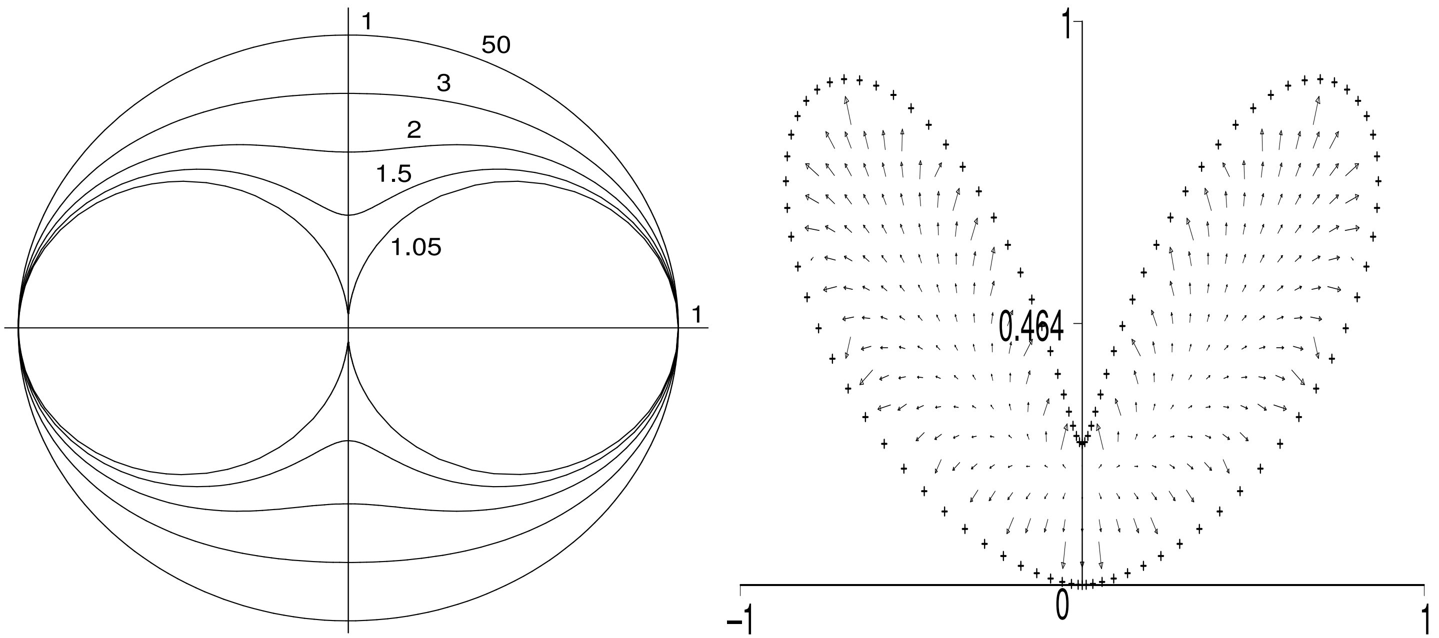
"Reduced Wave Green's Functions....", Kolokolnikov, MJW, EJAM (2003).

The proof uses residue theory and conformal mapping etc...

The Zeroes of ∇R_{m0}

Example 1: Let $f(z;a) = \frac{(1-a^2)z}{z^2-a^2}$. Then $f(B)$ is nonconvex for $1 < a < 1+\sqrt{3}$. For any $a > 1$, the complex variable formula can be used to show that $\nabla R_{m0} = 0$ has exactly one root at $z = 0$, which maximizes λ_0 for $\nu \ll 1$. This is qualitatively different than for the Dirichlet problem.

Example 2: A boundary integral is used to compute ∇R_{m0} for other nonconvex symmetric domains. The numerical results suggest that there is only one root to $\nabla R_{m0} = 0$. The boundary of the domain shown is $(x, y) = (\sin^2 2t + \frac{1}{4}\sin t)(\cos(t), \sin(t)), t \in [0, \pi]$. The resulting vector field ∇R_{m0} has only one equilibrium, at approximately $(0, 0.2)$.



A Non-Uniqueness Result I

Question: Let Ω be any simply connected domain, not necessarily convex. Is there a unique root of $\nabla R_{m0} = 0$?

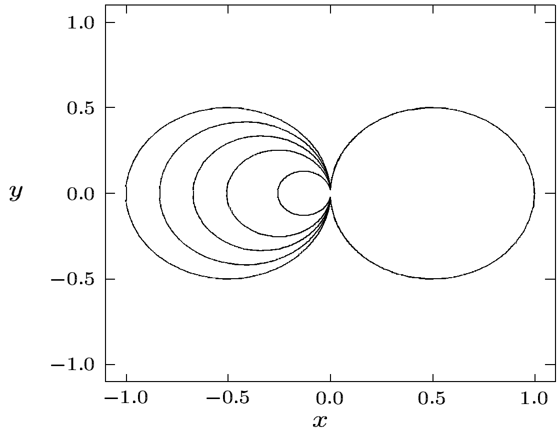
Not necessarily. Let B be the unit ball, and let $\Omega = f(B)$ where

$$f(z) = -\frac{\kappa z}{(z-a)(z+b)}\,, \quad a = 1+\varepsilon\,, \quad b = 1+\varepsilon\gamma\,,$$

with $\kappa = (a-1)(b+1)$ and $f(1) = 1$. Then, for $\varepsilon \to 0$, the area of Ω is

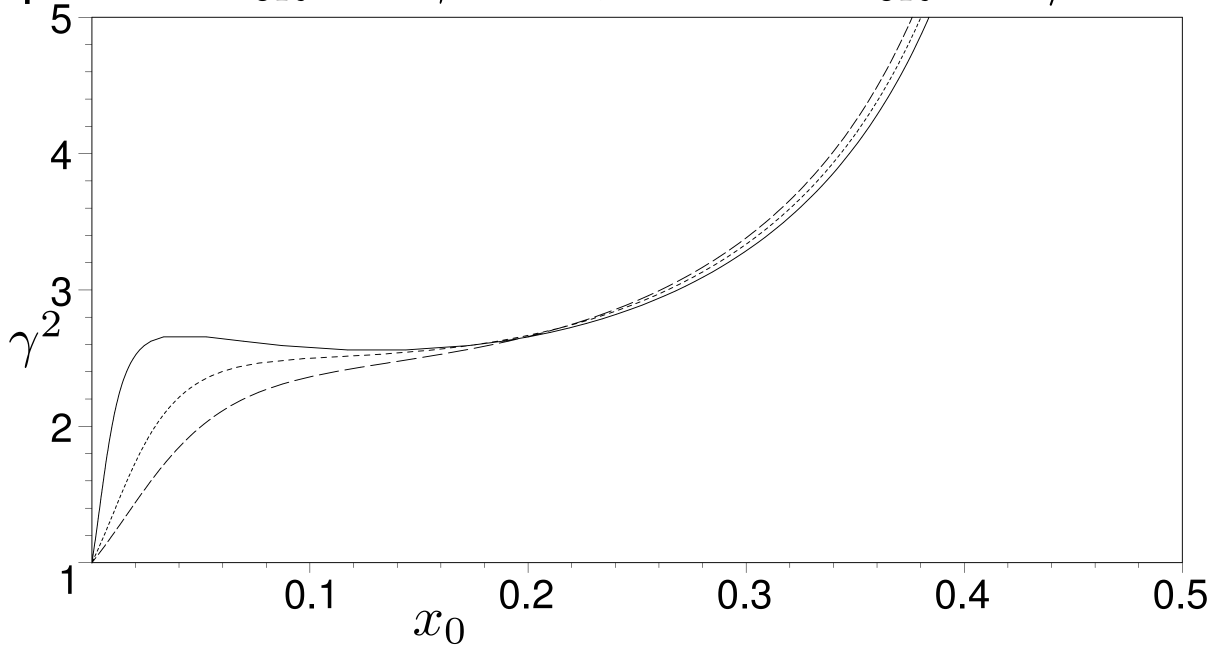
$$|\Omega| \sim \frac{\pi(1+\gamma^2)}{4\gamma^2}\,; \quad \gamma^2 = \text{ratio of area of big lobe/small lobe}$$

