AN ASYMPTOTIC ANALYSIS OF LOCALIZED SOLUTIONS FOR SOME REACTION-DIFFUSION MODELS IN MULTI-DIMENSIONAL DOMAINS

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Abstract

In the limit $\epsilon \to 0$, a spike-layer solution is constructed for the reaction-diffusion equation

$$\epsilon^2 \triangle u + Q(u) = 0\,, \qquad x \in D \subset \mathcal{R}^N\,,$$

$$\epsilon\, \partial_n u + bu = 0\,, \qquad x \in \partial D\,,$$

where $b > 0$ and D is a bounded convex domain. Here $Q(u)$ is such that there exists a unique radially symmetric function $u_c(\epsilon^{-1} r)$ satisfying $\epsilon^2 \triangle u_c + Q(u_c) = 0$ in all of $\mathcal{R}^N$, with $u_c(\rho)$ decaying exponentially at infinity. The spike-layer solution has the form $u \sim u_c[\epsilon^{-1}|x - x_0|]$, where the spike-layer location $x_0 \in D$ is to be found subject to the condition that $\text{dist}(x_0, \partial D) = O(1)$ as $\epsilon \to 0$. The determination of x_0 is shown to be exponentially ill-conditioned and asymptotic estimates for the exponentially small eigenvalues and the corresponding eigenfunctions associated with the linearized problem are obtained. These spectral results are used together with a limiting solvability condition to derive an equation for x_0. For a strictly convex domain, it is shown that there is an x_0 that is located at an $O(\epsilon)$ distance away from the point in D which is furthest from ∂D. Finally, hot-spot solutions to Bratu's equation are constructed asymptotically in a singularly perturbed limit.

Key Words: Spike-layers, exponentially small eigenvalues, projection method, hot-spot solutions.

1. Introduction

For certain autonomous nonlinear singular perturbation problems with internal layer behavior, the location of the internal layers can only be determined by incorporating the effect of exponentially small terms in the asymptotic expansion of the solution. For the case of one spatial dimension, boundary value problems of this type have been studied using various formal asymptotic methods in [11], [7], [17], [10] (see also the references therein). An example of such a problem is to construct an internal layer solution in the limit $\epsilon \to 0$ for

$$\epsilon^2 u'' + Q(u) = 0\,, \qquad -1 < x < 1\,; \qquad u'(\pm 1) = 0\,. \tag{1.1}$$

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For a given $Q(u)$ there are typically many solutions to (1.1). Depending on the form of $Q(u)$ these solutions can have either shock-type or spike-type internal layers.

For instance, if $Q(u) = -u + u^2/2$, then a solution to (1.1) with exactly one spike-layer has the form $u \sim u_c[\epsilon^{-1}(x - x_0)] \equiv 3\text{sech}^2[\epsilon^{-1}(x - x_0)/2]$, where $x_0 \in (-1, 1)$ is to be found. For (1.1), it follows by symmetry that $x_0 = 0$. However, since $u_c(z)$ decays exponentially as $z \to \pm\infty$, the problem of determining x_0 using asymptotic methods is exponentially ill-conditioned in the sense that $u_c[\epsilon^{-1}(x - x_0)]$ satisfies the boundary conditions in (1.1) to within exponentially small terms as $\epsilon \to 0$ for *any* $x_0 \in (-1, 1)$. An asymptotic method to determine x_0 in the presence of this ill-conditioning, which is based on a spectral decomposition of the solution to the linearized problem, was given in [17] (see also [7]) and was used there to treat various related internal layer problems.

In this paper we consider the following multi-dimensional analogue of (1.1):

$$\epsilon^2 \triangle u + Q(u) = 0\,, \qquad x \in D \subset \mathcal{R}^N\,, \tag{1.2a}$$

$$\epsilon \partial_n u + bu = 0\,, \qquad x \in \partial D\,. \tag{1.2b}$$

Here $\epsilon \ll 1$, D is a bounded convex domain with a smooth boundary ∂D, ∂_n denotes the outward normal derivative, and $b = b(s) \geq 0$, where $s \in R^{N-1}$ is a parameterization of ∂D. The nonlinearity $Q(u)$ is assumed to be such that there exists a unique radially symmetric solution $u_c(\epsilon^{-1}r)$ to (1.2a) in all of $\mathcal{R}^N$, which decays exponentially at infinity. This problem arises from a simple asymptotic reduction of a coupled system of reaction-diffusion equations of activator-inhibitor type (see [15], [16]). When $b = 0$, it was proved in [15] that the least-energy solution of (1.2) has the form $u \sim u_c[\epsilon^{-1}|x - x_0|]$, where x_0 is a point on ∂D. Subsequently, in [16] it was proved that x_0 is that point on ∂D which maximizes the mean curvature of ∂D. Results for (1.2) in the case where $Q = Q(u, x)$ were obtained in [8]. In analogy with (1.1), it is natural to expect that (1.2) has a spike-layer solution where the spike is not located on ∂D.

The main objective of this paper is to construct a spike-layer solution to (1.2) with exactly one spike, where the spike is *strictly contained* within D. Thus, we look for a solution to (1.2) in the form $u \sim u_c[\epsilon^{-1}|x - x_0|]$, where $x_0 \in D$ is to be found subject to the condition that $\text{dist}(x_0, \partial D) = O(1)$ as $\epsilon \to 0$. Here $\text{dist}(x_0, \partial D)$ denotes the distance between x_0 and ∂D. In analogy with (1.1), we show that the linearization of (1.2) about $u_c[\epsilon^{-1}|x - x_0|]$ is exponentially ill-conditioned. For $\epsilon \to 0$, we derive asymptotic estimates for the exponentially small eigenvalues and the corresponding eigenfunctions associated with the linearized problem. Then, by using these spectral estimates together with a limiting solvability condition for the solution to the linearized problem, we obtain an equation for the spike-layer location x_0. From this equation, we show that, for a strictly convex domain, there is an x_0 that is located at an $O(\epsilon)$ distance away from the point in D which is furthest from ∂D.

Our other objective is to construct hot-spot solutions for the Bratu problem

$$\begin{aligned} \triangle u + \lambda\, e^{u} &= 0\,, \qquad x \in D \in \mathcal{R}^2\,, \\ \partial_n u + bu &= 0\,, \qquad x \in \partial D\,. \end{aligned} \tag{1.3}$$

Here $b > 0$ and D is a bounded simply-connected domain. The qualitative feature of hot-spot solutions is that $u \to \infty$ as $\lambda \to 0$ in a localized region near some $x = x_j$, for $j = 1,..,m$, while $u = O(1)$ as $\lambda \to 0$ away from these points. When $b = \infty$, a system of equations for the hot-spot locations x_j, for $j = 1,..,m$, was derived in [13] and [14] using the Liouville transformation, which reduces (1.3) to the study of certain complex functions. For $b > 0$ and $\lambda \to 0$, we shall analyze (1.3) in a more straightforward way by using the method of matched asymptotic expansions, and we will derive a new result for the amplitudes of the hot-spots.

The outline of this paper is as follows. In §2 we review some known properties of solutions to (1.2) in $\mathcal{R}^N$ and we compute the canonical spike solution $u_c(\rho)$ numerically for a particular choice of $Q(u)$. In §3 we study the eigenvalue problem associated with linearizing (1.2) about $u_c[\epsilon^{-1}|x - x_0|]$. In §4 we derive an equation for the spike-layer location by extending the projection method of [17] to a multi-dimensional setting. This equation for x_0 is studied in §5.1 and §5.2 for the case $N = 2$ and $N = 3$, respectively. In §6 we construct hot-spot solutions for (1.3). Finally, in §7 we close with some remarks.

2. The Canonical Spike Solution

We assume that $Q(u)$ is such that the problem (1.2a), in all of $\mathcal{R}^N$, has a unique positive solution u_c with $u_c \to 0$ as $|x| \to \infty$. From [6] it follows that u_c is a radially symmetric monotone decreasing function that decays exponentially at infinity. This solution $u_c = u_c(\rho)$, where $\rho = \epsilon^{-1} r$ and $r = |x|$, is referred to as the canonical spike solution and it satisfies

$$u_c'' + \frac{(N-1)}{\rho} u_c' + Q[u_c(\rho)] = 0\,, \qquad \rho \geq 0\,, \tag{2.1a}$$

$$u_c'(0) = 0\,; \qquad u_c(\rho) \to 0\,, \quad \text{as} \quad \rho \to \infty\,; \qquad u_c > 0\,. \tag{2.1b}$$

Sufficient conditions on $Q(u)$ for the existence and uniqueness of the solution to (2.1) are given in [15] and [16] and proved in some of the references therein. Let $Q(u) = -\nu^2 u + f(u)$, where $\nu > 0$. Then, some of these sufficient conditions include the following:

- $u^{-1} f(u)$ is an increasing function of u,
- $f(u) = O(u^{\alpha})$ as $u \to 0$, where $\alpha > 1$,
- $f(u) = O(u^{m})$ as $u \to \infty$, where $1 < m < m_N$.

Here $m_N = (N+2)/(N-2)$ if $N \geq 3$ and $m_N = \infty$ if $N = 2$. For example, if $Q(u) = -u + u^m$, with $1 < m < m_N$, then (2.1) has a unique solution (see [9]).

Assuming that (2.1) has a unique, positive solution we can use the conditions $Q(0) = 0$ and $Q(u) = O(u)$ as $u \to 0$ to determine the precise far field behavior of $u_c(\rho)$. Using the method of dominant balance, we readily obtain that

$$u_c(\rho) \sim a\rho^{(1-N)/2} e^{-\nu\rho}\,, \quad \text{as} \quad \rho \to \infty\,, \qquad \text{where} \quad \nu \equiv \left[-Q^{'}(0)\right]^{1/2}. \tag{2.2}$$

Here a is a positive constant that is determined by the solution to (2.1). This exponential decay behavior is central to the analysis in §3 and §4.

