METASTABLE DYNAMICS AND SPATIALLY INHOMOGENEOUS EQUILIBRIA IN DUMBELL-SHAPED DOMAINS

Michael J. Ward[1] and Douglas Stafford
Department of Mathematics, University of British Columbia
Vancouver, B.C. V6T 1Z2, Canada

Abstract

The motion of internal layers for three singularly perturbed reaction diffusion problems, including the Allen-Cahn equation, is studied in a two-dimensional dumbell-shaped domain. The channel region that connects the two attachments, or lobes, of the dumbell is taken to be rectangular. The motion of straight-line internal layers in the channel region is analyzed using an asymptotic projection method. This motion is shown to be metastable and highly dependent on the local convexity properties of the boundary near the contact region between the ends of the channel and the two attachments. When the domain is non-convex it is shown that the metastable internal layers dynamics in the channel tends, as $t \to \infty$, to a limiting stable spatially inhomogeneous equilibrium solution.

1 Introduction

It is well-known that stable spatially inhomogeneous equilibrium solutions can not occur for a scalar reaction-diffusion equation in a convex domain with Neumann boundary conditions (see [7], [12]). However, this result does not hold in a non-convex setting. Some previous examples showing the existence of stable spatially inhomogeneous equilibria in a non-convex domain are given in [3], [10], [11], [12] and [18].

In particular, in [3] and [11], it was proved that a stable spatially inhomogeneous equilibrium solution can result from the long-time behavior of the metastable dynamics associated with the Allen-Cahn equation in a dumbell-shaped domain. Metastable dynamics, which refers more broadly to an asymptotically exponentially slow motion of localized structures to differential equations, has been studied extensively in a one-dimensional setting (eg. [3], [4], [6], [9], [14], [15], [19]). However, the occurrence of metastable dynamics in a multi-spatial dimensional setting has only been recently established in [2], [11] and [20].

Motivated by the work of [3] and [11], we consider three related singularly perturbed reaction diffusion equations of Allen-Cahn type in a two-dimensional dumbell-shaped domain (see Fig. 1).

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The channel region connecting the two attachments, or lobes, of the dumbell is assumed to be rectangular, with precise conditions stated below. For each problem, we study the time-dependent behavior of straight-line internal layer solutions that are confined to the channel region. The analysis of the motion of these internal layers requires exponential asymptotic precision and is done using the asymptotic projection method developed in [19]. This method, which exploits the existence of an exponentially small principal eigenvalue in the spectrum of the linearized problem, provides explicit differential equations for the internal layer locations in the channel region. The motion of these layers is shown to be metastable and highly dependent on the local convexity properties of the boundary near the contact region between the ends of the channel and the two attachments. When the domain is non-convex it is shown that the metastable internal layers dynamics in the channel region tends as $t \to \infty$ to a limiting stable spatially inhomogeneous equilibrium solution. In this way, we obtain further examples illustrating both the occurrence of metastable dynamics in a multi-dimensional setting and the existence of stable spatially inhomogeneous equilibrium solutions to scalar reaction-diffusion equations with Neumann boundary conditions in a non-convex domain.

In the analysis below, we assume that the two-dimensional dumbell-shaped domain D has the form $D = R \cup D_- \cup D_+$ where R is the rectangle $[0,1] \times [0,L]$, and D_- and D_+ are the two attachments on its sides (see Fig. 1). The points $(0,0)$, $(0,L)$, $(1,0)$, and $(1,L)$ are referred to as the corners of R. It is assumed that the domain boundary is smooth and that near the corners ∂D can be represented as $\partial D = \{(x,y) \,|\, y = \psi_i(x)\}$, where

$$y = \psi_1(x)\,, \quad \text{near } (0,0)\,; \qquad y = \psi_2(x) + L\,, \quad \text{near } (0,L), \tag{1a}$$

$$y = \psi_3(x)\,, \quad \text{near } (1,0)\,; \qquad y = \psi_4(x) + L\,, \quad \text{near } (1,L)\,. \tag{1b}$$

The local properties of the domain near the corners of R are central to the analysis below. Near each corner, it is assumed that there exist numbers $K_i \neq 0$, $\alpha_i > 0$, for $i = 1, \ldots, 4$ such that

$$\psi_1'(x) \sim -K_1(-x)^{\alpha_1}, \quad \text{as } x \to 0^-\,; \qquad \psi_2'(x) \sim K_2(-x)^{\alpha_2}, \quad \text{as } x \to 0^-, \tag{2a}$$

$$\psi_3'(x) \sim K_3(x-1)^{\alpha_3}, \quad \text{as } x \to 1^+\,; \qquad \psi_4'(x) \sim -K_4(x-1)^{\alpha_4}, \quad \text{as } x \to 1^+. \tag{2b}$$

When $\alpha_i = 2$, the constant K_i is proportional to the curvature of the i^{th} corner.

We now give the outline of the paper. In §2 we analyze the Allen-Cahn equation (AC) in D, which is the simplest model for the phase separation of a binary mixture (see [17]);

$$u_t = \epsilon^2 \Delta u + Q(u)\,, \qquad \mathbf{x} \in D\,, \tag{3a}$$

$$\partial_n u = 0\,, \qquad \mathbf{x} \in \partial D\,. \tag{3b}$$

Here $u(\mathbf{x},0)$ is given, $\mathbf{x} = (x,y)$, $\epsilon \ll 1$ and $Q(u)$ is a bistable nonlinearity having three zeroes located at $u = s_- < 0$, $u = 0$, and $u = s_+ > 0$, with

$$Q^{'}(s_\pm) < 0\,, \quad Q^{'}(0) > 0\,, \quad V(s_+) = 0\,, \quad V(u) = -\int_{s_-}^{u} Q(\eta)\, d\eta\,. \tag{4}$$

Thus, $V(u)$ is a double-well potential with wells of equal depth at s_+ and s_-. Following [3] and [11], we seek a time-dependent solution to (3) of the form

$$u \sim \begin{cases} s_- & \text{in } D_- \text{ and in } 0 < x < x_0\,, \\ s_+ & \text{in } D_+ \text{ and in } x_0 < x < 1\,, \end{cases} \tag{5}$$

which has an appropriate internal layer profile near $x_0 = x_0(t)$.

For $\epsilon \to 0$, it has been shown in [17] that the normal velocity v of a generic curved interface for (3) is given by the curvature flow

$$v \sim \epsilon^2 \kappa\,, \tag{6}$$

where κ is the curvature of the interface. In addition, if an interface intersects the boundary of the domain, then it must do so orthogonally. However, for straight line interfaces confined to the channel region it follows that $\kappa = 0$ and, hence, (6) gives no indication of the nature of the motion of the interface. In §2 we show that exponential precision is required to determine the time-dependent behavior of solutions to (3) of the form (5). In addition, we will use the asymptotic projection method to derive a differential equation for $x_0(t)$ and will establish criteria for which $x_0(t) \to x_0^e$ as $t \to \infty$ for some $x_0^e \in (0,1)$.

In §3 we study the constrained Allen-Cahn equation (CAC) in D, which is the simplest model for the phase separation of a binary mixture in the presence of a mass constraint (see [16]);

$$u_t = \epsilon^2 \Delta u + Q(u) - \sigma\,, \qquad \mathbf{x} \in D\,, \tag{7a}$$

$$\partial_n u = 0\,, \qquad \mathbf{x} \in \partial D\,, \tag{7b}$$

$$\int_D u\, d\mathbf{x} = m\,. \tag{7c}$$

Here $u(\mathbf{x},0)$ is given, $\mathbf{x} = (x,y)$, $\epsilon \ll 1$, and $Q(u)$ is as defined in (4). The unknown function $\sigma = \sigma(t)$ is a Lagrange multiplier parameter needed to satisfy the constant mass constraint (7c). A solution to (7) with two interfaces in the channel region R has the form

$$u \sim \begin{cases} s_- & \text{in } D_- \text{ and in } 0 < x < x_0\,, \\ s_+ & \text{in } x_0 < x < x_1\,, \\ s_- & \text{in } D_+ \text{ and in } x_1 < x < 1\,, \end{cases} \tag{8}$$

with appropriate internal layer profiles near $x_0 = x_0(t)$ and $x_1 = x_1(t)$.