Let $\gamma > 1$. For $\varepsilon \to 0$, $\Omega = f(B)$ approaches the union of two circles; a larger circle centred at $(1/2, 0)$ of radius $1/2$, and a smaller circle centred at $(-1/(2\gamma), 0)$ of radius $1/(2\gamma)$. This is an asymmetric dumbbell.



A Non-Uniqueness Result II

Principal Result For the asymmetric dumbbell mapping, suppose that $1 < \gamma < \gamma_c = 1.59657$. Then, for $\varepsilon \ll 1$, there is a unique root to $\nabla R_{m0} = 0$ in Ω. On the range $1.59657 < \gamma < \sqrt{3}$, then $\nabla R_{m0} = 0$ has three roots. The smallest root is $x_0 = O(\varepsilon)$ (in the neck of the dumbbell) and there are another two roots x_{0L} and x_{0R}, which satisfy $x_{0L} < 1 - \sqrt{3}/2$ and $x_{0R} > 1 - \sqrt{3}/2$. As $\gamma \to \sqrt{3}$ from below, the root x_{0L} tends to zero and annihilates the smallest root x_0 in a saddle-node bifurcation. For $\gamma > \sqrt{3}$, $\nabla R_{m0} = 0$ has a unique root x_{0R}. As $\gamma \to \infty$, we have $x_{0R} \to 1/2$.



γ^2 versus x_0 for different ε. The solid, dotted, and dashed curves are for $\varepsilon = 0.01,\ 0.03$, and 0.05, respectively.

A Non-Uniqueness Result III

Conclusion: This result shows that for a slightly asymmetric dumbbell-shaped domain, where $1 < \gamma < 1.59657$, the optimum place to maximize λ_0 is to put the trap in the channel region of the dumbbell, but shifted slightly towards the side of the largest lobe. For $\gamma \gg 1$, where the left lobe of the dumbbell is very small the optimum place to insert the trap is near the centre of the right lobe of the dumbbell. The result shows that the transition between these two regimes has a complicated bifurcation structure for $1.59657 < \gamma < \sqrt{3}$, where λ_0 has two local maxima and one local minimum.

Multiple Holes in the Unit Disk

Let Ω be the unit circle, so that $|\Omega| = \pi$. For this domain, we calculate G_m and R_m explicitly as

$$G_m(x;\xi) = -\frac{1}{2\pi}\log|x-\xi| + R_m(x;\xi)$$

$$R_m(x;\xi) = -\frac{1}{2\pi}\log\left|x|\xi| - \frac{\xi}{|\xi|}\right| + \frac{(|x|^2+|\xi|^2)}{2} - \frac{3}{4}\,.$$

For the unit disk, the problem of minimizing $p(x_1,\ldots,x_N)$ is equivalent to the problem of minimizing the function $\mathcal{F}(x_1,\ldots,x_N)$ defined by

$$\mathcal{F}(x_1,\ldots,x_N) = -\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\log|x_j - x_k| - \sum_{j=1}^{N}\sum_{k=1}^{N}\log|1 - x_j\bar{x}_k| + N\sum_{j=1}^{N}|x_j|^2\,,$$

for $|x_j| < 1$ and $x_j \neq x_k$ when $j \neq k$.

We consider the restricted optimization problem where $\mathcal{F}$ is optimized over certain ring-type configurations of holes. We then compare the results with those computed with optimization software from MATLAB.

One-Ring Configurations

Two Patterns: I (one ring), II (ring with a center hole). Specifically,

$$x_j = re^{2\pi ij/N}\,, \quad j = 1, \ldots, N\,, \quad \text{(P I)}\,,$$

$$x_j = re^{2\pi ij/(N-1)}\,, \quad j = 1, \ldots, N-1\,,\; x_N = 0\,, \quad \text{(P II)}\,.$$

More generally, we can construct m ring patterns with m rings of radii $r_1, \ldots, r_m$, with $r_j < r_{j+1}$, inside the unit disk. Assume that there are J_k holes on the ring of radius r_k. On the k^{th} ring, for $k = 1, \ldots, m$, the centres of the holes are assumed to satisfy

$$\xi_j^{(k)} = r_k e^{2\pi ij/J_k} e^{i\phi_k}\,, \quad j = 1, \ldots, J_k\,.$$

Here ϕ_k is a phase angle with $\phi_1 = 0$.

For each pattern we can calculate $p(x_1, \ldots, x_N)$ explicitly and then optimize over the ring radii.

Pattern I

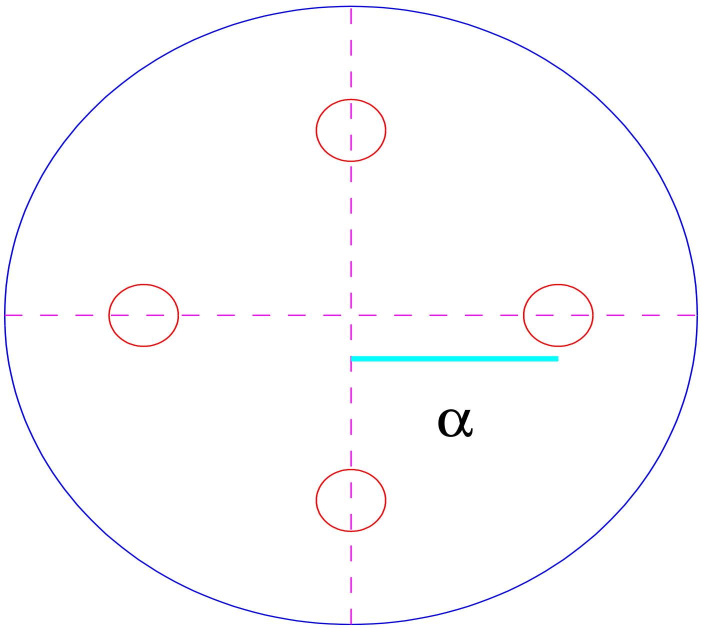
Principal Result: (Pattern I): Let $N > 1$, then $p = p_*/(2\pi)$ satisfies