For a given $Q(u)$, the solution to (2.1) is computed numerically using the boundary value solver COLSYS ([3]). The problem (2.1) is truncated to a large but finite domain $0 < \rho < \rho_m$ by using the following artificial boundary condition, which can be obtained from (2.2):

$$u_c^{'} = \left[\frac{(1-N)}{2\rho} - \nu\right] u_c\,, \qquad \text{at} \quad \rho = \rho_m\,. \tag{2.3}$$

To obtain numerical solutions for $u_c(\rho)$ when $N = 2$ and $N = 3$, we use a continuation procedure allowing N to take on real values. This procedure is initiated at $N = 1$, where an excellent initial guess for u_c is obtained by reducing (2.1a) to quadrature. To obtain explicit results for some asymptotic quantities in §3-5, we need numerical values for the constant a in (2.2) and the constant β defined by

$$\beta = \int_0^\infty \rho^{N-1} \left[u_c^{'}(\rho)\right]^2 \, d\rho\,. \tag{2.4}$$

The constant β is calculated by a numerical quadrature.

For $Q(u) = -u + u^m$ and $N = 2$, in Fig. 1 we plot $u_c(\rho)$ versus ρ for three values of m. Similar numerical results for $u_c(\rho)$ for the case $N = 3$ are shown in Fig. 2. In Table 1, we give numerical results for the constants a and β when $N = 2$ and $N = 3$. By varying the truncation value ρ_m, we believe that these results for a and β are correct to the number of digits shown.

3. Spectral Estimates for the Linearized Problem

We now study the spectral properties of the following eigenvalue problem that is associated with linearizing (1.2) about the canonical spike solution $u_c(\epsilon^{-1}r)$:

$$L_\epsilon \phi \equiv \epsilon^2 \Delta\phi + Q^{'}[u_c(\epsilon^{-1}r)]\phi = \lambda\phi\,, \qquad x \in D\,, \tag{3.1a}$$

$$\epsilon\, \partial_n \phi + b\phi = 0\,, \quad x \in \partial D\,; \qquad \big(\phi, \phi\big) = 1\,. \tag{3.1b}$$

Here $\big(u, v\big) \equiv \int_D uv\, dx$, $r = |x - x_0|$, and x_0 is the unknown location of the center of the spike. The eigenvalues and eigenfunctions of (3.1) are denoted by λ_j and ϕ_j for $j = 0, 1, \ldots$ The λ_j are real and satisfy $\lambda_j \to -\infty$ as $j \to \infty$. In the analysis below we assume that x_0 is strictly inside D so that $\text{dist}(x_0, \partial D) = O(1)$ as $\epsilon \to 0$.

3.1 Translation Eigenfunctions

Let x_j and x_{0j} denote the $j^{\rm th}$ coordinates of x and x_0, respectively. Since the solution to (1.2a) in $\mathcal{R}^N$ is $u_c(\epsilon^{-1}r)$, where $u_c(\rho)$ satisfies (2.1), we obtain upon differentiating (1.2a) with respect to x_j that

$$\epsilon^2 \triangle \left[\partial_{x_j} u_c(\epsilon^{-1}r)\right] + Q^{'} \left[u_c(\epsilon^{-1}r)\right] \left[\partial_{x_j} u_c(\epsilon^{-1}r)\right] = 0 . \qquad j = 1,..,N . \tag{3.2}$$

Therefore, by comparing (3.1a) with (3.2), it follows that the eigenvalue problem (3.1a) in all of $\mathcal{R}^N$ has N zero eigenvalues with corresponding eigenfunctions $\phi_j = B_j \partial_{x_j} u_c(\epsilon^{-1}r)$ for $j = 1,..,N$, where B_j is a normalization constant. These eigenfunctions result from the translation invariance of the problem (1.2a) when defined in $\mathcal{R}^N$.

Now consider the finite domain problem (3.1) with $\text{dist}(x_0, \partial D) = O(1)$. Since, from (2.2), $u_c[\epsilon^{-1}|x - x_0|]$ is exponentially small for $|x - x_0| = O(1)$, it is clear that $\partial_{x_j} u_c$ fails to satisfy the boundary condition (3.1b) by only exponentially small terms. This observation suggests that the translation eigenvalues and eigenfunctions get perturbed only very slightly by the presence of the finite domain. Specifically, we show that (3.1) has N exponentially small eigenvalues λ_j with corresponding eigenfunctions $\phi_j \sim B_j \left[\partial_{x_j} u_c(\epsilon^{-1}r) + \phi_{Lj}\right]$, for $j = 1,..,N$. Here, ϕ_{Lj} is a boundary layer function localized near ∂D, which allows (3.1b) to be satisfied.

Notice that each term $\partial_{x_j} u_c(\epsilon^{-1}r)$, for $j = 1,..,N$, has only one nodal line. Therefore, for the problem (3.1a) in all of $\mathcal{R}^N$ there must be exactly one positive eigenvalue λ_{0e}. The corresponding eigenfunction ϕ_{0e} is localized near $r = 0$ and decays exponentially as $\epsilon^{-1}r \to \infty$. The principal eigenpair λ_0, ϕ_0 for the finite domain problem (3.1a) should be exponentially close to the principal eigenpair λ_{0e}, ϕ_{0e} for (3.1a) in all of $\mathcal{R}^N$. In §3.2 we compute λ_{0e} and ϕ_{0e} numerically.

To estimate λ_j, for $j = 1,..,N$, we use Green's identity applied to (3.1) and $\partial_{x_j} u_c$ to derive

$$\lambda_j \left(\partial_{x_j} u_c, \phi_j\right) = -\epsilon \int_{\partial D} \phi_j \, \left(\epsilon \, \partial_n + b\right) \left[\partial_{x_j} u_c\right] \, dS , \tag{3.3}$$

where dS is the surface area element on ∂D. Substituting $\phi_j \sim B_j \left[\partial_{x_j} u_c + \phi_{Lj}\right]$ and (2.2) into (3.3), it follows that $\lambda_j = O(\epsilon^p e^{-2\epsilon^{-1}\nu})$ for $j = 1,..,N$, where p is a constant. We will estimate λ_j more precisely below once ϕ_{Lj} has been determined.

To calculate ϕ_{Lj} we need to write (3.1) in terms of a local coordinate system defined near ∂D. We set $\eta = n/\epsilon$, where $-n$ is the distance from $x \in D$ to ∂D and we let ξ denote $N-1$ coordinates orthogonal to n. Then, setting $\lambda = 0$ in (3.1a) and using $Q^{'}(u_c) \sim -\nu^2$, which is valid near ∂D, we find, to leading order, that ϕ_{Lj} satisfies

$$\begin{aligned} \partial_{\eta\eta}\phi_{Lj} - \nu^2 \phi_{Lj} = 0 \, , \quad \eta < 0 \, ; \qquad & \phi_{Lj} \to 0 \, , \quad \text{as} \;\; \eta \to -\infty \, , \\ \partial_{\eta}\phi_{Lj} + b\phi_{Lj} = h_j(\xi) \, , \quad \eta = 0 \, ; \qquad & h_j(\xi) \equiv -\left(\epsilon\partial_n + b\right) \left[\partial_{x_j} u_c\right] \Big|_{\eta = 0} \, . \end{aligned} \tag{3.4}$$

In deriving (3.4) we used the fact that $\partial_{x_j} u_c$ satisfies (3.2). The solution to (3.4) is $\phi_{Lj} = h_j(\xi) \left(\nu + b\right)^{-1} e^{\nu\eta}$. Therefore, in an $O(\epsilon)$ neighborhood of ∂D we have that

$$\phi_j = B_j \left[\partial_{x_j} u_c + h_j(\xi) (\nu + b)^{-1} \, e^{\nu n \epsilon^{-1}}\right] \, , \tag{3.5}$$

where $-n$ is the distance from $x \in D$ to ∂D.

In §4 we need an asymptotic formula for ϕ_j on ∂D. To obtain such a formula we first calculate $h_j(\xi)$ in (3.4) as

$$h_j(\xi) = -\epsilon^{-1} \frac{(x_j - x_{0j})}{r} \left[u_c^{''}(\epsilon^{-1} r) \hat{r} \cdot \hat{n} + b u_c^{'}(\epsilon^{-1} r) + O(\epsilon) \right] , \tag{3.6}$$

where $\hat{r} = (x - x_0) r^{-1}$ and $\hat{n}$ is the unit outward normal to ∂D. Then, by using the far field behavior (2.2) for u_c, we derive

$$h_j(\xi) \sim a\nu \, \epsilon^{(N-3)/2} \, (x_j - x_{0j}) r^{-(1+N)/2} e^{-\nu \epsilon^{-1} r} \left[b - \nu \hat{r} \cdot \hat{n} \right] . \tag{3.7}$$

Substituting (3.7) into (3.5), and using (2.2), we obtain the following asymptotic formula for ϕ_j on ∂D for $j = 1, .., N$:

$$\phi_j \sim -B_j \epsilon^{(N-3)/2} \frac{a\nu^2}{\nu + b} (x_j - x_{0j}) r^{-(1+N)/2} e^{-\nu \epsilon^{-1} r} \left[1 + \hat{r} \cdot \hat{n} \right] , \qquad x \in \partial D . \tag{3.8}$$

To calculate λ_j and B_j for $\epsilon \to 0$ we need to evaluate the inner product on the left side of (3.3). Since the dominant contribution to this inner product arises from the region near $x = x_0$ where $\phi_j \sim B_j \partial_{x_j} u_c$, a Laplace-type evaluation yields

$$\left(\partial_{x_j} u_c, \phi_j \right) \sim B_j \epsilon^{-2} \int_{\mathcal{R}^N} \left[u_c^{'}(\epsilon^{-1} r) \right]^2 \left(\frac{x_j - x_{0j}}{r} \right)^2 dx = \frac{B_j \epsilon^{N-2}}{N} \int_{\mathcal{R}^N} \left[u_c^{'}(\rho) \right]^2 \rho^{N-1} \, d\rho \, d\Omega_N . \tag{3.9}$$

Therefore, for $\epsilon \to 0$, we have

$$\left(\partial_{x_j} u_c, \phi_j \right) \sim B_j \epsilon^{N-2} \Omega_N N^{-1} \beta , \tag{3.10}$$

where β is defined in (2.4). Here Ω_N is the surface area of the unit ball in $\mathcal{R}^N$. To calculate B_j for $\epsilon \to 0$, which is defined by the normalization condition $\left(\phi_j, \phi_j \right) = 1$, we use $\phi_j \sim B_j \partial_{x_j} u_c$ and (3.10), to obtain

$$B_j \sim \left(\frac{N}{\beta \Omega_N} \right)^{1/2} \epsilon^{(2-N)/2} . \tag{3.11}$$