When the attachments D_+ and D_- are removed from D and are replaced by homogeneous Neumann conditions imposed on the segments $x = 0$ and $x = 1$, it is known that any two-layer equilibrium solution with asymptotic behavior (8) is unstable (see [5]). In this case, for the time-dependent problem, either $x_0 = 0$ or $x_1 = 1$ in finite time leaving behind a stable one-layer equilibrium solution. The metastable dynamics of such a solution was analyzed in [14]. In §3, we will show, by deriving a differential equation for the difference $x_1 - x_0$, that when the attachments $D_\pm$ are present, a stable two-layer equilibrium solution in R can exist. The existence of such a solution depends on the local convexity properties (2) at the corners R of the channel.

Finally, in §4 we study the tristable Allen-Cahn equation (TAC) in D, which arises in the study of liquid crystals (i. e. [8]);

$$u_t = \epsilon^2 \Delta u + Q(u)\,, \qquad \mathbf{x} \in D\,, \tag{9a}$$

$$\partial_n u = 0\,, \qquad \mathbf{x} \in \partial D\,. \tag{9b}$$

Here $u(\mathbf{x}, 0)$ is given, $\mathbf{x} = (x, y)$, $\epsilon \ll 1$, and $Q(u)$ is an odd periodic function satisfying,

$$Q(u) = -Q(-u)\,, \quad Q(u + 2s) = Q(u)\,; \qquad Q(-s) = 0\,, \quad Q^{'}(-s) < 0\,, \tag{10}$$

for some $s > 0$. We will look for a staircase solution to (9) with two interfaces of the form

$$u \sim \begin{cases} -s & \text{in } D_- \text{ and in } 0 < x < x_0\,, \\ s & \text{in } x_0 < x < x_1\,, \\ 3s & \text{in } D_+ \text{ and in } x_1 < x < 1\,, \end{cases} \tag{11}$$

with appropriate internal layer profiles near $x_0 = x_0(t)$ and $x_1 = x_1(t)$.

When the attachments D_+ and D_- are removed from D and are replaced by homogeneous Neumann conditions imposed on the sides $x = 0$ and $x = 1$ or R, it is known that any two-layer equilibrium solution of the form (11) is unstable. The metastable dynamics of such a staircase profile was studied in [19] (see also [8], [13]). In this situation, the inter-layer dynamics between x_0 and x_1 is repulsive, whereas the interaction between each layer and the walls at $x = 0$ and $x = 1$ is attractive. Thus, as $t \to \infty$ a stable, but spatially homogeneous, equilibrium solution results. However, as we show in §4, the presence of non-convex attachments $D_\pm$ will induce a repulsive force on x_0 and on x_1 that can balance the inter-layer repulsive force that exists between x_0 and x_1. As $t \to \infty$, this then leads to the existence of a stable spatially inhomogeneous equilibrium solution in R.

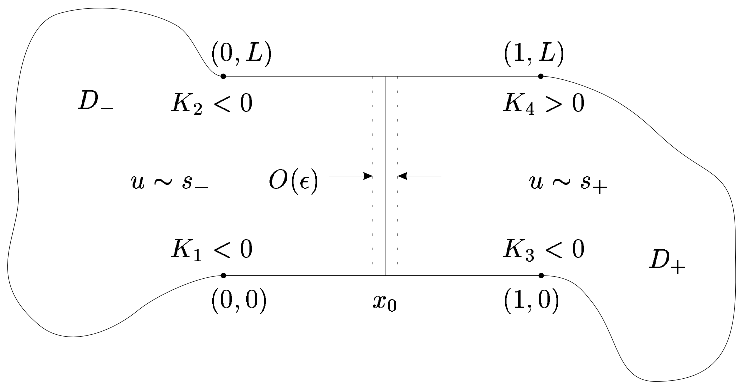

Figure 1: Plot of a typical domain D and an interface located at x_0.

2 The Allen-Cahn Equation

In this section we analyze the dynamics of a straight-line interface $x = x_0(t)$ for (3), which connects the two sides of the channel R as shown in Fig. 1.

For $\epsilon \ll 1$, we first construct a quasi-equilibrium solution to (3) in R that has the asymptotics given in (5). This solution is only a function of x and it has exactly one internal layer centered at $x = x_0$, where $0 < x_0 < 1$. The quasi-equilibrium solution is represented by $u = u_c\left[\epsilon^{-1}(x - x_0)\right]$, where $u_c(z)$ is the heteroclinic connection satisfying

$$u_c'' + Q(u_c) = 0\,, \qquad -\infty < z < \infty\,, \tag{12a}$$

$$u_c(0) = 0\,, \qquad u_c(\pm\infty) = s_\pm\,, \tag{12b}$$

where $Q(u)$ has the properties listed in (4). The far-field behavior of $u_c(z)$ is given by

$$u_c(z) \sim \begin{cases} s_+ - a_+ e^{-\nu_+ z}, & z \to +\infty\,, \\ s_- + a_- e^{\nu_- z}, & z \to -\infty\,, \end{cases} \tag{13}$$

where the positive constants $\nu_\pm$ and $a_\pm$ are defined by

$$\nu_\pm = \left[-Q'(s_\pm)\right]^{1/2}\,, \tag{14a}$$

$$\log a_\pm = \log(\pm s_\pm) + \int_0^{s_\pm} \left(\frac{\pm\nu_\pm}{[2V(\eta)]^{1/2}} + \frac{1}{\eta - s_\pm} \right) d\eta\,. \tag{14b}$$

Assume that the initial condition for (3) is $u(\mathbf{x}, 0) = u_c[\epsilon^{-1}(x - x_0^0)]$ for some $x_0^0 \in (0, 1)$. We then look for a solution to (3) in the channel region R in the form

$$u(\mathbf{x},t) \sim u_c\left[\epsilon^{-1}(x - x_0(t))\right] , \tag{15}$$

with $x_0(0) = x_0^0$, where $x_0 = x_0(t)$ represents the unknown trajectory of the level curve $u = 0$. Linearizing (3) around u_c by writing

$$u(\mathbf{x},t) = u_c[\epsilon^{-1}(x - x_0(t))] + v(\mathbf{x},t) , \tag{16}$$

where $v \ll u_c$, we obtain

$$L_\epsilon v \equiv \epsilon^2 \Delta v + Q^{'}(u_c) v = \partial_t u_c + v_t , \qquad \mathbf{x} \in D , \tag{17a}$$

$$\partial_n v = -\partial_n u_c, \qquad \mathbf{x} \in \partial D . \tag{17b}$$

In §2.1 we show that the eigenvalue problem associated with (17) has an exponentially small principal eigenvalue. In §2.2 we derive a differential equation for $x_0(t)$ by projecting v against the eigenspace associated with this exponentially small eigenvalue and in §2.3 we illustrate the result with an example.

2.1 An Exponentially Small Eigenvalue

The eigenvalue problem associated with (17) is

$$L_\epsilon \phi \equiv \epsilon^2 \Delta \phi + Q^{'}(u_c) \phi = \lambda \phi , \qquad \mathbf{x} \in D , \tag{18a}$$

$$\partial_n \phi = 0 , \qquad \mathbf{x} \in \partial D , \tag{18b}$$

$$(\phi, \phi) = \int_D \phi^2 \, d\mathbf{x} . \tag{18c}$$

Here, and below, the inner product is defined by $(u, v) \equiv \int_D uv \, d\mathbf{x}$. The eigenvalues and eigenfunctions of (18) are labeled by λ_j and ϕ_j respectively for $j = 0, 1, \ldots$, with $\lambda_j \to -\infty$ as $j \to \infty$.