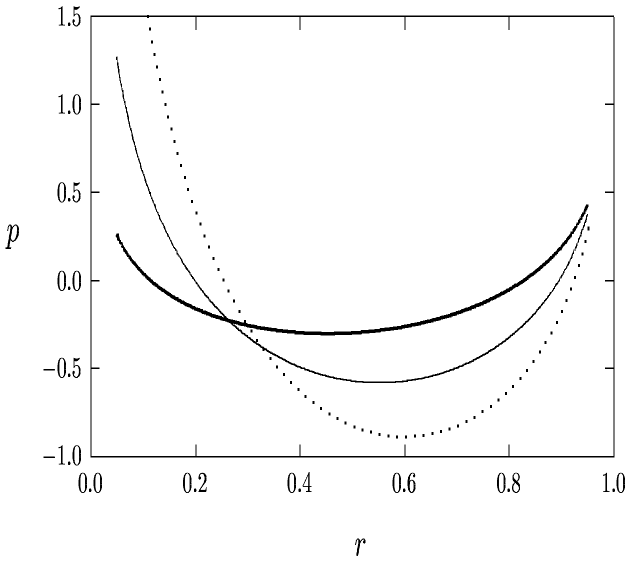
$$p_* = -N \log(N r^{N-1}) - N \log\left(1 - r^{2N}\right) + r^2 N^2 - \frac{3N^2}{4} \,.$$

Hence $p(r)$ has a unique minimum at $r = r_c$, where

$$\frac{r^{2N}}{1 - r^{2N}} = \frac{N-1}{2N} - r^2 \,.$$

Left: 4 holes on a ring. Right: p versus r for $N = 2, 3, 4$ holes on a ring.





Pattern II

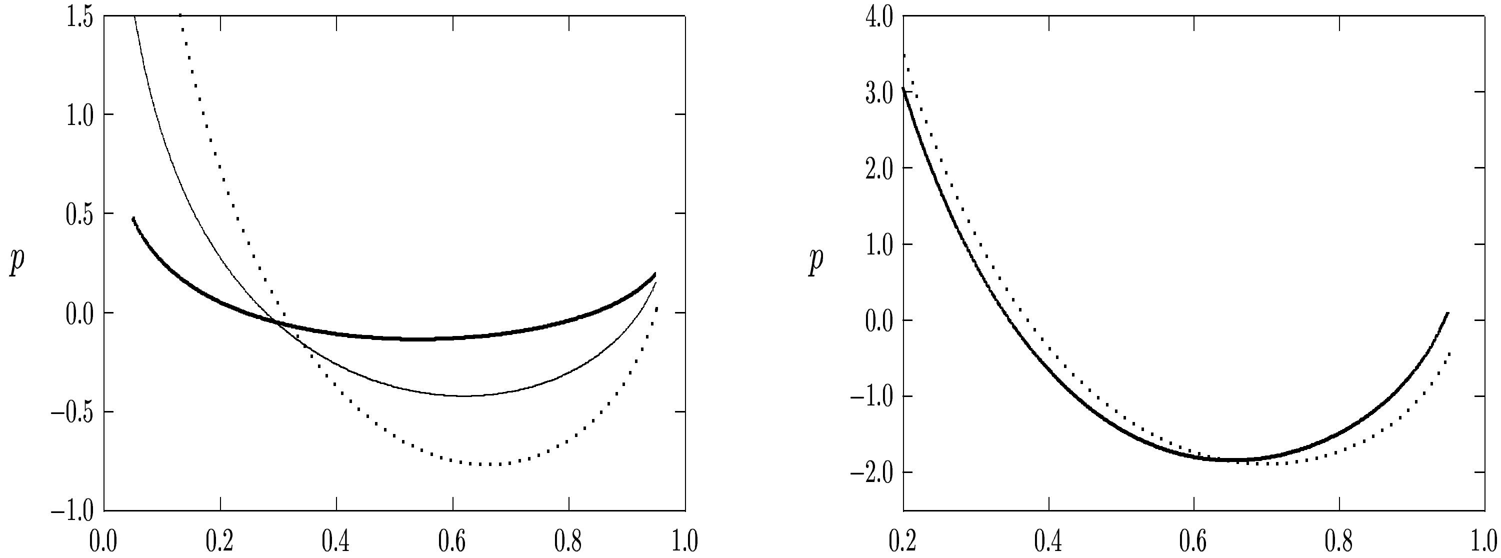
Principal Result: (Pattern II): Let $N > 1$, then $p = p_*(r)/(2\pi)$ satisfies

$$p_* = -(N-1)\log\left[(N-1)r^N\right] + r^2 N(N-1) - \frac{3N^2}{4} - (N-1)\log\left(1 - r^{2(N-1)}\right) .$$

Hence $p(r)$ has a unique minimum at $r = r_c$, where

$$\frac{r^{2N-2}}{1 - r^{2N-2}} = \frac{N}{N-1}\left(\frac{1}{2} - r^2\right) .$$

Left: $N = 2, 3, 4$ holes on a ring and a center hole. Right: 7 holes on a ring (heavy solid) and 6 holes on a ring with an extra center hole (dotted).