Next, to calculate λ_j for $j = 1, .., N$, we substitute (3.7), (3.8) and (3.10) into (3.3) to obtain the asymptotic formula

$$\lambda_j \sim -\frac{a^2 \nu^3 N}{\beta \Omega_N} \int_{\partial D} \frac{1}{\nu + b} \left(\frac{x_j - x_{0j}}{r} \right)^2 r^{1-N} e^{-2\nu \epsilon^{-1} r} \left[b - \nu \hat{r} \cdot \hat{n} \right] \left[1 + \hat{r} \cdot \hat{n} \right] \, dS . \tag{3.12}$$

Let $\xi = (\xi_1, .., \xi_{N-1})$ be a parameterization of ∂D and let x_0 be a given point in D with $\text{dist}(x_0, \partial D) = O(1)$. To obtain a precise asymptotic estimate for λ_j we can evaluate the surface integral in (3.12) using a multi-dimensional Laplace's method (see [5]). Assume that the distance between $x(\xi) \in \partial D$ and x_0 is minimized at a unique point $x(\xi_0) \in \partial D$ where $\xi = \xi_0$. Let $r = r_m$ denote the minimum distance. Then, the dominant contribution to the integral in (3.12) arises from the region near $\xi = \xi_0$. By choosing the parameterization of ∂D near ξ_0 such that each ξ_j corresponds to arclength along one of the principal directions through ξ_0, we obtain for any smooth $F(r)$ that

$$\int_{\partial D} r^{1-N}\, F(r) e^{-2\nu\epsilon^{-1} r}\, dS \sim \left(\frac{\pi\epsilon}{\nu r_m}\right)^{(N-1)/2} F(r_m)\, H(r_m)\, e^{-2\nu\epsilon^{-1} r_m} \,. \tag{3.13a}$$

Here $H(r_m)$ is defined by

$$H(r_m) \equiv (1 - r_m/R_1)^{-1/2} (1 - r_m/R_2)^{-1/2} \ldots (1 - r_m/R_{N-1})^{-1/2} \,, \tag{3.13b}$$

where R_j, for $j = 1, .., N-1$, are the principal radii of curvature of ∂D at $x(\xi_0)$. In obtaining (3.13) we have assumed that the non-degeneracy condition $R_j > r_m$ for $j = 1, .., N-1$ holds.

We can then use (3.13) to evaluate (3.12) asymptotically. This yields, for $j = 1, .., N$, that

$$\lambda_j \sim -\frac{2a^2\nu^3 N}{\beta\Omega_N} \left(\frac{b_m - \nu}{b_m + \nu}\right) \left(\frac{\pi\epsilon}{\nu r_m}\right)^{(N-1)/2} H(r_m) \left[\hat{r}_m \cdot \hat{i}_j\right]^2 e^{-2\nu\epsilon^{-1} r_m} \,. \tag{3.14}$$

Here $\hat{r}_m \equiv (x(\xi_0) - x_0) r_m^{-1}$, $b_m \equiv b(\xi_0)$ and $\hat{i}_j$ is the unit basis vector in the x_j^{th} direction. This asymptotic formula clearly shows that the problem of determining the location of the spike-layer solution for (1.2) is exponentially ill-conditioned. When $Q(u) = -u + u^m$, we can use the numerical results for a and β given in Table 1 to obtain numerical values for λ_j, $j = 1, .., N$.

3.2 The Principal Eigenfunction and Eigenvalue

We now compute the principal eigenpair λ_{0e}, ϕ_{0e} of (3.1a) in all of $\mathcal{R}^N$. The eigenfunction ϕ_{0e} is radially symmetric and is of one sign. Thus, we write $\phi_{0e} = \phi_{0e}(\rho)$, where $\rho = \epsilon^{-1} r$. In $\mathcal{R}^N$ we can re-scale (3.1a) to eliminate ϵ and, thereby, obtain that λ_{0e} is a positive constant independent of ϵ. Since $\phi_{0e}(\rho)$ decays exponentially as $\rho \to \infty$, the principal eigenpair λ_0, ϕ_0 for the finite domain problem (3.1) should be exponentially close to λ_{0e}, ϕ_{0e}. These perturbations will not be calculated here. However, since $\lambda_{0e} > 0$ is independent of ϵ, it follows that, for the finite domain problem, λ_0 must also be positive.

From (3.1a) in all of $\mathcal{R}^N$, we obtain that λ_{0e} and $\phi_{0e}(\rho)$ satisfy

$$\phi_{0e}'' + \frac{(N-1)}{\rho}\phi_{0e}' + Q'\left[u_c(\rho)\right] \phi_{0e} = \lambda_{0e}\phi_{0e} \,, \qquad \rho \geq 0 \,, \tag{3.15a}$$

$$\phi_{0e}'(0) = 0 \,; \qquad \phi_{0e}(\rho) \to 0 \,, \quad \text{as} \quad \rho \to \infty \,; \qquad \phi_{0e} > 0 \,, \tag{3.15b}$$

$$\int_0^\infty \rho^{N-1} \left[\phi_{0e}(\rho)\right]^2 d\rho = 1 \,. \tag{3.15c}$$

The boundary value solver COLSYS ([3]) is used to compute λ_{0e} and ϕ_{0e} after first re-writing (3.15) as a first order system of degree four. To compute solutions for $N = 2$ and $N = 3$ we let N

be real and we perform a continuation in N starting from the planar case $N = 1$. The condition $\phi_{0e}(\rho_m) = 0$ is used to truncate the domain. In the computations below we took $\rho_m = 12$.

For $Q(u) = -u + u^m$, in Table 2 we give numerical values for λ_{0e} for different values of m and N. For the case $N = 2$, in Fig. 3 we plot the eigenfunction $\phi_{0e}(\rho)$ for three different values of m. Similar computations can be done for other forms of $Q(u)$. We remark that when $N = 1$ and $Q(u) = -u + u^2/2$ we have that $u_c(\rho) = 3\text{sech}^2(\rho/2)$. In this case, we can obtain analytically that $\lambda_{0e} = 5/4$ and $\phi_{0e} = (\sqrt{15}/4)\text{sech}^3(\rho/2)$. This special solution was used as a partial check on our numerical procedure for computing λ_{0e}.

Since $\lambda_0 > 0$, we conclude that an equilibrium spike solution, where the spike is centered strictly inside the domain D, is linearly unstable for the parabolic problem associated with (1.2). Therefore, in contrast to the related metastable phase separation problems studied in [1], [2] and [18], which involve shock-type internal layers, the motion of spike layers for the associated parabolic problem will not be exponentially slow. This shows that the occurrence of exponentially small eigenvalues is necessary but not sufficient for the existence of an exponentially slow internal layer motion.

4. The Projection Method to Determine the Spike Location

We now use an extension of the projection method, developed in [17], to derive an equation for the spike location x_0. In (1.2), we set $u(x;\epsilon) = u_c[\epsilon^{-1}|x - x_0|] + w(x;\epsilon)$, where $u_c(\rho)$ is the canonical spike solution and x_0 is to be determined. Assuming that $w \ll u_c$, we obtain the following linearized problem for w:

$$L_\epsilon w \equiv \epsilon^2 \triangle w + Q'(u_c) w = 0\,, \qquad x \in D\,, \tag{4.1a}$$

$$\epsilon\, \partial_n w + bw = -\left(\epsilon\, \partial_n + b\right) u_c\,, \qquad x \in \partial D\,. \tag{4.1b}$$

Here $u_c = u_c(\epsilon^{-1} r)$ and $r = |x - x_0|$.

We then expand w in terms of the normalized eigenfunctions ϕ_j of (3.1) as

$$w(x;\epsilon) = \sum_{j=0}^{\infty} \frac{C_j}{\lambda_j} \phi_j\,, \qquad \text{where} \qquad C_j \equiv \lambda_j \big(w, \phi_j\big)\,. \tag{4.2}$$

The coefficients C_j are obtained by applying Green's identity to (3.1) and (4.1), which yields

$$C_j = \epsilon \int_{\partial D} \phi_j \left(\epsilon\, \partial_n + b\right) u_c \, dS\,. \tag{4.3}$$

For $\epsilon \to 0$ and $\text{dist}(x_0, \partial D) = O(1)$, the far field behavior (2.2) can be used to obtain

$$\left(\epsilon\, \partial_n + b\right) u_c(\epsilon^{-1} r) \sim a \epsilon^{(N-1)/2} r^{(1-N)/2} e^{-\nu \epsilon^{-1} r} \left[b - \nu \hat{r} \cdot \hat{n}\right]\,. \tag{4.4}$$

Substituting (4.4), (3.8) and (3.11) into (4.3), we find for $j = 1, .., N$ that

$$C_j \sim -\left(\frac{\epsilon^N N}{\beta \Omega_N}\right)^{1/2} a^2 \nu^2 \int_{\partial D} \frac{(x_j - x_{0j})}{\nu + b}\, r^{-N} e^{-2\nu\epsilon^{-1} r} \left[1 + \hat{r} \cdot \hat{n}\right] \left[b - \nu \hat{r} \cdot \hat{n}\right] dS\,. \tag{4.5}$$

The projection method to determine x_0 is based on imposing solvability conditions for (4.1) that must hold in the limit $\epsilon \to 0$. These conditions suppress the exponential ill-conditioning inherent in the linearized problem, thereby ensuring that the residual $w(x;\epsilon)$, given in (4.2), is exponentially small. To derive these conditions we note from (4.5) and (3.12) that $C_j/\lambda_j = O\left(\epsilon^{N/2}\right)$ for $j = 1,..,N$. It then follows from (4.2) that the exponentially small boundary residual in (4.1b) leads to an algebraically large response in $w(x;\epsilon)$. To eliminate this exponential sensitivity we impose that $C_j = 0$ for $j = 1,..,N$, which yields an equation for x_0. Since, there are no other λ_j that are exponentially small, it follows from (4.2), (4.3) and (4.4) that $w(x;\epsilon)$ is exponentially small. This leads to the following statement:

Proposition 1: (Spike Location): *For* $\epsilon \to 0$, *there is a spike-layer solution to* (1.2) *given by* $u \sim u_c[\epsilon^{-1}|x - x_0|]$, *where* $u_c(\rho)$ *satisfies* (2.1) *and* (2.2). *With the assumption that* $dist(x_0, \partial D) = O(1)$, *the location* x_0 *of the spike-layer is given by the root of* $I(x_0)$, *where the vector-valued function* $I(x_0)$ *is defined by*

$$I(x_0) \equiv \int_{\partial D} r^{1-N}\, e^{-2\nu\epsilon^{-1} r}\, [1 + \hat{r}\cdot\hat{n}] \left(\frac{b - \nu\hat{r}\cdot\hat{n}}{\nu + b}\right) \hat{r}\, dS\,. \tag{4.6}$$

In (4.6), ν *is defined in* (2.2), $b = b(\xi)$, $r = |x(\xi) - x_0|$, $\hat{r} = [x(\xi) - x_0]r^{-1}$ *and* $\hat{n} = \hat{n}(\xi)$ *is the unit outward normal to* ∂D *at the point* $x(\xi) \in \partial D$. *Here* $\xi = (\xi_1,..,\xi_{N-1})$ *are the surface coordinates that parametrize* ∂D.