We now obtain asymptotic estimates for the principal eigenpair of (18) under the assumption that $x_0 \in (0, 1)$ is fixed and that the distance from x_0 to the corners of R is $O(1)$. We first note that $L_\epsilon \left(u_c^{'}[\epsilon^{-1}(x - x_0)] \right) = 0$, that $u_c^{'}$ is of one sign, and that $u_c^{'}$ satisfies the boundary condition on R exactly. However, $u_c^{'}$ fails to satisfy the boundary condition on the attachments D_+ and D_- by exponentially small terms as $\epsilon \to 0$. Thus, we expect that the principal eigenfunction for (18) has the form

$$\phi_0 \sim M_0 \left(u_c^{'}[\epsilon^{-1}(x - x_0)] + \phi_{L0} \right) . \tag{19}$$

Here M_0 is a normalization constant and ϕ_{L0} is a boundary layer function localized near ∂D_- and ∂D_+ that is needed to satisfy the boundary condition (18b). Applying Green's identity to (18) and u_c' we obtain an equation for λ_0

$$\lambda_0 \left(u_c', \phi_0\right) = -\epsilon^2 \int_{\partial D} \phi_0 \partial_n u_c' \, ds \,. \tag{20}$$

To estimate λ_0 as $\epsilon \to 0$, we must evaluate each term in (20). First, to calculate ϕ_0 on $\partial D_\pm$ we must determine ϕ_{L0}. To do so we introduce a local coordinate system near ∂D_+ and ∂D_-. We set $\eta = n/\epsilon$, where $-n$ is the distance from $\mathbf{x} \in D_\pm$ to $\partial D_\pm$. Then, when $\eta = O(1)$ and in the left attachment D_- where $x < x_0$, we have $u_c \sim s_-$, $Q'(u_c) \sim -\nu_-^2$, so that (18a)-(18b) reduces to

$$\begin{gathered} \partial_{\eta\eta}\phi_{L0} - \nu_-^2 \phi_{L0} = 0 \,, \quad \eta < 0 \,; \quad \phi_{L0} \to 0 \,, \text{ as } \eta \to -\infty \,, \\ \partial_\eta \phi_{L0}|_{\eta=0} = -\epsilon \partial_n u_c'|_{\eta=0} \,. \end{gathered} \tag{21}$$

Similarly, in the right attachment D_+ where $x > x_0$, we have $u_c \sim s_+$, $Q'(u_c) \sim -\nu_+^2$ so that

$$\begin{gathered} \partial_{\eta\eta}\phi_{L0} - \nu_+^2 \phi_{L0} = 0 \,, \quad \eta < 0 \,; \quad \phi_{L0} \to 0 \,, \text{ as } \eta \to -\infty \,, \\ \partial_\eta \phi_{L0}|_{\eta=0} = -\epsilon \partial_n u_c'|_{\eta=0} \,. \end{gathered} \tag{22}$$

Solving these equations we get,

$$\phi_{L0} \sim \begin{cases} \left(-\epsilon \partial_n u_c'\right)|_{\eta=0} e^{\nu_- \eta} & \text{for } \eta = O(1) \text{ near } \partial D_- \\ \left(-\epsilon \partial_n u_c'\right)|_{\eta=0} e^{\nu_+ \eta} & \text{for } \eta = O(1) \text{ near } \partial D_+ \,. \end{cases} \tag{23}$$

Using the decay behavior (13), we obtain

$$\partial_n u_c' \sim \begin{cases} \epsilon^{-1} a_- \nu_-^2 e^{\nu_- \epsilon^{-1}(x - x_0)} n_x & \text{near } \partial D_- \,, \\ -\epsilon^{-1} a_+ \nu_+^2 e^{-\nu_+ \epsilon^{-1}(x - x_0)} n_x & \text{near } \partial D_+ \,. \end{cases} \tag{24}$$

Here n_x is the x component of the unit outward normal vector $\hat{\mathbf{n}} = (n_x, n_y)$ to ∂D_+ and ∂D_-. Substituting (24) into (23), and then using (19) and (13), we obtain an estimate for ϕ_0 near $\partial D_\pm$

$$\phi_0 \sim \begin{cases} M_0 a_- \nu_- e^{\nu_- \epsilon^{-1}(x - x_0)} \left(1 - \nu_- n_x e^{\nu_- \epsilon^{-1} \eta}\right) & \text{near } \partial D_- \,, \\ M_0 a_+ \nu_+ e^{-\nu_+ \epsilon^{-1}(x - x_0)} \left(1 + \nu_+ n_x e^{\nu_+ \epsilon^{-1} \eta}\right) & \text{near } \partial D_+ \,. \end{cases} \tag{25}$$

Setting $\eta = 0$ in (25) gives an estimate for ϕ_0 on $\partial D_\pm$.

Next, we calculate $\left(u_c', \phi_0\right)$. The dominant contribution to this inner product arises from the region near $x = x_0$ where $\phi_0 \sim M_0 u_c'$. We estimate,

$$\left(u_c', \phi_0\right) \sim M_0 \left(u_c', u_c'\right) \sim \epsilon L M_0 \beta \,, \qquad \beta \equiv \int_{-\infty}^{\infty} \left[u_c'(z)\right]^2 dz \,, \tag{26}$$

where L is the width of the channel. Now we calculate the normalization constant M_0 from $(\phi_0, \phi_0) = 1$. Since $M_0^2 \left(u_c^{'}, u_c^{'}\right) \sim 1$, (26) yields

$$M_0 \sim (\epsilon L \beta)^{-1/2} \ . \tag{27}$$

To determine λ_0 as $\epsilon \to 0$, we substitute (24), (25) and (26) into (20) to obtain

$$\lambda_0 \sim \frac{1}{L\beta} \left\{ I_+ - I_- \right\} \ , \tag{28}$$

where I_+ and I_- are defined by

$$I_- \equiv \int_{\partial D_-} a_-^2 \nu_-^3 e^{2\nu_- \epsilon^{-1}(x - x_0)} (1 - \nu_- n_x) n_x \, ds \, , \tag{29}$$

$$I_+ \equiv \int_{\partial D_+} a_+^2 \nu_+^3 e^{-2\nu_+ \epsilon^{-1}(x - x_0)} (1 + \nu_+ n_x) n_x \, ds \, . \tag{30}$$

Since the integrands in (29) and (30) are exponentially decreasing away from the interface $x = x_0$, the dominant contribution to these integrals arises from the $O(\epsilon)$ regions near the corners of R. In these corner regions, we can calculate n_x in terms of the local properties given in (2). We find,

$$n_x = \frac{-K_1(-x)^{\alpha_1}}{\left[K_1^2(-x)^{2\alpha_1}+1\right]^{1/2}} \qquad \text{near } (0,0) \text{ as } x \to 0^- \, , \tag{31a}$$

$$n_x = \frac{-K_2(-x)^{\alpha_2}}{\left[K_2^2(-x)^{2\alpha_2}+1\right]^{1/2}} \qquad \text{near } (0,b) \text{ as } x \to 0^- \, , \tag{31b}$$

$$n_x = \frac{K_3(x-1)^{\alpha_3}}{\left[K_3^2(x-1)^{2\alpha_3}+1\right]^{1/2}} \qquad \text{near } (1,0) \text{ as } x \to 1^+ \, , \tag{31c}$$

$$n_x = \frac{K_4(x-1)^{\alpha_4}}{\left[K_4^2(x-1)^{2\alpha_4}+1\right]^{1/2}} \qquad \text{near } (1,b) \text{ as } x \to 1^+ \, . \tag{31d}$$