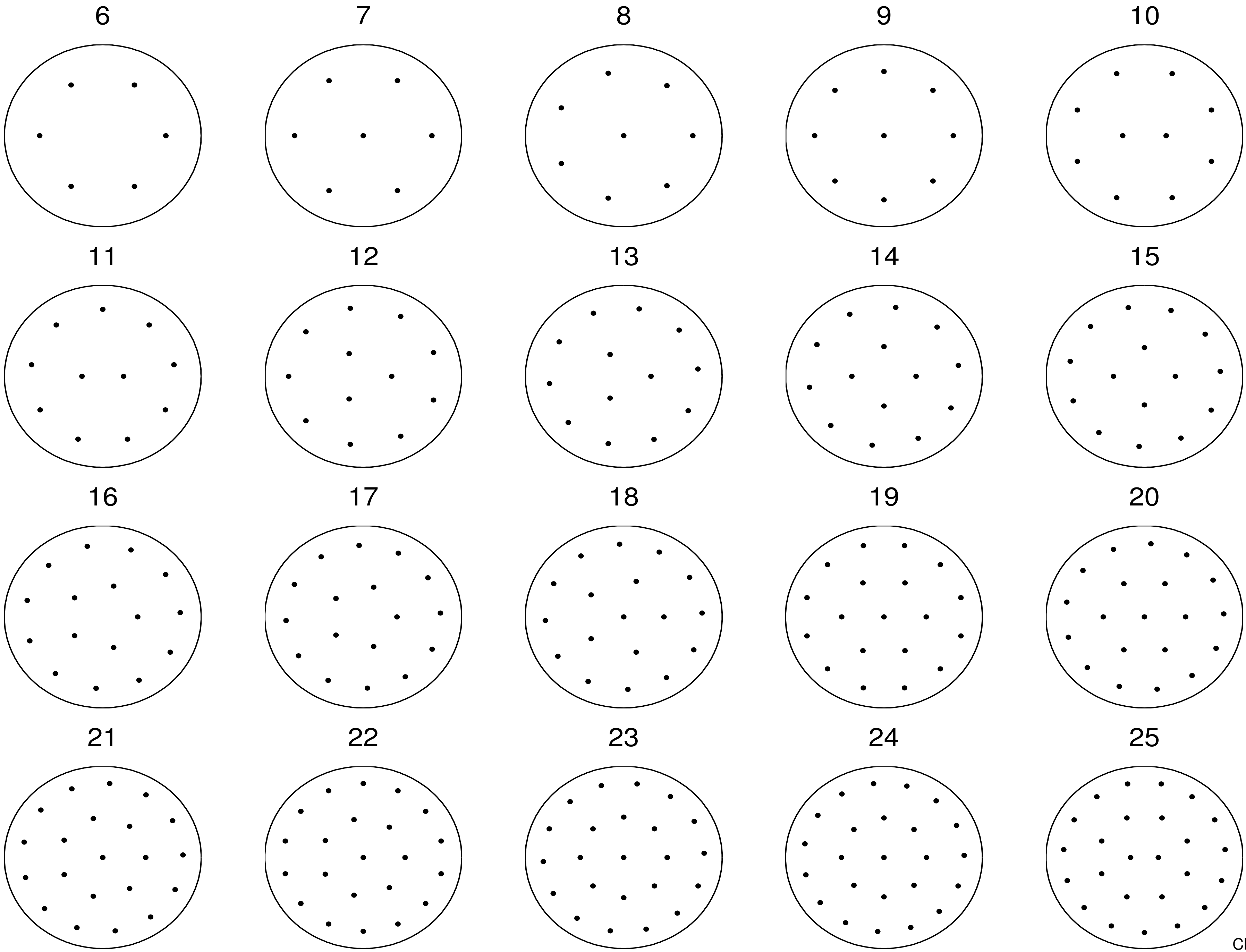


Restricted Optimization: m-ring Patterns

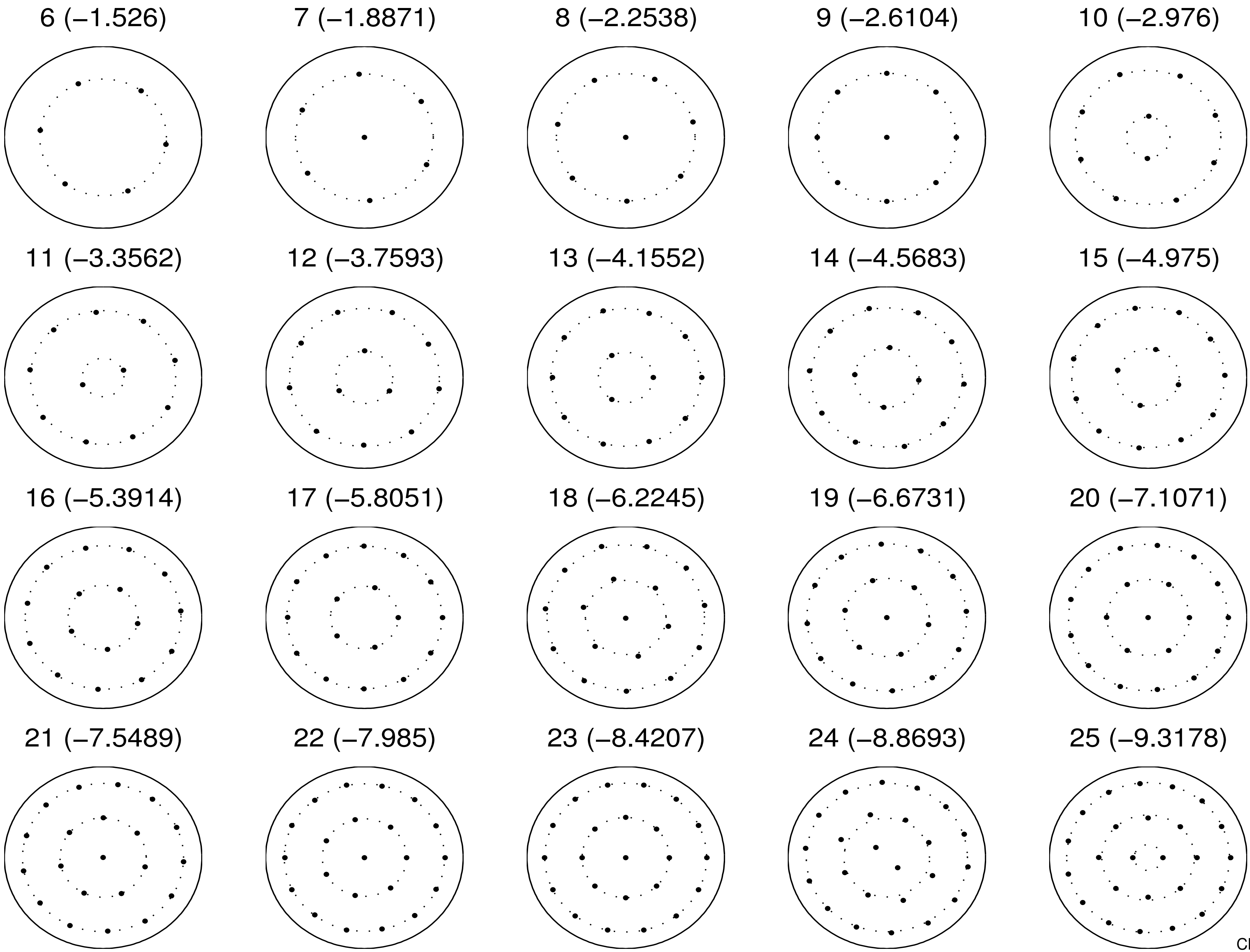
N	optimal pattern	$p_{\min}$	optimum r_j	second best pattern	p
6	(6)	-1.5260	0.642	[1](5)	-1.5134
7	[1](6)	-1.8871	0.698	(7)	-1.8398
8	[1](7)	-2.2538	0.702	(2,6)	-2.1732
9	[1](8)	-2.6104	0.705	(2,7)	-2.5754
10	(2,8)	-2.9686	0.222, 0.737	[1](9)	-2.9549
11	(2,9)	-3.3498	0.212, 0.736	(3,8)	-3.3449
12	(3,9)	-3.7546	0.288, 0.760	(2,10)	-3.7175
13	(3,10)	-4.1511	0.277, 0.758	(4,9)	-4.1457
14	(4,10)	-4.5660	0.327, 0.776	(3,11)	-4.5336
15	(4,11)	-4.9728	0.316, 0.773	(5,10)	-4.9636
16	(5,11)	-5.3903	0.354, 0.788	(4,12)	-5.3652
17	(5,12)	-5.8040	0.343, 0.785	[1](5,11)	-5.7921
18	[1](5,12)	-6.2242	0.408, 0.797	(6,12)	-6.2195
19	[1](6,12)	-6.6713	0.429, 0.809	[1](5,13)	-6.6422
20	[1](6,13)	-7.1052	0.418, 0.805	[1](7,12)	-7.0983
21	[1](7,13)	-7.5480	0.436, 0.815	[1](6,14)	-7.5257
22	[1](7,14)	-7.9844	0.426, 0.811	[1](6,15)	-7.9313
23	[1](8,14)	-8.4204	0.442, 0.819	[1](7,15)	-8.4058
24	[1](8,15)	-8.8566	0.433, 0.816	(2,8,14)	-8.8561
25	(2,8,15)	-9.3056	0.141, 0.469, 0.824	(3,8,14)	-9.3020