5. The Spike Location

The spike-layer location x_0 satisfies $I(x_0) = 0$, where I is defined in (4.6). In §5.1 we consider the case $N = 2$ and in §5.2 we briefly consider the case $N = 3$.

5.1 The Two-Dimensional Case N = 2

We begin by qualitatively examining the geometrical implications of (4.6). For simplicity, we assume for the moment that $b(\xi) - \nu$ does not change sign as $x(\xi)$ is varied over ∂D. In (4.6), ξ is arclength along ∂D.

Let $r(\xi) = |x(\xi) - x_0|$ where $x(\xi) \in \partial D$. For a given $x_0 \in D$ and when $\epsilon \ll 1$, the dominant contribution to $I(x_0)$ in (4.6) arises from those points $x(\xi_i) \in \partial D$, with $i = 1,..,M$, which are 'asymptotically' closest to x_0. By 'asymptotically' closest, we mean that $r'(\xi_i) = 0$ for $i = 1,..,M$, that $r(\xi_i) - r(\xi_j) = O(\epsilon)$ for $i,j = 1,..,M$, and that $r(\xi) - r(\xi_i) > \delta > 0$ holds for any point $x(\xi) \in \partial D$ which is not contained in the union of the $O(\epsilon)$ disks centered at $x(\xi_i)$ for $i = 1,..,M$. Here δ is a constant independent of ϵ. The condition $r'(\xi_i) = 0$ yields that $\hat{r}_i\cdot\hat{n}_i = 1$ for $i = 1,..,M$, where $\hat{r}_i \equiv [x(\xi_i) - x_0]/r_i$, $r_i \equiv r(\xi_i)$ and $\hat{n}_i \equiv \hat{n}(\xi_i)$ is the unit outward normal to ∂D at ξ_i. We let $\kappa_i \leq 0$ denote the curvature of ∂D at ξ_i and we assume that $\kappa_i r_i > -1$ so that the strict inequality $r''(\xi_i) < 0$ holds for $i = 1,..,M$. Then, by using Laplace's method on (4.6), we readily calculate that

$$I(x_0) \sim 2\left(\frac{\pi\epsilon}{\nu}\right)^{1/2} \sum_{i=1}^{M} \left(\frac{b_i - \nu}{b_i + \nu}\right) \frac{r_i^{-1/2}}{(1 + \kappa_i r_i)^{1/2}}\, e^{-2\nu\epsilon^{-1} r_i}\, \hat{r}_i\,. \tag{5.1}$$

Here $b_i \equiv b(\xi_i)$. From our definition of 'asymptotically' closest, it follows that the terms in (5.1) have the same order as $\epsilon \to 0$.

Several geometrical features are readily apparent from (5.1). Suppose that $x_0 \in D$ is such that there is only one asymptotically closest point on ∂D (i. e. $M = 1$). Then, from (5.1), the vector condition $I(x_0) = 0$ cannot be satisfied and thus such a value of x_0 cannot correspond to the location of the spike-layer solution. Therefore, to satisfy $I(x_0) = 0$, x_0 must be such that there are at least two points on ∂D which are asymptotically closest to x_0. Suppose that there are *exactly* two such points (i. e. $M = 2$ in (5.1)). Then, since $b - \nu$ is assumed to be of one sign, it follows from (5.1) that the condition $\hat{r}_1 = -\hat{r}_2$ is required for $I(x_0)$ to vanish. This implies that $\hat{n}_1 = -\hat{n}_2$. Now consider the case when x_0 is such that there are *exactly* three asymptotically closest points on ∂D. Then, since $b - \nu$ is of one sign, it is clear from (5.1) that to satisfy $I(x_0) = 0$ there cannot exist any straight line passing through x_0 for which the three asymptotically closest points lie on only one side of this line. Since an arbitrary point x_0 in a convex domain will not typically have more than three asymptotically closest points on ∂D, we will not consider this case here.

We now show that these geometrical requirements imposed by (4.6) for determining the spike-layer location are closely related to the problem of determining the center of the largest circle that can be inscribed within D. We first suppose that there exists a unique largest inscribed circle $\mathcal{B}$ for D. Let r_{in} and x_{in} denote the radius and center of $\mathcal{B}$, respectively. For instance, x_{in} is always uniquely determined whenever D is strictly convex (i. e. $\kappa < 0$ on ∂D). Assuming that $\mathcal{B}$ is uniquely determined, we now show that (5.1) is asymptotically satisfied when x_0 is such that $|x_0 - x_{in}| = O(\epsilon)$. Thus, (1.2) has a spike-layer solution that is $O(\epsilon)$ close to the center of $\mathcal{B}$.

To show this, we first suppose that $\mathcal{B}$ is tangent to ∂D at *exactly* two distinct points $x(\xi_1) \in \partial D$ and $x(\xi_2) \in D$. Then, the following local and global conditions are satisfied:

$$\hat{n}_1 = -\hat{n}_2\,; \qquad \hat{n}_i \cdot \frac{(x(\xi_i) - x_{in})}{|x(\xi_i) - x_{in}|} = 1\,, \qquad i = 1,2\,, \tag{5.2a}$$

$$|x(\xi) - x_{in}| - r_{in} > 0\,, \qquad \forall\; x(\xi) \in \partial D\,, \quad \text{with} \quad \xi \neq \xi_1\,,\; \xi \neq \xi_2\,. \tag{5.2b}$$

Here $\hat{n}_i \equiv \hat{n}(\xi_i)$ for $i = 1,2$. Suppose that $\kappa_i r_{in} > -1$ holds for $i = 1,2$, where $\kappa_i \equiv \kappa(\xi_i)$. Then it follows from (5.1) and (5.2) that $x_0 = x_{in}$ when $\kappa_1 = \kappa_2$. This case, where the spike-layer location coincides with the center of $\mathcal{B}$, is illustrated in Fig. 4. If $\kappa_1 \neq \kappa_2$, then x_0 lies on the chord $\mathcal{C}$ joining $x(\xi_1)$ and $x(\xi_2)$ but is offset by an $O(\epsilon)$ amount from x_{in}. Setting $I(x_0) = 0$ in (5.1) with $M = 2$ we obtain

$$\left(\frac{b_1 - \nu}{b_1 + \nu}\right) \frac{r_1^{-1/2}}{(1 + \kappa_1 r_1)^{1/2}}\, e^{-2\nu\epsilon^{-1} r_1} \sim \left(\frac{b_2 - \nu}{b_2 + \nu}\right) \frac{r_2^{-1/2}}{(1 + \kappa_2 r_2)^{1/2}}\, e^{-2\nu\epsilon^{-1} r_2}\,. \tag{5.3}$$

Here $r_i = |x(\xi_i) - x_0|$ for $i = 1,2$. Since x_0 lies on $\mathcal{C}$, then $r_1 + r_2 = 2r_{in}$. This leads to the following result:

Proposition 2: (Two-Point Contact): *Assume that there exists a unique largest inscribed circle $\mathcal{B}$ for D and that $\mathcal{B}$ makes exactly two-point contact with ∂D at $x(\xi_i) \in \partial D$ for $i = 1,2$. Let x_{in} and r_{in} denote the center and radius of $\mathcal{B}$, respectively. Suppose that $b - \nu$ has the same sign at each contact point and that $\kappa_i r_{in} > -1$ for $i = 1,2$. Then, x_0 lies on the chord $\mathcal{C}$ joining $x(\xi_1)$ and $x(\xi_2)$, which necessarily must pass through x_{in}. Moreover, for $\epsilon \to 0$, the spike-layer location x_0 satisfies*

$$x_0(\epsilon) = x_{in} + \epsilon x_0^1 + O(\epsilon^2) \,, \tag{5.4a}$$

where

$$x_0^1 = \frac{1}{8\nu} \left\{ 2 \log \left[\left(\frac{b_1 - \nu}{b_1 + \nu} \right) \left(\frac{b_2 + \nu}{b_2 - \nu} \right) \right] + \log \left(\frac{1 + \kappa_2 r_{in}}{1 + \kappa_1 r_{in}} \right) \right\} \hat{n}_2 \,, \qquad \hat{n}_2 \equiv \hat{n}(\xi_2) \,. \tag{5.4b}$$

To illustrate this result, suppose that $b_1 = b_2 \neq \nu$ and that $|\kappa_1| > |\kappa_2|$. Then, from (5.4), it follows that x_0 is located on $\mathcal{C}$ at an $O(\epsilon)$ distance from x_{in} in the direction of the contact point where the magnitude of the curvature is smaller. As a remark, we note that if $(b_1 - \nu)(b_2 - \nu) < 0$, then (5.3) has no solution and thus there is no root to $I(x_0) = 0$ near x_{in}.