Since $n_x \ll n_x^2$ near $x = 0$ and $x = 1$ we can, to leading order, neglect the terms in (29) and (30) proportional to n_x^2. Then, using Laplace's method, we get

$$I_- \sim a_-^2 \nu_-^3 \int_0^{-\infty} \left(K_1(-x)^{\alpha_1} + K_2(-x)^{\alpha_2}\right) e^{2\nu_- \epsilon^{-1}(x - x_0)} \, dx \, , \tag{32}$$

$$I_+ \sim a_+^2 \nu_+^3 \int_1^{\infty} \left(K_3(x-1)^{\alpha_3} + K_4(x-1)^{\alpha_4}\right) e^{-2\nu_+ \epsilon^{-1}(x - x_0)} \, dx \, . \tag{33}$$

These integrals can be evaluated explicitly to yield

$$I_- \sim -a_-^2 \nu_-^3 \left[K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2)\right] e^{-2\nu_- \epsilon^{-1} x_0} \, , \tag{34}$$

$$I_+ \sim a_+^2 \nu_+^3 \left[K_3 \zeta_+(\alpha_3) + K_4 \zeta_+(\alpha_4)\right] e^{-2\nu_+ \epsilon^{-1}(1 - x_0)} \, , \tag{35}$$

where $\zeta_\pm(\alpha)$ is defined by

$$\zeta_\pm(\alpha) \equiv \left(\frac{\epsilon}{2\nu_\pm}\right)^{\alpha+1} \Gamma(\alpha+1)\,, \tag{36}$$

and $\Gamma(z)$ is the Gamma function. Substituting (34) and (35) into (28) we obtain the following key result:

Proposition 1: (Eigenvalue for (AC)) *For $\epsilon \to 0$, the principal eigenvalue λ_0 associated with linearizing (3) about the quasi-equilibrium solution (15) with a fixed $x_0 \in (0,1)$ has the asymptotic estimate*

$$\lambda_0 \sim \frac{1}{L\beta}\left\{a_-^2\nu_-^3\left[K_1\zeta_-(\alpha_1)+K_2\zeta_-(\alpha_2)\right]e^{-2\nu_-\epsilon^{-1}x_0} + a_+^2\nu_+^3\left[K_3\zeta_+(\alpha_3)+K_4\zeta_+(\alpha_4)\right]e^{-2\nu_+\epsilon^{-1}(1-x_0)}\right\}. \tag{37}$$

Here $\nu_\pm$, $a_\pm$, β, $\zeta_\pm(\alpha)$ and α_i, K_i, are defined in (14a), (14b), (26), (36), and (2), respectively.

Notice that the sign of λ_0 is determined by the signs of K_i for $i = 1, .., 4$ defined in (2). Therefore, the local convexity property (2) of the domain D near the corners of R will determine the stability or instability of the quasi-equilibrium solution.

2.2 Metastable Dynamics and Equilibria

We now determine the dynamics of the interface $x_0 = x_0(t)$ for (15) in the channel R. Since the principal eigenvalue for (18) is exponentially small, we can proceed with a quasi-steady approximation $v_t \ll \partial_t u_c$ in (17). The solution to (17) is then expanded in terms of the eigenpairs ϕ_j, λ_j for $j \geq 0$ of (18) as

$$v(\mathbf{x},t) = \sum_{j=0}^{\infty} \frac{c_j(t)}{\lambda_j}\phi_j(\mathbf{x})\,. \tag{38}$$

Using orthogonality of the ϕ_j and integration by parts we derive

$$c_j = (\phi_j, \partial_t u_c) + \epsilon^2 \int_{\partial D} \phi_j \partial_n u_c \, ds\,. \tag{39}$$

Since λ_0, as estimated in (37), is exponentially small, we require that $c_0 = 0$. This projection step yields the slow motion equation

$$(\phi_0, \partial_t u_c) = -\epsilon^2 \int_{\partial D_+} \phi_0 \partial_n u_c \, ds - \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n u_c \, ds\,. \tag{40}$$

Here we have used the result that $\partial_n u_c^{'} = 0$ on the boundary of R.

Next the terms in (40) are evaluated to determine the motion of the interface. To calculate $(\phi_0, \partial_t u_c)$ we use (15) and (26) to get

$$(\phi_0, \partial_t u_c) \sim -LM_0 \beta x_0^{'} \,. \tag{41}$$

Now we use (13) and (25) to evaluate the right side of (40) as $\epsilon \to 0$. This yields,

$$-\epsilon^2 \int_{\partial D_+} \phi_0 \partial_n u_c \, ds - \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n u_c \, ds \sim - \left(\frac{\epsilon}{\nu_-} I_- + \frac{\epsilon}{\nu_+} I_+ \right) M_0 \,. \tag{42}$$

Here $I_\pm$ are defined in (29) and (30). Substituting (41) and (42) into (40), and using (34) and (35), we obtain the following main result for the slow dynamics of $x_0(t)$:

Proposition 2: (Metastability for (AC)) *For $\epsilon \to 0$, the metastable dynamics of the location $x_0(t)$ of the straight-line interface solution (15) in the channel R, where $x_0 \in (0,1)$, satisfies the asymptotic nonlinear differential equation*

$$x_0^{'} \sim \frac{\epsilon}{L\beta} \left\{ a_+^2 \nu_+^2 \left[K_3 \zeta_+(\alpha_3) + K_4 \zeta_+(\alpha_4) \right] e^{-2\nu_+ \epsilon^{-1}(1-x_0)} - a_-^2 \nu_-^2 \left[K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2) \right] e^{-2\nu_- \epsilon^{-1} x_0} \right\} . \tag{43}$$

Here the constants are as defined in proposition 1.

Thus the motion of the interface location $x_0(t)$ is determined by the constants associated with the tail behavior of u_c, by the shape of the boundary at the corners of R, and by the distance from the interface to these corners. The interface will move according to (43) until either a steady state is attained or the interface has moved to one of the vertical sides of R. In the latter case, the subsequent evolution of the interface is determined by (6). This result agrees with [3] and [11] obtained using a different method.

Define $A_\pm$ by

$$A_+ = K_3 \zeta_+(\alpha_3) + K_4 \zeta_+(\alpha_4) \,, \qquad A_- = K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2) \,. \tag{44}$$

Then (43) has a unique steady-state solution x_0^e given by

$$x_0^e \sim \frac{\nu_+}{\nu_+ + \nu_-} + \frac{\epsilon}{2(\nu_+ + \nu_-)} \log \left(\frac{a_-^2 \nu_-^2 A_-}{a_+^2 \nu_+^2 A_+} \right) , \tag{45}$$

only when $A_+ A_- > 0$. As seen from (43), this equilibrium solution is stable when $A_+ < 0$ and $A_- < 0$, and is unstable when $A_+ > 0$ and $A_- > 0$. Sufficient conditions for the existence of x_0^e are that either $K_i > 0$ for all i or $K_i < 0$ for all i.

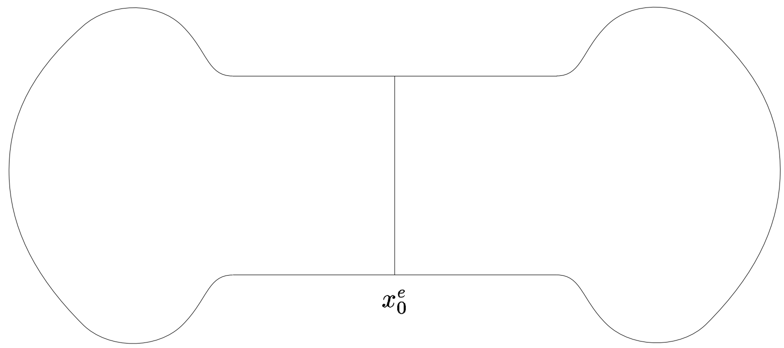

Figure 2: Plot of a domain, D, that exhibits a stable steady state interface location. For this domain $K_i < 0$ for $i = 1, \ldots, 4$ and the steady state interface location, x_0^e is given by (45).