Table 1: Numerical results for the optimum configuration within the class of two and three-ring patterns with or without a centre hole. The first three columns indicate the optimum configuration, the minimum value of p, and the optimum ring radii. The last two columns correspond to the second best pattern. The notation $[1](5,12)$ indicates a two-ring pattern with a centre hole, which has 5 and 12 holes on the inner and outer rings, respectively.

Restricted Optimization: m-ring Patterns II

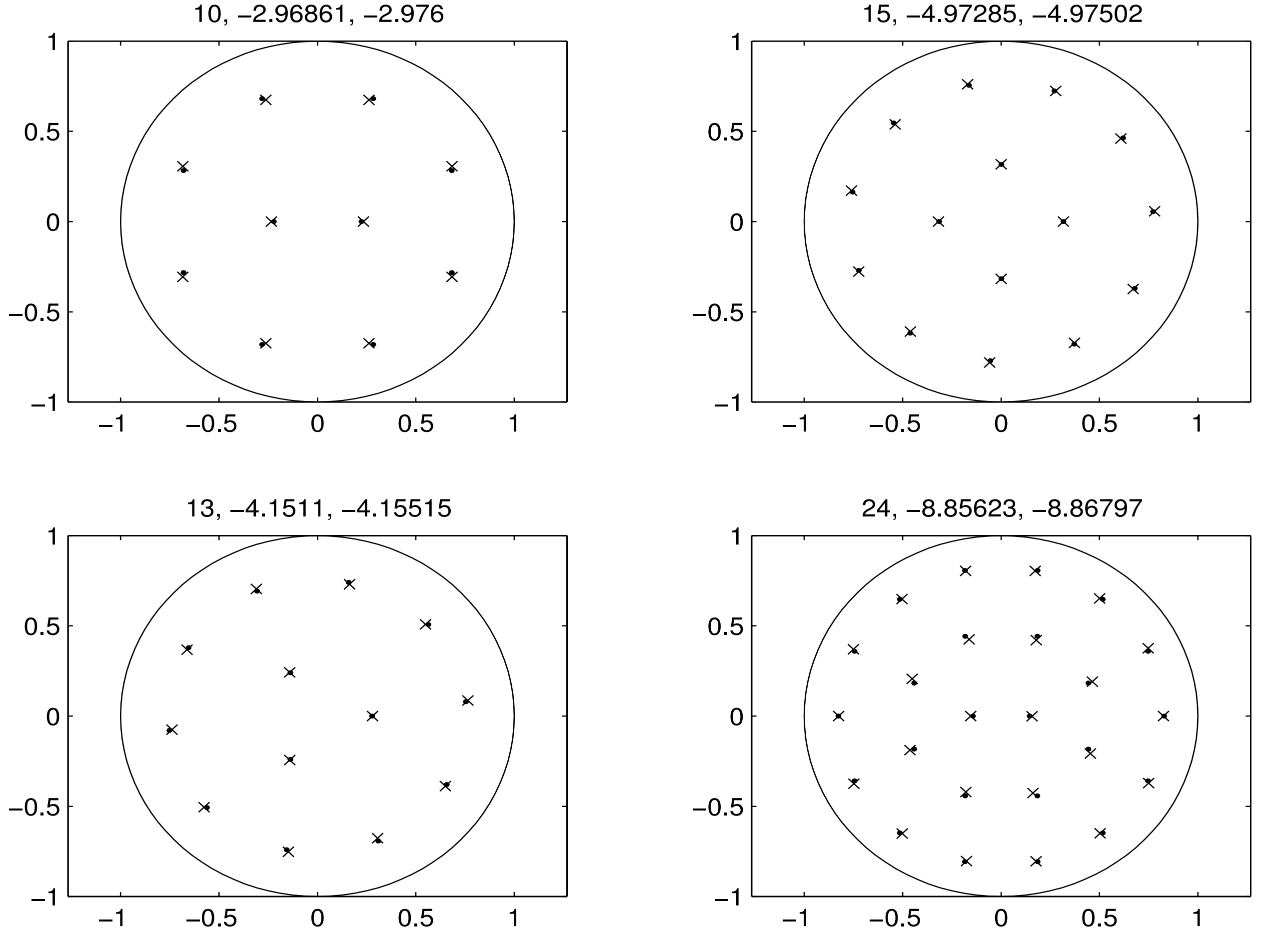


Full Optimization: m-ring Patterns II



Comparison: Restricted and Full Optimization

Optimization with respect to radii (dots) is compared with a MATLAB optimization with respect to $2N$ variables