A similar result for the spike-layer location can be obtained in the case where $\mathcal{B}$ is uniquely determined and makes exactly three-point contact with ∂D. In this case we obtain the following result:

Proposition 3: (Three-Point Contact): *Assume that the largest inscribed circle $\mathcal{B}$ for D is uniquely determined and that $\mathcal{B}$ makes exactly three-point contact with ∂D at $x(\xi_i^0) \in \partial D$ for $i = 1,2,3$. Let x_{in} and r_{in} denote the center and radius of $\mathcal{B}$. Suppose that $b - \nu$ has the same sign at each contact point and that $\kappa_i^0 r_{in} > -1$ for $i = 1,2,3$, where $\kappa_i^0 \equiv \kappa(\xi_i^0)$. Then, for $\epsilon \to 0$, the spike-layer location $x_0(\epsilon)$ satisfies*

$$x_0(\epsilon) = x_{in} + \epsilon x_0^1 + O(\epsilon^2) \,, \tag{5.5a}$$

where x_0^1 is the solution to the linear system

$$\begin{aligned} \left(\hat{n}_3^0 - \hat{n}_1^0\right) \cdot x_0^1 &= (2\nu)^{-1} \left\{ \log\left(A_1^0 / A_3^0\right) + \log\left(-\hat{n}_1^0 \cdot \hat{t}_2^0 / \hat{n}_3^0 \cdot \hat{t}_2^0\right) \right\} \,, \\ \left(\hat{n}_1^0 - \hat{n}_2^0\right) \cdot x_0^1 &= (2\nu)^{-1} \left\{ \log\left(A_2^0 / A_1^0\right) + \log\left(-\hat{n}_2^0 \cdot \hat{t}_3^0 / \hat{n}_1^0 \cdot \hat{t}_3^0\right) \right\} \,. \end{aligned} \tag{5.5b}$$

Here $\hat{n}_i^0$ and $\hat{t}_i^0$ are the unit outward normal vector and the unit tangent vector (oriented in the counter-clockwise sense) at $x(\xi_i^0) \in \partial D$, respectively, for $i = 1,2,3$. In addition, A_i^0 for $i = 1,2,3$ is defined by

$$A_i^0 = \left(1 + \kappa_i^0 r_{in}\right)^{-1/2} \left(\frac{b_i^0 - \nu}{b_i^0 + \nu} \right) \,, \quad \text{where} \qquad b_i^0 = b(\xi_i^0) \,, \quad \kappa_i^0 = \kappa(\xi_i^0) \,. \tag{5.5c}$$

The derivation of this result is given in Appendix A.

As a special case, suppose that $b_1^0 = b_2^0 = b_3^0 \neq \nu$ and $\kappa_1^0 = \kappa_2^0 = \kappa_3^0$ and that the angles between the adjacent line segments joining x_{in} to the contact points at $x(\xi_i^0)$ are 120° apart. Then, from (5.5), it follows that $x_0^1 = 0$ and thus $x_0(\epsilon) - x_{in} = o(\epsilon)$ as $\epsilon \to 0$. In Fig. 5 we illustrate this special case for a triangular-shaped domain.

Finally, when the center x_{in} of the largest inscribed circle $\mathcal{B}$ for D is not unique, the determination of the spike-layer location from (4.6) is more subtle. For instance, consider the rectangular domain $|x_1| < a_1$, $|x_2| < a_2$ with $a_1 > a_2$ and suppose that $b - \nu$ is of one sign on ∂D. Then, $r_{in} = a_2$ and x_{in} is any point on the line segment $x_2 = 0$ with $|x_1| < a_1 - a_2$. In this degenerate case, a simple application of Laplace's method on (4.6) shows only that the spike-layer location will lie on this line segment. However, by examining the subdominant contributions to the integral in (4.6) away from the contact points it is clear that (4.6) will be asymptotically satisfied only when x_0 is located at the origin.

5.2 The Three-Dimensional Case N = 3

Let $x_0 \in D$ be fixed and let $x(\xi_i)$ for $i = 1, .., M$ be the points on ∂D which are 'asymptotically' closest to x_0. By using Laplace's method on (4.6) we obtain in place of (5.1) that

$$I(x_0) \sim 2\pi\epsilon\nu^{-1} \sum_{i=1}^{M} \left(\frac{b_i - \nu}{b_i + \nu} \right) r_i^{-1} \, H_i(r_i) \, e^{-2\nu\epsilon^{-1} r_i} \, \hat{r}_i \, , \tag{5.6a}$$

where $r_i = |x(\xi_i) - x_{in}|$ and $H_i(r_i)$ is defined by

$$H_i(r_i) \equiv \left(1 - r_i/R_1^i\right)^{-1/2} \, \left(1 - r_i/R_2^i\right)^{-1/2} \, . \tag{5.6b}$$

Here R_j^i is the j^{th} principal radius of curvature of ∂D at $x(\xi_i) \in \partial D$.

We now give a partial result for the precise determination of the location x_0 for which the condition $I(x_0) = 0$ is asymptotically satisfied. It follows from (4.6) that, when $b - \nu$ is of one sign on ∂D, x_0 is $O(\epsilon)$ close to the center x_{in} of the largest inscribed sphere $\mathcal{B}$ for D, whenever this sphere is uniquely determined. It is in general more difficult to derive an *explicit* two-term expansion for $x_0(\epsilon)$ when $N = 3$ than it was when $N = 2$. However, in the simple case where $\mathcal{B}$ makes two-point contact with ∂D, we can obtain the following result from (5.6), which is analogous to (5.4).

Proposition 4. (Two-Point Contact): *Assume that there exists a unique largest inscribed sphere $\mathcal{B}$ for D and that $\mathcal{B}$ makes exactly two-point contact with ∂D at $x(\xi_i) \in \partial D$ for $i = 1, 2$. Let x_{in} and r_{in} denote the center and radius of $\mathcal{B}$, respectively. Suppose that $b - \nu$ has the same sign at each contact point and that $r_{in}/R_j^i < 1$ for $i, j = 1, 2$. Then, x_0 lies on the chord $\mathcal{C}$ joining $x(\xi_1)$ and $x(\xi_2)$. For $\epsilon \to 0$, x_0 satisfies (5.4a), where x_0^1 is now given by*

$$x_0^1 = \frac{1}{4\nu} \left\{ \log \left[\left(\frac{b_1 - \nu}{b_1 + \nu} \right) \left(\frac{b_2 + \nu}{b_2 - \nu} \right) \right] + \log \left[\frac{H_1(r_{in})}{H_2(r_{in})} \right] \right\} \hat{n}_2 \, , \qquad \hat{n}_2 \equiv \hat{n}(\xi_2) \, . \tag{5.7}$$

6. Hot-Spot Solutions for Bratu's Equation

We now construct hot-spot solutions for (1.3) in the limit $\lambda \to 0$. We first derive a canonical hot-spot solution that will be used in §6.1 as the inner solution. The canonical hot-spot solution $u_c(r;a)$ is taken to satisfy

$$u_c'' + \frac{1}{r}u_c' + \lambda e^{u_c} = 0, \qquad r \geq 0, \tag{6.1a}$$

$$u_c(0;a) = a, \tag{6.1b}$$

$$u_c(r;a) = O(1) \quad \text{as } \lambda \to 0, \quad \text{for } \sigma \ll r \ll 1. \tag{6.1c}$$

Here $a = a(\lambda)$ is the amplitude of the hot-spot and $\sigma = \sigma(\lambda) \equiv \lambda^{-1/2}e^{-a/2}$ is the length scale of the core of the hot-spot. We assume that $a(\lambda) \to \infty$ and $\sigma(\lambda) \to 0$ as $\lambda \to 0$. The condition (6.1c), which is needed below in §6.1, determines the leading order behavior of $a(\lambda)$ as $\lambda \to 0$ and ensures that u_c is finite as $\lambda \to 0$ in an overlap region away from the core of the hot-spot.

The solution to (6.1), which satisfies $u_c' < 0$, is

$$u_c(r;a) = a - 2\log\left[1 + r^2/(8\sigma^2)\right], \qquad \sigma = \lambda^{-1/2}e^{-a/2}. \tag{6.2}$$

In the far field, where $\sigma \ll r \ll 1$, we can expand (6.2) for $\lambda \to 0$ to obtain

$$u_c(r;a) = -a - 2\log\lambda + 2\log 8 - 4\log r - 16\sigma^2 r^{-2} + O\left(\sigma^4 r^{-4}\right). \tag{6.3}$$

To satisfy (6.1c) we require that $a = a(\lambda) = -2\log\lambda + O(1)$ as $\lambda \to 0$. Thus, from (6.2), $\sigma = O(\lambda^{1/2})$ as $\lambda \to 0$.

6.1 The Asymptotic Construction of a Solution with m Hot-Spots

For $\lambda \to 0$, we now use the method of matched asymptotic expansions to construct a solution to (1.3) that has m hot-spots. For $j = 1,..,m$, the location $x_j \in D$ of the j^{th} hot-spot and its associated amplitude $a_j(\lambda)$ are to be determined. We assume that the hot-spots are widely separated in the sense that $|x_j - x_k| = O(1)$ as $\lambda \to 0$ for $j \neq k$. In the inner region near the j^{th} hot-spot, defined by $|x - x_j| = O(\lambda^{1/2})$, we have that $u \sim u_c[|x - x_j|; a_j(\lambda)]$, where $u_c(r;a)$ is given in (6.2). For each $j = 1,..,m$, we expand $a_j(\lambda)$ for $\lambda \to 0$ as

$$a_j(\lambda) = -2\log\lambda + a_{j0} + \lambda a_{j1} + \cdots. \tag{6.4}$$

Substituting $a = a_j(\lambda)$ and $r = |x - x_j|$ into (6.3), we obtain the following far field form:

$$u_c[|x - x_j|; a_j(\lambda)] = -a_{j0} + 2\log 8 - 4\log|x - x_j| - \lambda\left(a_{j1} + \frac{16e^{-a_{j0}}}{|x - x_j|^2}\right) + O(\lambda^2), \tag{6.5}$$

which holds for $\lambda^{1/2} \ll |x - x_j| \ll 1$.