Corollary 1: (Stable Equilibria) *Assume that D is locally non-convex near each corner so that in (2) we have $K_i < 0$ for all i. Then, for $\epsilon \ll 1$, (3) admits a stable spatially inhomogeneous equilibrium solution u^e of the form*

$$u^e(\mathbf{x};\epsilon) \sim u_c\left[\epsilon^{-1}(x - x_0^e)\right] , \tag{46}$$

where x_0^e is given in (45). An example of such a domain is shown in Fig. 2.

2.3 An Example

We illustrate these results for the specific example $Q(u) = 2(u - u^3)$. Then, from (12), (14a), (14b), and (26), we have $u_c(z) = \tanh(z)$, $\nu_\pm = 2$, $a_\pm = 2$, and $\beta = 4/3$. Suppose in addition that $\alpha_i = \alpha$ for all i. Then, the dynamics (43) reduces to

$$x_0' \sim \frac{12\epsilon^{\alpha+2}\Gamma(\alpha+1)}{4^{\alpha+1}L}\left[(K_3+K_4)e^{-4\epsilon^{-1}(1-x_0)} - (K_1+K_2)e^{-4\epsilon^{-1}x_0}\right] . \tag{47}$$

When $(K_1 + K_2)(K_3 + K_4) > 0$, the unique steady-state solution for (47) is

$$x_0^e \sim \frac{1}{2} + \frac{\epsilon}{8}\log\left(\frac{K_1+K_2}{K_3+K_4}\right) . \tag{48}$$

If $(K_1 + K_2) < 0$ and $(K_3 + K_4) < 0$, then $x_0(t) \to x_0^e$ for any initial condition $x_0(0) \in (0, 1)$. In particular, this occurs when $K_i < 0$ for all i.

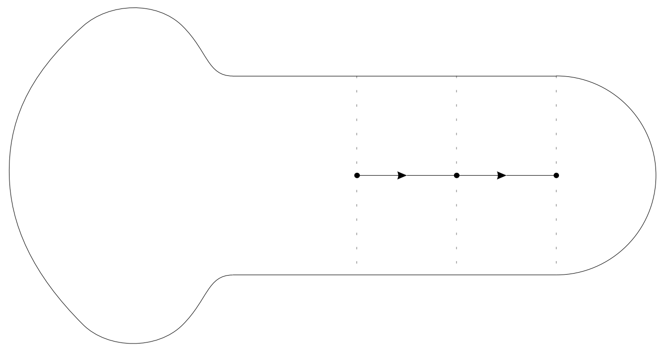

Figure 3: Plot of a domain, D, in which the interface moves toward the right. Note that $K_1 < 0$, $K_2 < 0$ and $K_3 > 0$, $K_4 > 0$ for this domain.

Now suppose $K_1 < 0$, $K_2 < 0$ and $K_3 > 0$, $K_4 > 0$. An example of such a domain is shown in Fig. 3. In this case, $x_0^{'} > 0$ and hence $x_0(t)$ monotonically approaches the end of the channel at $x = 1$ according to (47). Then, the interface becomes curved and its dynamics evolves by the curvature flow (6) until the interface collapses against the boundary of the right attachment D_+ leaving the spatially homogeneous steady-state solution $u \sim s_-$. An analogous situation occurs when $K_1 > 0$, $K_2 > 0$ and $K_3 < 0$, $K_4 < 0$. In this case the interface moves monotonically to the left and enters the left attachment D_- and collapses against ∂D_- leaving the constant steady-state solution $u \sim s_+$.

3 The Constrained Allen-Cahn Equation

In this section we analyze the dynamics of a solution to (7) of the form given in (8), which has two straight-line interfaces in the channel region R located at $x = x_0(t)$ and $x = x_1(t)$. We assume that t is such that these interfaces lie within R so that

$$0 < x_0(t) < x_1(t) < 1 \,. \tag{49}$$

We then define $d_i(t)$ by

$$d_0(t) = x_0(t) \,, \qquad d_1(t) = x_1(t) - x_0(t) \,, \qquad d_2(t) = 1 - x_1(t) \,. \tag{50}$$

Thus, d_0 and d_2 are the distances from x_0 and x_1 to the left and right attachments D_- and D_+, respectively.

For a solution to (7) of the form (8), it follows that σ in (7a) is exponentially small and that u is given asymptotically by (see [14])

$$u(\mathbf{x},t) \sim \tilde{u}^\epsilon[x;x_0(t),x_1(t)] \equiv u_c\left[\epsilon^{-1}(x-x_0(t))\right] + u_c\left[\epsilon^{-1}(x_1(t)-x)\right] - s_+ \,. \tag{51}$$

Here $u_c(z)$ is the solution to (12) with the asymptotic behavior (13). The mass m in (7c) must be consistent with this solution so that for $\epsilon \ll 1$ we require

$$m = \int_D \tilde{u}^\epsilon \, d\mathbf{x} \sim s_-\left(V_- + V_+\right) + s_-\left(1-d_1\right) + s_+ d_1 \,, \tag{52}$$

where $V_\pm$ are the volumes of $D_\pm$. We assume that m is such that (52) has a solution d_1 with $0 < d_1 < 1$. Since mass is conserved for (7), it follows that on time intervals where $d_j > 0$ for $j = 0,1,2$, we have

$$d_0^{'} = -d_2^{'} \qquad d_1^{'} = 0 \,. \tag{53}$$

We assume that the initial condition for (7) has the form (51) for some $x_0(0)$ and $x_1(0)$ satisfying $0 < x_0(0) < x_1(0) < 1$.

We proceed by deriving a differential equation for $d_0(t)$. Then $d_2(t)$ is determined by

$$d_2(t) = 1 - d_1 - d_0(t) \,, \tag{54}$$

where d_1 is given in terms of m by (52). In the analysis below we omit the details of the calculation that are similar to those in §2. We first linearize (7) around $\tilde{u}^\epsilon$ by writing

$$u(\mathbf{x},t) = \tilde{u}^\epsilon[x;x_0(t),x_1(t)] + v(\mathbf{x},t) \,, \tag{55}$$

where $v \ll \tilde{u}^\epsilon$. This yields the linearized problem

$$L_\epsilon v \equiv \epsilon^2 \Delta v + Q^{'}\left(\tilde{u}^\epsilon\right) v = \partial_t \tilde{u}^\epsilon + v_t - E + \sigma \,, \qquad \mathbf{x} \in D \,, \tag{56a}$$

$$\partial_n v = -\partial_n \tilde{u}^\epsilon \,, \qquad \mathbf{x} \in \partial D \,. \tag{56b}$$

Here E is the layer interaction term defined by

$$E = E(x;x_0,x_1) \equiv Q\left(\tilde{u}^\epsilon\right) - Q\left(u_c[\epsilon^{-1}(x-x_0)]\right) - Q\left(u_c[\epsilon^{-1}(x_1-x)]\right) \,. \tag{57}$$

The corresponding eigenvalue problem is (18) where u_c in (18a) is replaced by $\tilde{u}^\epsilon$. By similar reasoning as in §2.1, the first two eigenvalues of this problem are exponentially small and the corresponding eigenfunctions ϕ_0 and ϕ_1 are given asymptotically by

$$\phi_j \sim M_j \left(u_c^{'}[(-1)^j \epsilon^{-1}(x - x_j)] + \phi_{Lj} \right) , \qquad j = 0, 1 . \tag{58}$$

Here M_j is a normalization constant and ϕ_{Lj} is a boundary layer function localized near $\partial D_{\pm}$ that is needed to satisfy the boundary condition (18b) exactly. However, since the support of ϕ_0 is concentrated in an an $O(\epsilon)$ region near x_0, which lies to the left of x_1, in the analysis below we need only calculate ϕ_{L0} on ∂D_{-}. Similarly, we need only calculate ϕ_{L1} on the boundary ∂D_{+} of the right attachment D_{+}.