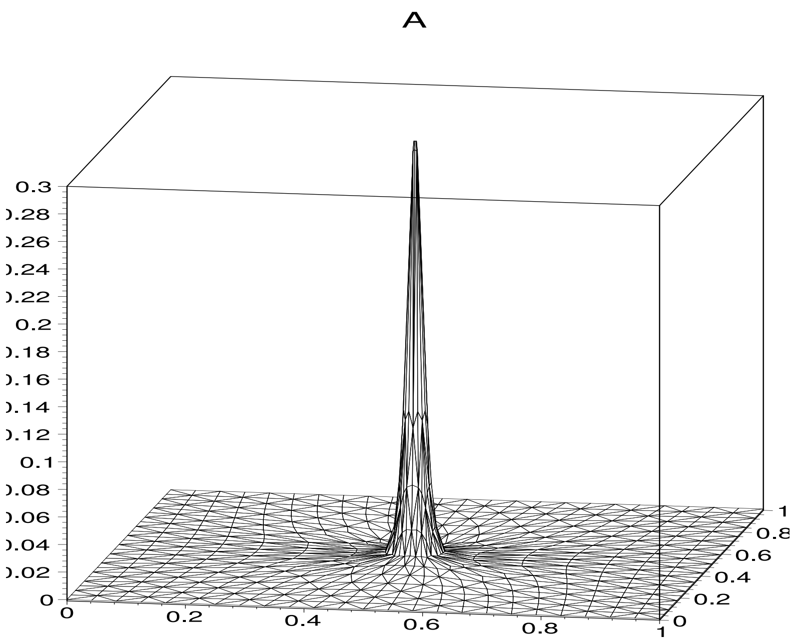


Spikes and the Gierer-Meinhardt Model

The GM model for an activator a and an inhibitor h in a 2-D bounded domain Ω, with $\varepsilon \ll 1$ is

$$a_t = \varepsilon^2 \triangle a - a + \frac{a^2}{h}\,, \quad \partial_n a = 0\,, \quad x \in \partial\Omega$$
$$\tau h_t = D \triangle h - h + \varepsilon^{-2} a^2\,, \quad \partial_n h = 0\,, \quad x \in \partial\Omega\,.$$

There is no variational structure for this problem. However, there are particle-like solutions for a, called spikes, when $\varepsilon \ll 1$. The behavior of these solutions depends strongly on the inhibitor diffusivity D.



Spike Dynamics, Equilibria: $D \gg O(-\ln \varepsilon)$

Principal Result: (KW) Provided that a stability condition on the spike profile is satisfied, then for $D \gg O(-\ln \varepsilon)$ and $\varepsilon \ll 1$ the spike dynamics is

$$\frac{dx_0}{dt} \sim -4\pi\varepsilon^2 \left(\frac{1}{-\ln \varepsilon + 2\pi \frac{D}{|\Omega|}} \right) \nabla R_{m0} \, ,$$

where R_{m0} is the regular part of the Neumann Green's function.

$$R_{m0} \equiv R_m(x_0, x_0), \quad \nabla R_{m0} = \nabla_x R_m(x, x_0)|_{x=x_0} \, .$$

For an N-spike solution (with spikes of equal height), the spike locations x_j satisfy

$$\frac{dx_j}{dt} \sim -\frac{4\pi\varepsilon^2 \nu}{1 + 2\pi\nu D N |\Omega|^{-1}} \left(\nabla R_m(x; x_j)|_{x=x_j} + \sum_{\substack{k=1 \\ k \neq j}}^{N} \nabla G_m(x; x_k)|_{x=x_j} \right) \, .$$

Key Correspondence Principal: The optimum hole locations for the eigenvalue problem correspond to equilibrium spike locations. A one-spike equilibrium is located at the minimum of R_{m0}. An N-spike equilibrium is located at the minimum of the function $p(x_1, \ldots, x_N)$.

Spike Dynamics and Equilibria: $D = O(1)$

Principal Result: (KW) Provided that a stability condition on the spike profile is satisfied, then for $D = O(1)$ and $\varepsilon \to 0$ the dynamics of a spike satisfies

$$\frac{dx_0}{dt} \sim -\left(\frac{4\pi q}{p-1}\right) \frac{\varepsilon^2}{\ln(\frac{1}{\varepsilon}) + 2\pi R_0} \nabla R_0 \,, \quad \text{as } \varepsilon \to 0 \,,$$

where $R_0 \equiv R(x_0, x_0)$ and $\nabla R_0 \equiv \nabla_x R(x, x_0)|_{x=x_0}$.

Here R_0 is the regular part of the reduced wave Green's function

$$\triangle G - \frac{1}{D} G = -\delta(x - x_0) \,, \quad x \in \Omega \,; \quad \partial_n G = 0 \,, \quad x \in \partial\Omega \,,$$

$$R(x, x_0) = G(x, x_0) + \frac{1}{2\pi} \ln |x - x_0| \,.$$

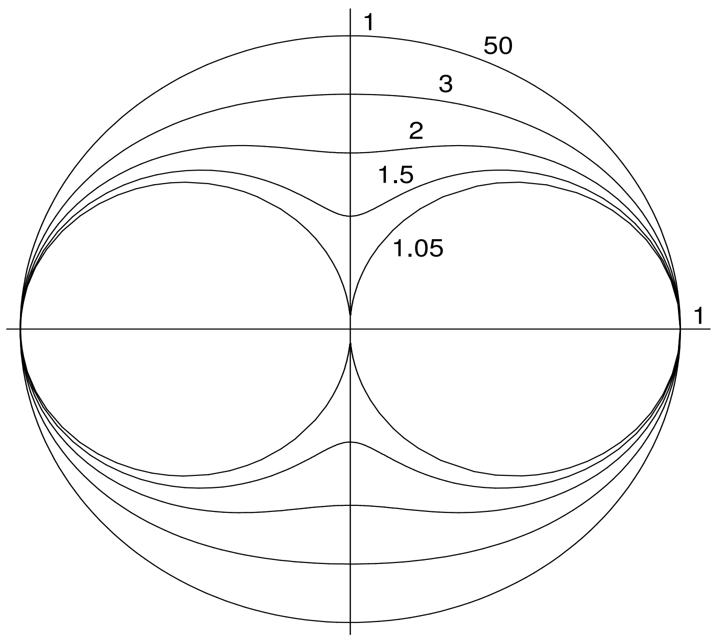
In a symmetric dumbbell-shaped domain:

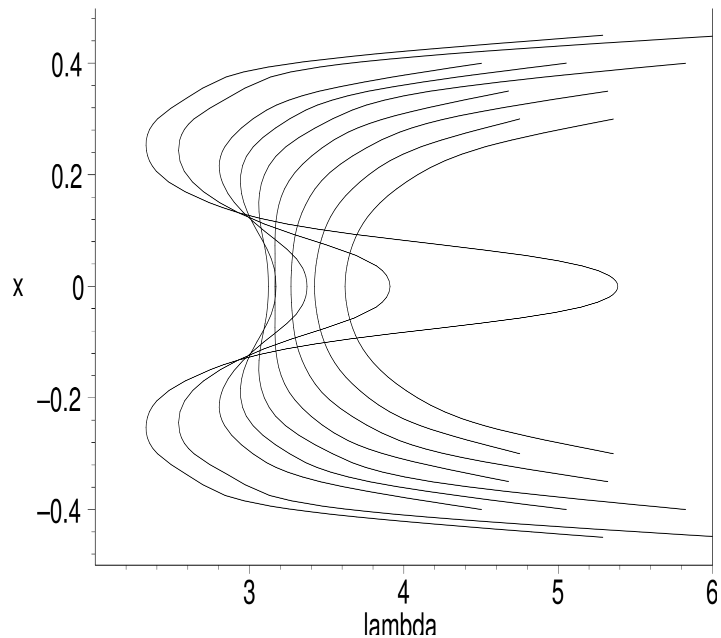
- For D small enough, R is determined in terms of the distance function. Hence, $\nabla R_0 = 0$ has a root in each lobe of a dumbbell.
- For $D \gg 1$, ∇R_0 can be approximated by ∇R_m, the Neumann regular part, which has a root only at the origin.
- So what happens to the roots as D is varied? (Bifurcation)

The Zeroes of ∇R_0

A boundary integral method is used to compute the roots of $\nabla R_0 = 0$ for the class of mappings $f(z;a) = \frac{(1-a^2)z}{z^2-a^2}$ of the unit disk. For $a \gg 1$, the mapping is a near identity and we get (approximately) a circle. For $a \to 1^+$ we get two (approximately) disjoint circles, connected by a thin neck.