In the outer region away from the hot-spots, where $|x - x_j| = O(1)$ for $j = 1,..,m$, we expand the solution to (1.3) as

$$u = u_0 + \lambda u_1 + \cdots . \tag{6.6}$$

Substituting (6.6) into (1.3), and matching (6.6) to the far field form (6.5) of each inner solution, we obtain the following problem for u_0:

$$\triangle u_0 = 0\,, \qquad x \in D\backslash\{x_1,..,x_m\}\,, \tag{6.7a}$$

$$\partial_n u_0 + b u_0 = 0\,, \qquad x \in \partial D\,, \tag{6.7b}$$

$$u_0 \sim -4\log|x - x_j| - a_{j0} + 2\log 8\,, \quad \text{as} \quad x \to x_j\,, \quad j = 1,..,m\,. \tag{6.7c}$$

Now let $G(x; x_k)$ be the Green's function for the Laplacian with boundary condition (6.7b) and with singularity at $x = x_k \in D$. Then, we can decompose G as

$$G(x; x_k) = g_s(x; x_k) + g_r(x; x_k)\,, \qquad g_s(x; x_k) \equiv \frac{1}{2\pi}\log|x - x_k|\,, \tag{6.8}$$

where g_r is regular at $x = x_k$. The solution to (6.7) can then be conveniently written as

$$u_0(x) = -4\log|x - x_j| + \Phi_j(x)\,, \tag{6.9a}$$

where

$$\Phi_j(x) \equiv -8\pi g_r(x; x_j) - 8\pi \sum_{\substack{k=1 \\ k\neq j}}^{m} \left[g_s(x; x_k) + g_r(x; x_k)\right]. \tag{6.9b}$$

Since $\Phi_j(x)$ is regular as $x \to x_j$, we obtain for $x \to x_j$ that $u_0(x)$ has the form

$$u_0(x) \sim -4\log|x - x_j| + \Phi_j(x_j) + \nabla\Phi_j(x_j)\cdot(x - x_j) + \cdots\,, \quad \text{as} \quad x \to x_j\,, \quad j = 1,..,m\,. \tag{6.10}$$

Since each hot-spot is radially symmetric about $x = x_j$, we obtain upon comparing (6.7c) with (6.10) that $\nabla\Phi_j(x_j) = 0$ for $j = 1,..,m$. Thus, from (6.9b), the hot-spot locations x_j, for $j = 1,..,m$, are to be determined from the following system:

$$\nabla g_r(x_j; x_j) + \sum_{\substack{k=1 \\ k\neq j}}^{m} \left[\nabla g_s(x_j; x_k) + \nabla g_r(x_j; x_k)\right] = 0\,, \qquad j = 1,..,m\,. \tag{6.11}$$

This result was derived previously in [13] and [14] using a complex variable method. Next, by comparing (6.7c) with (6.10), we find that

$$a_{j0} = 2\log 8 - \Phi_j(x_j)\,, \qquad j = 1,..,m\,. \tag{6.12}$$

Substituting (6.12) into (6.4) then yields a two-term expansion for $a_j(\lambda)$. This result for $a_j(\lambda)$, which was not obtained in [13] or [14], shows that the amplitude of a particular hot-spot depends weakly, as $\lambda \to 0$, on the x_j and on the *global* properties of the domain D contained in the Green's function. For a fixed value of m, it appears to be difficult to determine whether the system (6.11) has solutions for an arbitrary domain.

For simplicity, consider the case of one hot-spot where $m = 1$. Then, from (6.4), (6.11) and (6.12) we have that

$$a_1(\lambda) \sim -2\log\lambda + a_{10}\,; \qquad a_{10} = 2\log 8 - 4\log[R(x_1)]\,; \qquad \nabla g_r(x_1; x_1) = 0\,. \tag{6.13}$$

Here $R(x_1)$ is the conformal radius of D at $x = x_1$, defined by $g_r(x_1; x_1) = -(2\pi)^{-1}\log[R(x_1)]$. Substituting (6.13) into (6.2), we find that the inner solution becomes

$$u \sim u_c[|x - x_1|; a_1] = 2\log\gamma - 2\log\left(1 + \gamma r^2 [R(x_1)]^{-2}\right), \qquad \gamma \equiv 8\lambda^{-1}[R(x_1)]^{-2}\,. \tag{6.14}$$

In the case when $b = 0$, $R(x_1)$ can be found in terms of the mapping function that takes D to the unit disk, where $x_1 \in D$ is mapped to the origin of the disk.

As a special case, suppose that D is convex and is symmetric with respect to the coordinate axes. Let D' be a new domain that contains D and which has the same axes of symmetry. Then, from [14], $x_1 = 0$ for both domains. Also, from an inequality in [4], the conformal radius $R(0)$ is larger for D' than for D. Thus, from (6.13) and (6.14), we conclude that the amplitude of the hot-spot is smaller but its spatial extent is larger for D' than for D.

6.2 An Explicit Example with one Hot-Spot

As an example, consider (1.3) in a circular cylindrical domain of radius one. Then, by symmetry, $\nabla g_r(0;0) = 0$, so that the hot-spot is located at $x_1 = 0$. By calculating $g_r(x;0)$, we find that $R(0) = e^{1/b}$. Therefore, from (6.13), $a_{10} = 2\log 8 - 4b^{-1}$. For this geometry, we can also calculate the next term a_{11} in the expansion for $a_1(\lambda)$ given in (6.4). To do so, we must first obtain the problem for the outer correction u_1 given in (6.6). From (6.6), (6.5) and (1.3), we obtain that u_1 satisfies

$$u_1'' + \frac{1}{r}u_1' = -e^{u_0}\,, \qquad 0 < r < 1\,, \tag{6.15a}$$

$$u_1'(1) + bu_1(1) = 0\,; \qquad u_1 \sim -a_{11} - 16e^{-a_{10}}r^{-2}\,, \quad \text{as} \quad r \to 0\,. \tag{6.15b}$$

From (6.7), $u_0 = 4b^{-1} - 4\log r$. Then, solving for u_1 we obtain $u_1 = -\frac{1}{4}e^{4/b}\left(2b^{-1} - 1 + r^{-2}\right)$. Thus, a_{11} can be found from (6.15b). This leads to the following three-term expansion for $a_1(\lambda)$:

$$a_1(\lambda) \sim -2\log\lambda + 2\log 8 - 4b^{-1} + \lambda\, e^{4/b}\left[(2b)^{-1} - 1/4\right], \quad \text{as} \quad \lambda \to 0. \tag{6.16}$$

For this example, (6.16) can be verified by using the exact hot-spot solution to (1.3), which is given by

$$u = 2\log\left[\frac{1+\alpha}{1+\alpha r^2}\right] + \frac{4\alpha}{b(1+\alpha)}\,, \qquad a_1 = 2\log(1+\alpha) + \frac{4\alpha}{b(1+\alpha)}\,. \tag{6.17a}$$

Here $\alpha = \alpha(\lambda)$ is the solution branch of

$$\lambda = \frac{8\alpha}{(1+\alpha)^2} \exp\left[-\frac{4\alpha}{b(1+\alpha)}\right] , \tag{6.17b}$$

for which $\alpha = O(\lambda^{-1})$ as $\lambda \to 0$. Substituting $\alpha = \alpha_0 \lambda^{-1} + \alpha_1 + \cdots$ into (6.17b) and collecting powers of λ, we obtain α_0 and α_1. This gives,

$$\alpha \sim 8e^{-4/b}\lambda^{-1} + \left(4b^{-1} - 2\right) + \cdots , \qquad \text{as} \;\; \lambda \to 0 . \tag{6.18}$$

Then, substituting (6.18) into the expression for $a_1 = a_1(\lambda)$ given in (6.17a) we readily recover (6.16).

7. Discussion

The asymptotic analysis used to construct a hot-spot solution for (1.3) is very different from the analysis used in §2-5 to construct a spike solution for (1.2). This difference results from the fact that the canonical spike solution decays exponentially in the far field, whereas the canonical hot-spot solution has a logarithmic singularity in the far field.

For the spike problem (1.2), it is the *combined* effect of the translation invariance and of the exponentially decay of the canonical spike solution that leads to the existence of exponentially small eigenvalues for the linearized problem. Our result for a strictly convex domain that the spike solution is located, to leading order, at the point furthest from the boundary of D is directly related to the existence of these small eigenvalues. Exponentially small eigenvalues also occur, for similar reasons as for (1.2), for certain linearizations of some phase transition models. In particular, it was shown in [1] and [2] that exponentially small eigenvalues arise when linearizing the Cahn-Hilliard equation about equilibrium internal layer solutions with spherical interfaces. Similar results were obtained in [18] for a non-local Allen-Cahn equation. For these special internal layer solutions, called bubble solutions, the center of the bubble can only be determined by retaining certain exponentially small terms in the analysis. For the non-local Allen-Cahn equation in a strictly convex domain it was shown in [18] that the center of the equilibrium bubble solution is located asymptotically at the furthest point from the boundary of the domain. More generally, we conjecture that this same geometric criterion will determine the location of a localized structure for other singularly perturbed problems where exponentially small eigenvalues arise from the combined effect of translation invariance and exponential decay.

Although (1.3) and (1.2) are both translation invariant, the logarithmic far field behavior of the canonical hot-spot solution precludes the existence of any exponentially small eigenvalues for (1.3). This logarithmic behavior has the effect of replacing the nonlinearity in (1.3) by a delta function singularity. Therefore, it is natural that it is the Green's function for D that plays a pivotal role in determining the location of a hot-spot solution. The resulting condition (6.13) for the hot-spot location is qualitatively very different, and more difficult to analyze, than the analogous result for the spike problem.

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Appendix A. The Spike Location for N = 2 in the Case of Three-Point Contact

We now derive the result (5.5) for the spike-layer location x_0 in the case where the two-dimensional domain D has a unique largest inscribed circle $\mathcal{B}$ which makes exactly three-point contact with ∂D at $x(\xi_i^0) \in \partial D$ for $i = 1,2,3$. Here $\xi = \xi_i^0$ is the arclength coordinate at the i^{th} contact point. The center and radius of $\mathcal{B}$ are denoted by x_{in} and r_{in}, respectively. It is clear that the following conditions are satisfied:

$$\begin{gathered} \hat{r}_i^0 \cdot \hat{n}_i^0 = 1\,, \qquad \text{at} \;\; \xi = \xi_i^0\,, \;\; i = 1,2,3\,, \\ r_1^0 = r_2^0 = r_3^0 = r_{in}\,; \qquad |x(\xi) - x_{in}| > r_{in}\,, \quad \forall \; x(\xi) \in \partial D\,, \quad \text{with} \quad \xi \neq \xi_i^0\,, \;\; i = 1,2,3\,. \end{gathered} \tag{A.1}$$

Here $r_i^0 \equiv |x(\xi_i^0) - x_{in}|$ and $\hat{r}_i^0 \equiv \left(x(\xi_i^0) - x_{in}\right)/r_{in}$ for $i = 1,2,3$.