By using a boundary layer analysis as in §2.1, we derive the following boundary formulae for ϕ_0 and ϕ_1:

$$\phi_0 \sim M_0 a_{-} \nu_{-} e^{\nu_{-} \epsilon^{-1}(x - x_0)} \left(1 - \nu_{-} n_x\right) , \qquad \text{on } \partial D_{-} , \tag{59a}$$

$$\phi_1 \sim M_1 a_{-} \nu_{-} e^{\nu_{-} \epsilon^{-1}(x_1 - x)} \left(1 + \nu_{-} n_x\right) , \qquad \text{on } \partial D_{+} , \tag{59b}$$

where n_x is the x-component of the unit outward normal vector $\mathbf{n}$ to ∂D. Here the normalization constants M_j can be evaluated as

$$M_0 = M_1 = M \equiv (\epsilon L \beta)^{-1/2} , \tag{60}$$

where β is defined in (26). The corresponding exponentially small eigenvalues λ_0 and λ_1 can be estimated as in §2.1.

Since λ_0 and λ_1 are exponentially small we can make the quasi-steady approximation $v_t \ll \partial_t \tilde{u}^{\epsilon}$ in (56a). We then expand the solution to (56) in terms of the eigenfunctions ϕ_j as in (38). In place of (39), the coefficients c_j in the expansion are now

$$c_j = \left(\phi_j, \partial_t \tilde{u}^{\epsilon}\right) - (E, \phi_j) + \sigma\,(\phi_j, 1) + \epsilon^2 \int_{\partial D} \phi_j \partial_n \tilde{u}^{\epsilon} \, ds , \qquad j = 0, 1, \ldots . \tag{61}$$

Upon imposing the limiting solvability conditions that $c_0 = 0$ and $c_1 = 0$, we get the coupled equations

$$\left(\phi_0, \partial_t \tilde{u}^{\epsilon}\right) \sim (E, \phi_0) - \sigma\,(\phi_0, 1) - \epsilon^2 \int_{\partial D} \phi_0 \partial_n \tilde{u}^{\epsilon} \, ds , \tag{62a}$$

$$\left(\phi_1, \partial_t \tilde{u}^{\epsilon}\right) \sim (E, \phi_1) - \sigma\,(\phi_1, 1) - \epsilon^2 \int_{\partial D} \phi_1 \partial_n \tilde{u}^{\epsilon} \, ds . \tag{62b}$$

These equations will determine $\sigma(t)$ and an ordinary differential equation for $d_0(t)$.

Next, we evaluate the terms in (62). A simple calculation using (51) and (58) shows that the following formulae for the inner products are valid to within negligible exponentially small terms:

$$\left(\partial_t \tilde{u}^\epsilon, \phi_j\right) \sim -(-1)^j L\beta M x_j^{'} \,, \qquad (\phi_j, 1) \sim \epsilon L(s_+ - s_-) \,, \qquad j = 0, 1 \,. \tag{63}$$

Therefore, since $d_0^{'} = -d_2^{'}$, $d_0^{'} = x_0^{'}$, $d_2^{'} = -x_1^{'}$ we can subtract (62a) and (62b) and use (63) to get a differential equation for $d_0(t)$

$$2d_0^{'} LM\beta \sim (E, \phi_1) - (E, \phi_0) - \epsilon^2 \int_{\partial D} \phi_1 \partial_n \tilde{u}^\epsilon \, ds + \epsilon^2 \int_{\partial D} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,. \tag{64}$$

Similarly, by adding (62a) and (62b) we get an equation for $\sigma(t)$

$$2\epsilon LM(s_+ - s_-)\sigma \sim (E, \phi_0) + (E, \phi_1) - \epsilon^2 \int_{\partial D} \phi_1 \partial_n \tilde{u}^\epsilon \, ds - \epsilon^2 \int_{\partial D} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,. \tag{65}$$

The interaction terms (E, ϕ_j) in (64) and (65) can be evaluated asymptotically for $\epsilon \to 0$ as in [14] with the result

$$(E, \phi_j) \sim -2\epsilon M a_+^2 \nu_+^2 e^{-\nu_+ \epsilon^{-1} d_1} \,, \qquad j = 0, 1 \,. \tag{66}$$

In addition, upon comparing the asymptotic orders of the two boundary integrals on the right sides of (64) and (65), we conclude that we can asymptotically approximate the boundary integral for ϕ_0 over ∂D with one only over ∂D_-. Similarly, the integral involving ϕ_1 can be asymptotically approximated by one only over ∂D_+. From this observation, and by using (66), we can reduce (64) and (65) asymptotically to

$$2d_0^{'} LM\beta \sim -\epsilon^2 \int_{\partial D_+} \phi_1 \partial_n \tilde{u}^\epsilon \, ds + \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,, \tag{67}$$

$$2\epsilon LM(s_+ - s_-)\sigma \sim -4\epsilon a_+^2 \nu_+^2 e^{-\nu_+ \epsilon^{-1} d_1} - \epsilon^2 \int_{\partial D_+} \phi_1 \partial_n \tilde{u}^\epsilon \, ds - \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,. \tag{68}$$

The final step in the derivation is to asymptotically calculate the boundary integrals in (67) and (68). This is done as in §2.1. We first asymptotically evaluate the integrands in these expressions by using (13), (51), and (59). Then, the resulting Laplace-type integrals are evaluated asymptotically as $\epsilon \to 0$ in terms of the local properties (2) of D at the corners of the channel R. In this way, we obtain

$$\epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \sim -\epsilon a_-^2 \nu_-^2 M \left(K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2)\right) e^{-2\nu_- \epsilon^{-1} d_0} \,, \tag{69}$$

$$\epsilon^2 \int_{\partial D_+} \phi_1 \partial_n \tilde{u}^\epsilon \, ds \sim -\epsilon a_-^2 \nu_-^2 M \left(K_3 \zeta_-(\alpha_3) + K_4 \zeta_-(\alpha_4)\right) e^{-2\nu_- \epsilon^{-1} d_2} \,. \tag{70}$$

Here K_i and α_i are defined in (2) and $\zeta_-(\alpha)$ is defined by

$$\zeta_-(\alpha) \equiv \left(\frac{\epsilon}{2\nu_-}\right)^{\alpha+1} \Gamma(\alpha+1)\,. \tag{71}$$

Finally, by substituting (69) and (70) into (67) and (68), we get the key metastability result:

Proposition 3: (Metastability for (CAC)) *For $\epsilon \to 0$ consider a two-layer solution for (7) of the form (51). Define the distances $d_i(t)$ as in (50). Then, when x_0 and x_1 are in the channel R, the distance $d_0(t)$ between x_0 and the edge of the attachment D_- satisfies the following asymptotic nonlinear differential equation*

$$d_0' \sim \frac{\epsilon a_-^2 \nu_-^2}{2L\beta} \left\{ \left[K_3 \zeta_- \alpha_3) + K_4 \zeta_-(\alpha_4)\right] e^{-2\nu_- \epsilon^{-1} d_2} - \left[K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2)\right] e^{-2\nu_- \epsilon^{-1} d_0} \right\}. \tag{72}$$

Here d_2 is determined in terms of d_0 by (54). For such a solution, the function $\sigma(t)$ in (7) is given asymptotically by

$$\begin{aligned} \sigma \quad \sim \quad & \frac{a_-^2 \nu_-^2}{2L(s_+ - s_-)} \left\{ \left[K_3 \zeta_-(\alpha_3) + K_4 \zeta_-(\alpha_4)\right] e^{-2\nu_- \epsilon^{-1} d_2} + \left[K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2)\right] e^{-2\nu_- \epsilon^{-1} d_0} \right. \\ & \left. - \frac{4 a_+^2 \nu_+^2}{a_-^2 \nu_-^2} e^{-\nu_+ \epsilon^{-1} d_1} \right\}. \end{aligned} \tag{73}$$

Here ν_-, a_-, β, $\zeta_-(\alpha)$ and α_i, K_i, are defined in (14a), (14b), (26), (71), and (2), respectively.