Left: the image domain for different a. Right: the zeroes of $\nabla R_0 = 0$ along the real axis x versus $\lambda \equiv D^{-1/2}$. There is a subcritical pitchfork bifurcation for two nearly disjoint circles (a near one), and a supercritical pitchfork when $a \gg 1$.





Spot Stability for the GM model: I

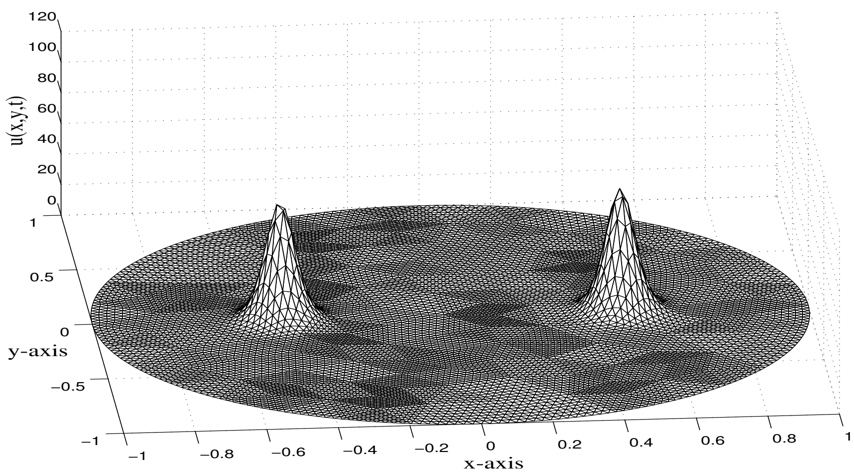
By analyzing a nonlocal eigenvalue problem in two space dimensions:

Theorem: [Winter Wei, JNLS 2001] For $\tau = 0$, $\varepsilon \to 0$, and $D \geq O(-\ln \varepsilon)$, an N-spike equilibrium solution is stable on an $O(1)$ time-scale iff

$$D < D_N \sim \frac{|\Omega|}{2\pi \nu N}, \qquad \nu \equiv -1/\ln \varepsilon .$$

- leading-order predicts D_N is independent of the spike locations x_i.
- need higher order terms in the logarithmic series for D_N. As for the Neumann eigenvalue problem with traps, we anticipate

$$D_N \sim \frac{|\Omega|}{2\pi \nu N} + F(x_1, \ldots, x_N) + O(\nu), \quad \nu \equiv -1/\ln \varepsilon .$$



For a movie showing a spike collapse due to overcrowding click here.

Spot Stability for the GM model: II

By incorporating the next term in the analysis:

Principal Result: [KW, 2006] Let $\tau = 0$, $\varepsilon \to 0$, $D \geq O(\nu^{-1})$ where $\nu \equiv -1/\ln\varepsilon$. Then, an N-spike quasi-equilibrium solution is stable on an $O(1)$ time-scale iff

$$D < D_N \sim \frac{|\Omega|}{2\pi\nu N} + |\Omega| \left(-p(x_1, \ldots, x_N) + \frac{2}{N} \min_{j=1,\ldots,N-1} \boldsymbol{c}_j^t \mathcal{G} \boldsymbol{c}_j \right) + O(\nu) \,.$$

Here $\boldsymbol{e}^t = (1, \ldots, 1)$ and the $\boldsymbol{c}_j$ correspond to an $N-1$ dimensional subspace perpendicular to e: i.e. $\boldsymbol{c}_j^t e = 0$ for with $\boldsymbol{c}_j^t \boldsymbol{c}_j = 1$.

Sketch: Let w be the radially symmetric ground state. The NLEP problems for $\tau = 0$ is

$$\triangle\Phi - \Phi + 2w\Phi - \chi_j w^2 \frac{\int_{\mathbb{R}^2} w\Phi \, dy}{\int_{\mathbb{R}^2} w^2 \, dy} = \lambda\Phi \,, \quad j = 1, \ldots, N \,,$$

$$\chi_j \equiv \frac{2N\mu_j}{\boldsymbol{e}^t \mathcal{G} \boldsymbol{e}} \,, \qquad \mathcal{C}\boldsymbol{c}_j = \mu_j \boldsymbol{c}_j \,, \qquad \mathcal{C} \equiv I + \frac{2\pi\nu D}{|\Omega|} \boldsymbol{e}\boldsymbol{e}^t + 2\pi\nu\mathcal{G} \,.$$

To calculate the stability threshold set $\min \chi_j = 1$ and solve for $D = D_N$.

Related Concentration Problems: Unit Disk

Our eigenvalue optimization problem is equivalent to minimizing

$$\mathcal{F}(x_1,\ldots,x_N) = -\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\log|x_j - x_k| - \sum_{j=1}^{N}\sum_{k=1}^{N}\log|1 - x_j\bar{x}_k| + N\sum_{j=1}^{N}|x_j|^2\,,$$

for $|x_j| < 1$, and $x_j \neq x_k$ when $j \neq k$.

In contrast, upon taking a certain limit of a variational formulation of the GL model of superconductivity in the unit disk, it was shown in Lefter, Radulescu (1996) and Sandier, Soret (2000) that, for an equilibrium vortex configuration $x_1,\ldots,x_N$ with vortices of a common winding number and $|x_j| < 1$, the vortex locations correspond to a minimum point of the renormalized energy $\mathcal{W}$ defined by

$$\mathcal{W}(x_1,\ldots,x_N) = -\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\log|x_j - x_k| - \sum_{j=1}^{N}\sum_{k=1}^{N}\log|1 - x_j\bar{x}_k|\,.$$

This problem differs from that of the eigenvalue problem only by the confinement potential $N\sum_{j=1}^{N}|x_j|^2$.