It follows from (5.1) (with $M = 3$) and $(A.1)$ that $I(x_{in})$ typically does not vanish. Therefore, $x_0 \neq x_{in}$ in general. However, x_0 and x_{in} do coincide in the special case when $b_1^0 = b_2^0 = b_3^0 \neq \nu$, $\kappa_1^0 = \kappa_2^0 = \kappa_3^0$ and when the angles between the adjacent line segments joining x_{in} to the contact points at $x(\xi_i^0)$ are 120° apart.

In the more typical case where $I(x_{in}) \neq 0$, we now seek a solution to $I(x_0) = 0$ in an $O(\epsilon)$ neighborhood of x_{in}. However, in using Laplace's method on (4.6) we must allow that the dominant contribution to the integral in (4.6) is now obtained from arclength coordinates $\xi_i(\epsilon)$ that are $O(\epsilon)$ close to ξ_i^0. The problem for $x_0(\epsilon)$ and $\xi_i(\epsilon)$ for $i = 1,2,3$ is given by

$$\hat{r}_i^\epsilon \cdot \hat{n}_i^\epsilon = 1\,, \qquad \text{at} \;\; \xi = \xi_i(\epsilon)\,, \;\; i = 1,2,3\,, \tag{A.2}$$

$$\hat{n}_1^0\, A_1^0\, e^{-2\epsilon^{-1}\nu r_1^\epsilon} + \hat{n}_2^0\, A_2^0\, e^{-2\epsilon^{-1}\nu r_2^\epsilon} \sim -\hat{n}_3^0\, A_3^0\, e^{-2\epsilon^{-1}\nu r_3^\epsilon}\,. \tag{A.3}$$

For $i = 1,2,3$ we have defined $\hat{n}_i^\epsilon \equiv \hat{n}[\xi_i(\epsilon)]$, $r_i^\epsilon \equiv |x[\xi_i(\epsilon)] - x_0(\epsilon)|$ and $\hat{r}_i^\epsilon \equiv \left(x[\xi_i(\epsilon)] - x_0(\epsilon)\right)/r_i^\epsilon$. In addition, A_i^0 is defined in $(5.5c)$. The condition $(A.2)$ can also be expressed as

$$\hat{r}_i^\epsilon \cdot \hat{t}_i^\epsilon = 0\,, \qquad \text{at} \;\; \xi = \xi_i(\epsilon)\,, \;\; i = 1,2,3\,, \tag{A.4}$$

where $\hat{t}_i^\epsilon$ is the unit tangent vector to ∂D at $\xi = \xi_i(\epsilon)$.

The system $(A.3)$, $(A.4)$ gives five equations for the unknowns $\xi_i(\epsilon)$ for $i = 1,2,3$ and the two-vector $x_0(\epsilon) \in D$. For $\epsilon \to 0$, we seek a solution to this system in the form

$$\xi_i(\epsilon) = \xi_i^0 + \epsilon\,\xi_i^1 + \cdots\,, \qquad x_0(\epsilon) = x_{in} + \epsilon\, x_0^1 + \cdots\,, \tag{A.5}$$

where x_0^1 and ξ_i^1 for $i = 1,2,3$ are to be determined. A straightforward calculation leads to the following result:

$$r_i^\epsilon = r_{in} - \epsilon\, \hat{n}_i^0 \cdot x_0^1 + O(\epsilon^2)\,, \tag{A.6a}$$

$$\hat{r}_i^\epsilon = \hat{r}_i^0 + \frac{\epsilon}{r_{in}} \left[\hat{t}_i^0 \xi_i^1 - x_0^1 + \left(\hat{n}_i^0 \cdot x_0^1\right) \hat{n}_i^0\right] + O(\epsilon^2)\,, \tag{A.6b}$$

$$\hat{t}_i^\epsilon = \hat{t}_i^0 + \epsilon\, \xi_i^1 \kappa_i^0 \hat{n}_i^0 + O(\epsilon^2)\,. \tag{A.6c}$$

The superscript 0 indicates that the corresponding quantity is to be evaluated at the contact point $\xi = \xi_i^0$.

Substituting $(A.6b)$ and $(A.6c)$ into the orthogonality relation $(A.4)$, and setting the coefficient of $O(\epsilon)$ in the resulting expression to zero, we obtain

$$\xi_i^1 = [x_0^1 \cdot \hat{t}_i^0] \left(1 + \kappa_i^0 r_{in}\right)^{-1} , \qquad i = 1,2,3 . \tag{A.7}$$

Then, substituting $(A.6a)$ into $(A.3)$ we get the following equation for x_0^1:

$$\hat{n}_1^0 \, A_1^0 \, e^{2\nu \hat{n}_1^0 \cdot x_0^1} + \hat{n}_2^0 \, A_2^0 \, e^{2\nu \hat{n}_2^0 \cdot x_0^1} \sim -\hat{n}_3^0 \, A_3^0 \, e^{2\nu \hat{n}_3^0 \cdot x_0^1} . \tag{A.8}$$

To solve for x_0^1 in $(A.8)$, we take the dot product of $(A.8)$ with $\hat{t}_i^0$ for $i = 2,3$. Since $\hat{n}_i^0 \cdot \hat{t}_i^0 = 0$ for $i = 1,2,3$, this procedure yields two equations for x_0^1, which can be solved explicitly. In this way we obtain that x_0^1 satisfies the linear system given in $(5.5b)$. In terms of x_0^1, a two-term expansion for the arclength coordinates $\xi_i(\epsilon)$ for $i = 1,2,3$ are obtained by substituting $(A.7)$ into $(A.5)$.

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m	β $(N=2)$	a $(N=2)$	β $(N=3)$	a $(N=3)$
2	2.47	10.80	10.42	16.07
3	1.86	3.50	4.51	2.71
4	1.50	2.12	2.29	0.84

Table 1: For $Q(u) = -u + u^m$, numerical values for β and a are given for various m and for $N = 2, 3$.

m	λ_{0e} $(N=2)$	λ_{0e} $(N=3)$
2	1.65	2.36
3	5.41	15.29
4	13.23	144.18

Table 2: For $Q(u) = -u + u^m$, numerical values for λ_{0e} are given for various m and for $N = 2, 3$.

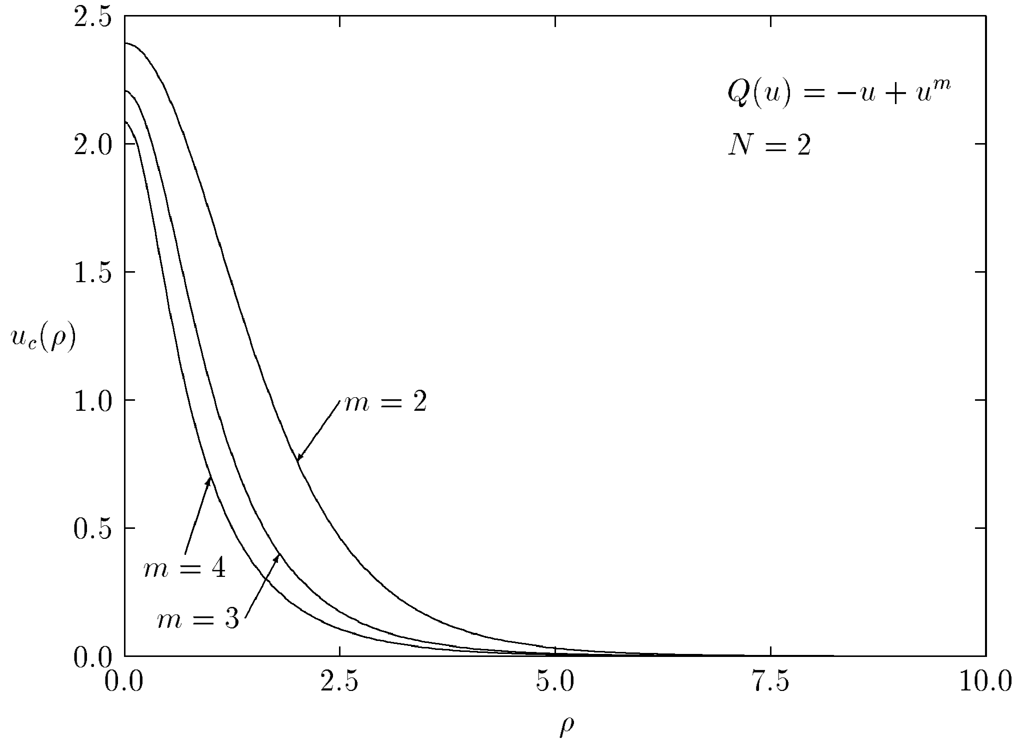

Figure 1: For $Q(u) = -u + u^m$ and $N = 2$, we plot the numerically computed canonical spike solution $u_c(\rho)$ for three different values of m.

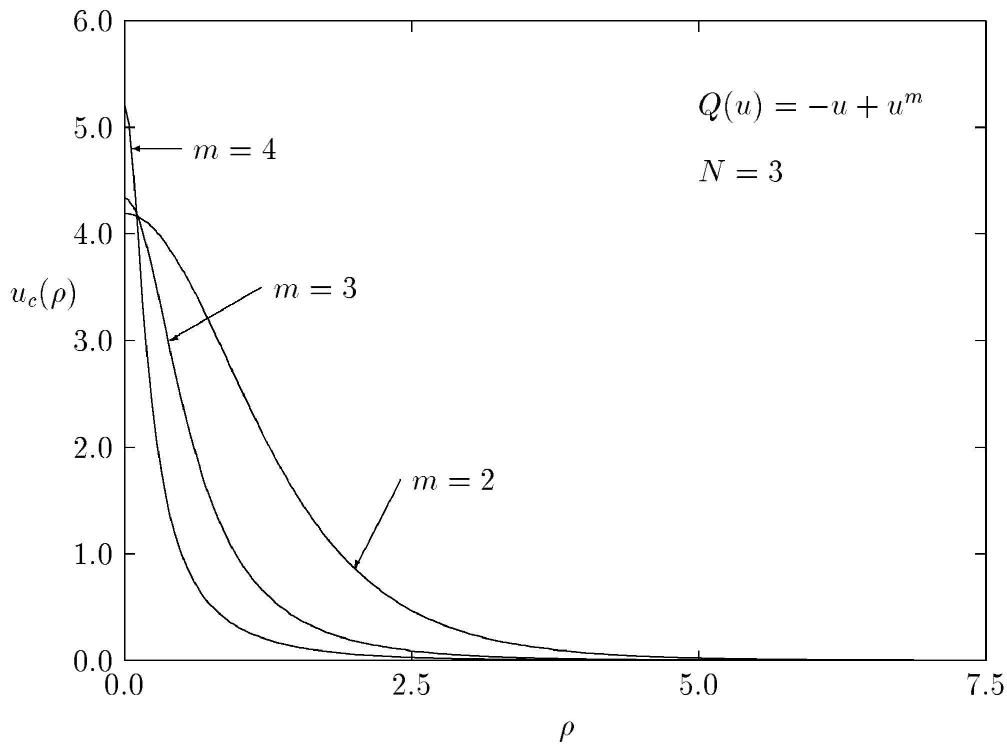

Figure 2: For $Q(u) = -u + u^m$ and $N = 3$, we plot the numerically computed canonical spike solution $u_c(\rho)$ for three different values of m.