We now determine the equilibrium solution. Define A and B by

$$A = K_3 \zeta_-(\alpha_3) + K_4 \zeta_-(\alpha_4)\,, \qquad B = K_1 \zeta_-(\alpha_1) + K_2 \zeta_-(\alpha_2)\,. \tag{74}$$

Then (72) has a unique steady-state solution d_0^e when $AB > 0$. In this case, the equilibrium values x_0^e and x_1^e are readily found to be

$$x_0 \sim \frac{(1-d_1)}{2} + \frac{\epsilon}{4\nu_-} \log\left(\frac{B}{A}\right), \qquad x_1 \sim \frac{(1+d_1)}{2} + \frac{\epsilon}{4\nu_-} \log\left(\frac{B}{A}\right). \tag{75}$$

This equilibrium is stable when $A < 0$ and $B < 0$ and is unstable when $A > 0$ and $B > 0$. This leads to the equilibrium result:

Corollary 2: (Stable Equilibria) *Assume that D is locally non-convex near each corner so that in (2) we have $K_i < 0$ for all i. Then, for $\epsilon \ll 1$, (7) admits a stable spatially inhomogeneous equilibrium solution u^e of the form*

$$u^e(\mathbf{x};\epsilon) \sim u_c\left[\epsilon^{-1}(x - x_0^e)\right] + u_c\left[\epsilon^{-1}(x_1^e - x)\right] - s_+\,. \tag{76}$$

Here u_c satisfies (12) and x_0^e and x_1^ϵ are defined in (75).

For the specific example where $Q(u) = 2(u - u^3)$ and $\alpha_i = \alpha$ for all i, the dynamics (72) reduces for $\epsilon \ll 1$ to

$$d_0^{'} \sim \frac{6\epsilon^{\alpha+2}\Gamma(\alpha+1)}{4^{\alpha+1}L}\left[(K_3+K_4)e^{-4\epsilon^{-1}d_2} - (K_1+K_2)e^{-4\epsilon^{-1}d_0}\right]. \tag{77}$$

When $(K_1 + K_2)(K_3 + K_4) > 0$, the equilibrium values of x_0 and x_1 are

$$x_0^e \sim \frac{(1-d_1)}{2} + \frac{\epsilon}{8}\log\left[\frac{K_1+K_2}{K_3+K_4}\right], \qquad x_1^e \sim \frac{(1+d_1)}{2} + \frac{\epsilon}{8}\log\left[\frac{K_1+K_2}{K_3+K_4}\right]. \tag{78}$$

This solution is stable when $(K_1 + K_2) < 0$ and $(K_3 + K_4) < 0$.

4 The Tristable Allen-Cahn Equation

In this section we analyze the dynamics of a staircase-type solution to (9) of the form (11). Let the interfaces in R be located at $x = x_0(t)$ and $x = x_1(t)$ and assume that t is such that

$$0 < x_0(t) < x_1(t) < 1. \tag{79}$$

The distances d_i are as defined in (50).

The staircase solution (11) can be represented as

$$u(\mathbf{x},t) \sim \tilde{u}^\epsilon[x; x_0(t), x_1(t)] \equiv u_c\left[\epsilon^{-1}(x - x_0(t))\right] + u_c\left[\epsilon^{-1}(x - x_1(t))\right] + s. \tag{80}$$

Here $u_c(z)$ is the heteroclinic connection satisfying

$$u_c^{''} + Q(u_c) = 0, \qquad -\infty < z < \infty, \tag{81a}$$

$$u_c(0) = 0, \qquad u_c(\pm\infty) = \pm s, \tag{81b}$$

where $Q(u)$ has the properties listed in (10). The far-field behavior of $u_c(z)$ is given by

$$u_c(z) \sim \pm s \mp a e^{\mp\nu z}, \quad \text{as} \quad z \to \pm\infty; \qquad \nu \equiv \left[-Q^{'}(-s)\right]^{1/2}, \tag{82}$$

where a is a positive constant. We assume that the initial condition for (9) has the form (80) for some $x_0(0)$ and $x_1(0)$ satisfying $0 < x_0(0) < x_1(0) < 1$.

We proceed by deriving a coupled set of differential equations for $d_0(t)$ and $d_1(t)$. We first linearize (9) by introducing v as in (55). In place of (56), we find that v satisfies the linearized problem

$$L_\epsilon v \equiv \epsilon^2 \Delta v + Q^{'}\left(\tilde{u}^\epsilon\right) v = \partial_t \tilde{u}^\epsilon + v_t - E, \qquad \mathbf{x} \in D, \tag{83a}$$

$$\partial_n v = -\partial_n \tilde{u}^\epsilon, \qquad \mathbf{x} \in \partial D. \tag{83b}$$

Here E is the layer interaction term defined by

$$E = E(x; x_0, x_1) \equiv Q\left(\tilde{u}^{\epsilon}\right) - Q\left(u_c[\epsilon^{-1}(x - x_0)]\right) - Q\left(u_c[\epsilon^{-1}(x - x_1)]\right) . \qquad (84)$$

The corresponding eigenvalue problem is (18) where u_c in (18a) is replaced by $\tilde{u}^{\epsilon}$. The first two eigenvalues of this problem are exponentially small and the corresponding eigenfunctions ϕ_0 and ϕ_1 are given asymptotically by

$$\phi_j \sim M\left(u_c^{'}[\epsilon^{-1}(x - x_j)] + \phi_{Lj}\right) , \qquad j = 0, 1 , \qquad (85)$$

where the normalization constant M satisfies

$$M \sim (\epsilon L \beta)^{-1/2} , \qquad \beta \equiv \int_{-\infty}^{\infty} \left[u_c^{'}(z)\right]^2 dz . \qquad (86)$$

Here ϕ_{Lj} is a boundary layer function localized near $\partial D_{\pm}$. For similar reasons as in §3, we need only calculate ϕ_{L0} on ∂D_- and ϕ_{L1} on ∂D_+.

By using a boundary layer analysis as in §2.1, we derive in place of (59),

$$\phi_0 \sim M a \nu e^{\nu \epsilon^{-1}(x - x_0)} \left(1 - \nu n_x\right) , \qquad \text{on } \partial D_- , \qquad (87a)$$

$$\phi_1 \sim M a \nu e^{\nu \epsilon^{-1}(x_1 - x)} \left(1 + \nu n_x\right) , \qquad \text{on } \partial D_+ . \qquad (87b)$$

The corresponding exponentially small eigenvalues λ_0 and λ_1 can be estimated as in §2.1.