Related Concentration Problems II

Gueron, Shafrir (1999), studied the discrete variational problem of minimizing $\mathcal{H}(x_1, \ldots, x_N)$, for $x_j \in \mathbb{R}^2$, where

$$\mathcal{H}(x_1, \ldots, x_N) = -\frac{1}{2} \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \log |x_j - x_k| + \frac{1}{4} N(N-1) \log \left(\sum_{j=1}^{N} |x_j|^2 \right) .$$

This problem is equivalent to the constrained minimization problem

$$\min \mathcal{F}(x_1, \ldots, x_N) = -\sum_{j \neq k} \log |x_j - x_k| \, , \qquad \sum_{k=1}^{N} |x_j|^2 = 1 \, .$$

The minimizers of $\mathcal{H}$ correspond to stable configurations of point vortices in an ideal fluid which rotate with uniform angular velocity corresponding to prescribed angular momentum.

Further papers on interacting particle systems under Coulombic interaction fields and constrained by a confinement potential are in the Physics literature.

Persistence in Patchy Landscapes I

Let $\mu(x)$ be the intrinsic population growth rate. Assuming logistic growth and with no emigration of the population, the Fisher-Kolmogorov model is

$$u_t = \triangle u + u(\nu(x) - u)\,, \quad x \in \Omega\,; \quad \partial_n u = 0\,, \quad x \in \partial\Omega\,.$$

Hence, there is no immigration.

To study the stability of the $u \equiv 0$, extinct population solution, we set $u = e^{-\lambda t}\phi$ to get

$$\triangle\phi + \lambda\phi = -\nu(x)\phi\,, \quad x \in \Omega\,; \quad \partial_n \phi = 0\,, \quad x \in \partial\Omega\,.$$

If the first eigenvalue is $\lambda_1 > 0$ then $u = 0$ is stable and we have extinction. If $\lambda_1 < 0$ then $u = 0$ is unstable. Cantrell and Cosner (1991) proved that when $u = 0$ is unstable there must exist a unique nontrivial equilibrium solution $u > 0$ corresponding to the persistence of the species.

Persistence in Patchy Landscapes II

The patchiness of the environment is modeled by assuming that the "growth rate" of the extinct population solution satisfies

$$\nu = 0\,, \quad x \in D \backslash D_\varepsilon^- \,; \quad \text{neutral background}\,.$$

$$\nu = -\frac{\nu_-}{\varepsilon^2}\,, \quad x \in D_\varepsilon^- \,; \quad \text{localized patches of extinction}\,,$$

Assume the "localized patches of extiction" have a fixed small area Area$(D_\varepsilon^-) = O(\varepsilon^2) \ll 1$ with D_ε^- possibly multiply-connected (i.e. disjoint patches of extinction). Clearly $\lambda_1 > 0$ for $\varepsilon \ll 1$ so that $u = 0$ is stable.

We then re-seed the population in localized patches by letting

$$\nu = \frac{\nu_+}{\varepsilon^2}\,, \quad x \in D_\varepsilon^+ \,; \quad \text{localized population re-seeding}\,.$$

Assume Area$(D_\varepsilon^+) = O(\varepsilon^2) \ll 1$ and fixed and D_ε^+ possibly disjoint.

Can we now make $\lambda_1 < 0$? If so, where should the localized re-seeding patches be located to maximize $-\lambda_1$. In particular, is a fragmented re-seeding (D_ε^+ multiply-connected) preferable to a concentrated re-seeding? Berestycki et al.

Open: Eigenvalue Optimization in 3-D

In a three-dimensional bounded domain we consider

$$\Delta u + \lambda u = 0\,, \quad x \in \Omega \backslash \Omega_p\,; \quad \int_{\Omega \backslash \Omega_p} u^2\, dx = 1\,,$$

$$\partial_n u = 0 \quad x \in \partial \Omega\,, \quad u = 0\,, \quad x \in \partial \Omega_p\,.$$

Here $\Omega_p = \cup_{i=1}^{N} \Omega_{\varepsilon_i}$ is a collection of N small interior non-overlapping holes, with $\Omega_{\varepsilon_i} \to x_i$ as $\varepsilon \to 0$ and the centers x_i are arbitrary.
The expansion for $\varepsilon \ll 1$ of the principal eigenvalue is

$$\lambda_0 \sim \varepsilon \lambda_0 + \varepsilon^2 \lambda_1 + \cdots\,, \qquad \lambda_0 = \frac{4\pi}{|\Omega|} \sum_{j=1}^{N} C_j\,.$$

Here C_j is the capacitance of the j^{th} hole. In terms of the magnified hole $\Omega_j \equiv \Omega_{\varepsilon_j}/\varepsilon$, C_j is to be found from

$$\triangle_y v = 0\,, \quad y \notin \Omega_j\,; \quad v = 1\,, \quad y \in \partial \Omega_j\,; \quad v \sim -\frac{C_j}{|y|}\,, \quad |y| \to \infty\,.$$

To optimize λ wrt hole locations we need the λ_1 term. This term should involve the 3-D Neumann Green's function with singularity of $1/r$.

Open: Eigenvalues and Boundary Patches I

Consider the eigenvalues of the Laplacian in a domain Ω with no traps and assume that the boundary $\partial\Omega$ of the domain is almost entirely reflecting, but that there are N small patches of length ε or surface area ε^2 where $u = 0$ that are centered at $x_1, \ldots, x_N \in \partial\Omega$. In other words, the walls of the room are essentially insulated but there are N small patches where there is leakage of heat.

Then, with $\nu = -1/\log\varepsilon$, the leading order eigenvalue asymptotics are

$$\lambda \sim \frac{\pi N \nu}{|\Omega|} + \nu^2 \lambda_1 \,, \quad \text{2-Dimensions}\,,$$

$$\lambda \sim \frac{2\pi\varepsilon N C}{|\Omega|} + \varepsilon^2 \lambda_1 \,, \quad \text{3-Dimensions}\,.$$

Now suppose that we want to choose the locations x_j of the N patches to maximize λ (i.e. to maximize λ_1). These are the points which optimize the leakage rate. Intuitively, we would expect that the x_j should be as far as part as possible or equivalently be “equidistributed” on the boundary of the domain.

Open: Eigenvalues and Boundary Patches II

One might guess that this problem is very similar to putting N mutually repelling point charges on the boundary of the domain and then finding the equilibrium location of “minimal energy”.

In other words, in 3-dimensions when $\partial\Omega$ is a sphere, the optimal patch locations should be closely related to minimizing the following function with constraint:

$$\mathcal{E}(x_1, \ldots, x_N) = \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{|x_j - x_k|}, \quad |x_j| = 1 \, .$$

This is a famous dicrete optimization problem of finding the minimal discrete energy of “electrons” confined to the sphere. People that have looked at this are E. Saff, A. Kuijlaars, N. J. Sloane etc... It is also related to the discovery of new atomic configurations.

The difficulty with this problem is that the number of local minima grow exponentially with N, and so finding the global minimum is not trivial computationally.

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