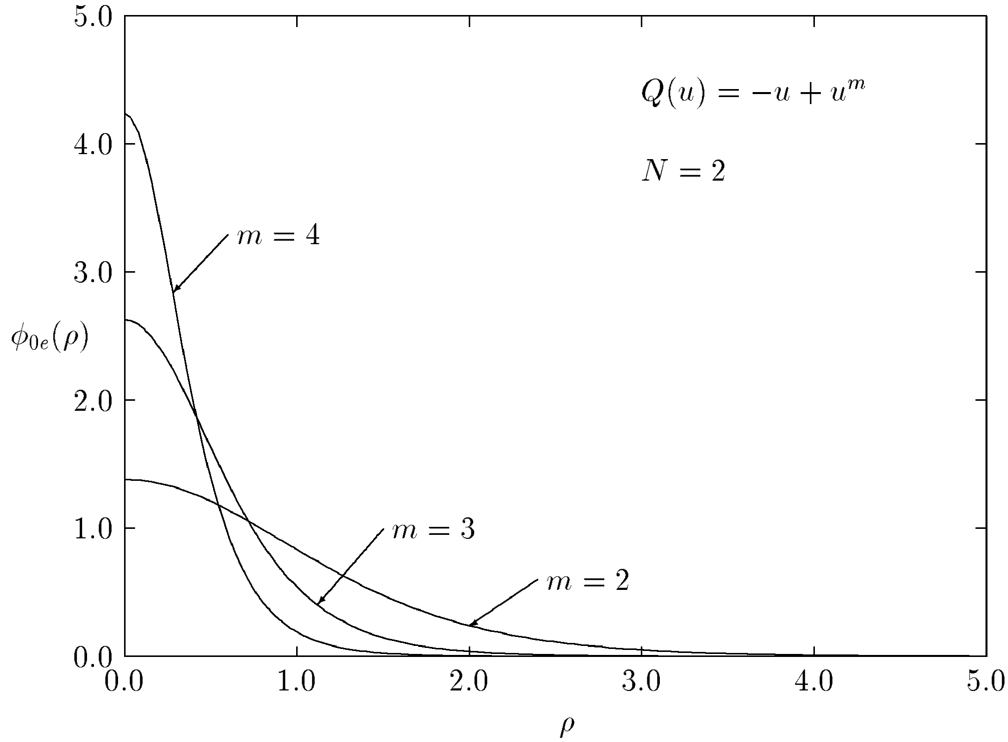

Figure 3: For $Q(u) = -u + u^m$ and $N = 2$, we plot the numerically computed first eigenfunction $\phi_{0e}(\rho)$ for three different values of m.

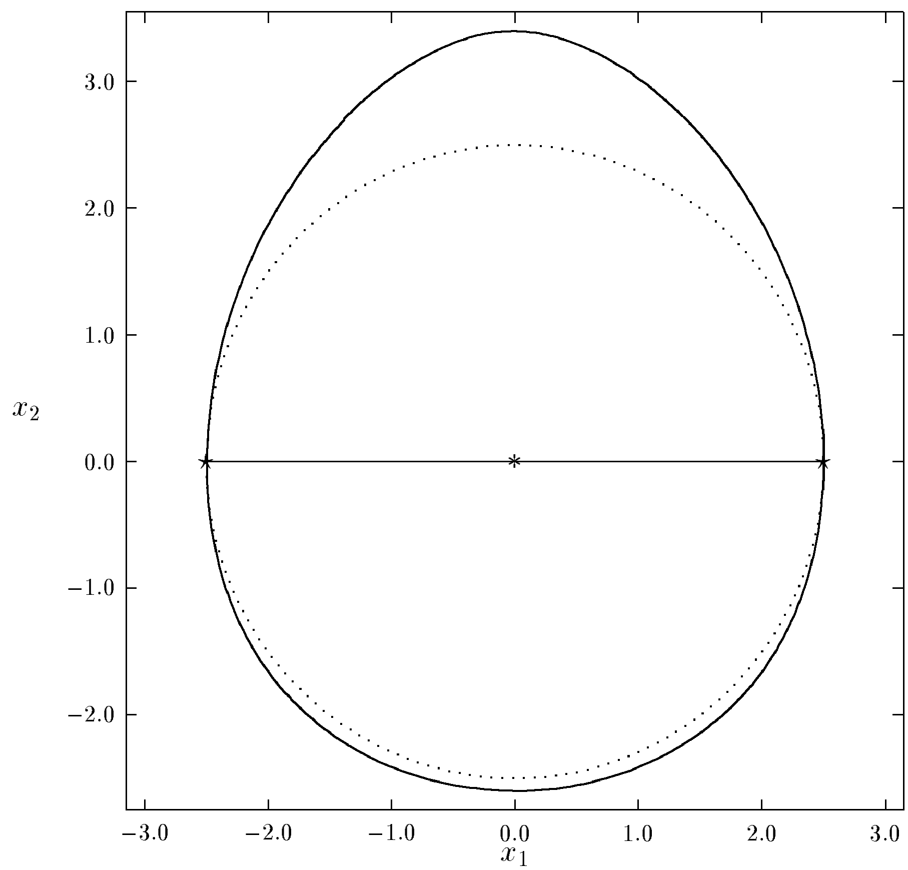

Figure 4: We plot the largest inscribed circle (dotted curve) for the domain whose boundary is given parametrically by $x_1(\theta) = p(\theta)\cos\theta - p^{'}(\theta)\sin\theta$ and $x_2(\theta) = p(\theta)\sin\theta + p^{'}(\theta)\cos\theta$ where $p(\theta) = 3 + 0.4\sin^3\theta - 0.5\cos^2\theta$ and $0 \leq \theta \leq 2\pi$. The largest inscribed circle, with center at $*$, makes two-point contact with ∂D at the points labelled by $\star$. Since the curvatures of ∂D at the two contact points are equal, x_0 is also located at $*$ when $b_1 = b_2 \neq \nu$.

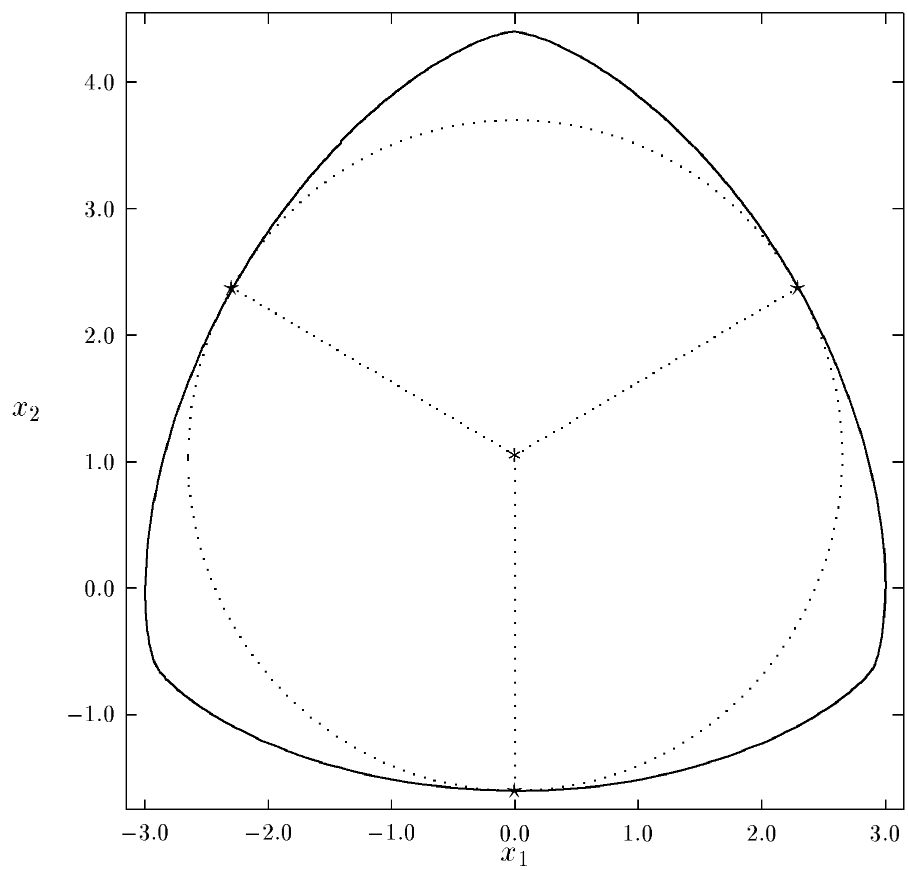

Figure 5: We plot the largest inscribed circle (dotted curve) for the domain whose boundary is given parametrically by $x_1(\theta) = p(\theta)\cos\theta - p'(\theta)\sin\theta$ and $x_2(\theta) = p(\theta)\sin\theta + p'(\theta)\cos\theta$ where $p(\theta) = 3 + 1.4\sin^3\theta$ and $0 \leq \theta \leq 2\pi$. The largest inscribed circle, with center at $*$, makes three-point contact with ∂D at the points labelled by $\star$. Since the curvatures of ∂D at the three contact points are equal and since the dotted lines shown are 120° apart, x_0 is also located at $*$ when $b_1^0 = b_2^0 = b_3^0 \neq \nu$.