Since λ_0 and λ_1 are exponentially small we can make the quasi-steady approximation $v_t \ll \partial_t \tilde{u}^{\epsilon}$ in (83a). We then expand the solution to (83) in terms of the eigenfunctions ϕ_j as in (38). In place of (39), the coefficients c_j in the expansion are now

$$c_j = \left(\phi_j, \partial_t \tilde{u}^{\epsilon}\right) - (E, \phi_j) + \epsilon^2 \int_{\partial D} \phi_j \partial_n \tilde{u}^{\epsilon} \, ds , \qquad j = 0, 1, \ldots . \qquad (88)$$

Upon imposing the limiting solvability conditions that $c_0 = 0$ and $c_1 = 0$, we get the coupled equations

$$\left(\phi_0, \partial_t \tilde{u}^{\epsilon}\right) \sim (E, \phi_0) - \epsilon^2 \int_{\partial D} \phi_0 \partial_n \tilde{u}^{\epsilon} \, ds , \qquad (89a)$$

$$\left(\phi_1, \partial_t \tilde{u}^{\epsilon}\right) \sim (E, \phi_1) - \epsilon^2 \int_{\partial D} \phi_1 \partial_n \tilde{u}^{\epsilon} \, ds . \qquad (89b)$$

These equations provide differential equations for $d_0(t)$ and $d_1(t)$. Then $d_2(t)$ is found from $d_2(t) = 1 - d_0(t) - d_1(t)$.

Next, we evaluate the terms in (89). A simple calculation using (80) and (85) yields

$$\left(\partial_t \tilde{u}^\epsilon, \phi_j\right) \sim -L\beta M x_j^{'} \,, \qquad j = 0, 1 \,. \tag{90}$$

In addition, it was shown in [14] that for $\epsilon \ll 1$,

$$(E, \phi_j) \sim 2\epsilon M (-1)^j a^2 \nu^2 e^{-\nu\epsilon^{-1} d_1} \,, \qquad j = 0, 1 \,. \tag{91}$$

By combining (89a) and (89b) and using (90) and (91), we obtain differential equations for $d_0(t)$ and $d_1(t)$,

$$d_1^{'} L M \beta \sim 4a^2\nu^2 \epsilon e^{-\nu\epsilon^{-1} d_1} + \epsilon^2 \int_{\partial D_+} \phi_1 \partial_n \tilde{u}^\epsilon \, ds - \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,, \tag{92a}$$

$$d_0^{'} L M \beta \sim -2a^2\nu^2 \epsilon e^{-\nu\epsilon^{-1} d_1} + \epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \,. \tag{92b}$$

Next, we estimate the boundary integrals in (92). We use (80), (82), and (87) to determine the integrands and then use Laplace's method to asymptotically evaluate the integrals in terms of the local properties at the corners of the channel R given in (2). We obtain,

$$\epsilon^2 \int_{\partial D_-} \phi_0 \partial_n \tilde{u}^\epsilon \, ds \sim -\epsilon a^2 \nu^2 M \left(K_1 \zeta(\alpha_1) + K_2 \zeta(\alpha_2)\right) e^{-2\nu\epsilon^{-1} d_0} \,, \tag{93}$$

$$\epsilon^2 \int_{\partial D_+} \phi_1 \partial_n \tilde{u}^\epsilon \, ds \sim \epsilon a^2 \nu^2 M \left(K_3 \zeta(\alpha_3) + K_4 \zeta(\alpha_4)\right) e^{-2\nu\epsilon^{-1}(1-d_0-d_1)} \,. \tag{94}$$

Here K_i and α_i are defined in (2) and $\zeta(\alpha)$ is defined by

$$\zeta(\alpha) \equiv \left(\frac{\epsilon}{2\nu}\right)^{\alpha+1} \Gamma(\alpha+1) \,. \tag{95}$$

Finally, substituting (93) and (94) into (92) we obtain the metastability result:

Proposition 4: (Metastability for (TAC)) *For $\epsilon \to 0$ consider a two-layer solution for (9) of the form (80). Define the distances $d_i(t)$ as in (50). Then, when x_0 and x_1 are in the channel R, the distances $d_0(t)$ and $d_1(t)$ satisfy the coupled set of asymptotic nonlinear differential equations*

$$d_1^{'} \sim \frac{\epsilon a^2 \nu^2}{L\beta} \left\{ \left[K_3 \zeta(\alpha_3) + K_4 \zeta(\alpha_4)\right] e^{-2\nu\epsilon^{-1}(1-d_0-d_1)} + \left[K_1 \zeta(\alpha_1) + K_2 \zeta(\alpha_2)\right] e^{-2\nu\epsilon^{-1} d_0} + 4 e^{-\nu\epsilon^{-1} d_1} \right\} . \tag{96a}$$

$$d_0^{'} \sim -\frac{\epsilon a^2 \nu^2}{L\beta} \left\{ \left[K_1 \zeta(\alpha_1) + K_2 \zeta(\alpha_2)\right] e^{-2\nu\epsilon^{-1} d_0} + 2 e^{-\nu\epsilon^{-1} d_1} \right\} . \tag{96b}$$

Here ν, a, β, $\zeta(\alpha)$ and α_i, K_i, are defined in (82), (82), (86), (95), and (2), respectively.

Define A and B by

$$A = K_3\zeta_-(\alpha_3) + K_4\zeta_-(\alpha_4)\,, \qquad B = K_1\zeta_-(\alpha_1) + K_2\zeta_-(\alpha_2)\,. \tag{97}$$

Then (96) has a unique steady-state solution d_0^e, d_1^e when $AB > 0$. When $AB > 0$, we can easily find that x_0^e and x_1^e are given for $\epsilon \ll 1$ by

$$x_0^e \sim \frac{1}{4} - \frac{\epsilon}{8\nu}\log\left(\frac{4A}{B^3}\right)\,, \qquad x_1^e \sim \frac{3}{4} + \frac{\epsilon}{8\nu}\log\left(\frac{4B}{A^3}\right)\,. \tag{98}$$

This equilibrium is stable when $A < 0$ and $B < 0$ and is unstable when $A > 0$ and $B > 0$. This leads to the equilibrium result:

Corollary 3: (Stable Equilibria) *Assume that D is locally non-convex near each corner so that in (2) we have $K_i < 0$ for all i. Then, for $\epsilon \ll 1$, (9) admits a stable spatially inhomogeneous equilibrium solution u^e of the form*

$$u^e(\mathbf{x};\epsilon) \sim u_c\left[\epsilon^{-1}(x - x_0^e)\right] + u_c\left[\epsilon^{-1}(x - x_1^e)\right] + s\,. \tag{99}$$

Here u_c satisfies (81) and x_0^e and x_1^ϵ are defined in (98).

As an example, let $Q(u) = -\sin[\pi(u+1)]$. From (81) and (82) we calculate

$$u_c(z) = \frac{4}{\pi}\tan^{-1}\left[e^{\sqrt{\pi}z}\right] - 1\,, \quad a = \frac{4}{\pi}\,, \quad \nu = \sqrt{\pi}\,, \quad \beta = \frac{8}{\pi^{3/2}}\,. \tag{100}$$

Assume that $\alpha_i = \alpha$ and $K_i < 0$ for $i = 1, .., 4$. Then, from (98), the locations of the internal layers for the stable equilibrium solution are

$$x_0^e \sim \frac{1}{4} - \frac{\epsilon}{8\sqrt{\pi}}\log\left(\frac{4(K_3+K_4)}{\zeta^2(K_1+K_2)^3}\right)\,, \qquad x_1^e \sim \frac{3}{4} + \frac{\epsilon}{8\sqrt{\pi}}\log\left(\frac{4(K_1+K_2)}{\zeta^2(K_3+K_4)^3}\right)\,, \tag{101}$$

where

$$\zeta = \left(\frac{\epsilon}{2\sqrt{\pi}}\right)^{\alpha+1}\Gamma(\alpha+1)\,. \tag{102}$$

Thus, we have $x_0^e - 1/4 = O(\epsilon\log\epsilon)$ and $x_1^e - 3/4 = O(\epsilon\log\epsilon)$.

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