The Stability of Stripes and Spots for Some Reaction-Diffusion Models

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Outline of the Talk

1. Terminology: homoclinic vs. mesa stripes; weak vs. semi-strong interactions; breakup and zigzag instabilities of stripes

2. Homoclinic stripes and rings for the GM and GS models (semi-strong):
   - Rigorous analysis of the spectrum of certain NLEPs.
   - Stripe and pulse-splitting (simultaneous) in the GS Model.

3. Homoclinic stripes for the GM and GS models (weak interaction):
   - Zigzag and breakup instabilities of stripes.
   - Labyrinthine patterns and self-replication (edge-type) is possible.

4. A stripe for the GM model with saturation:
   - “Fat” homoclinic stripes under small saturation.
   - Mesa stripes under larger saturation.

5. Spots for the GM and GS Models: Examples
   - Dynamic bifurcations of spots (semi-strong regime)
   - Equilibrium spot locations correspond to optimal trap locations for a certain Laplacian eigenvalue problem (semi-strong regime)
   - Spot-splitting (weak interaction regime)

6. Open problems
Singularity Perturbed RD Models: Localization

Spatially localized solutions, with strong deviations from the uniform state, occur for singularly perturbed RD models of the form:

\[ v_t = \varepsilon^2 \Delta v + g(u, v); \quad \tau u_t = D \Delta u + f(u, v), \quad \partial_n u = \partial_n v = 0 \quad x \in \partial \Omega. \]

Since \( \varepsilon \ll 1 \), then \( v \) is the component that can be localized in space as a stripe or spot. The classical Gierer-Meinhardt (GM) and Gray-Scott (GS) models represent particular choices of \( f, g \).

GM Model:

\[ g(u, v) = -v + v^p/u^q \quad f(u, v) = -u + v^r/u^s. \]


GS Model:

\[ g(u, v) = -v + Auv^2, \quad f(u, v) = (1 - u) - uv^2. \]

Homoclinic Stripe

**Homoclinic Stripe:** occurs when $v$ localizes on a planar curve $C$ (a pulse or spike corresponds to localization on an interval). The stripe cross-section for $v$ is approximated by a homoclinic orbit of the 1-D system in the direction perpendicular to the stripe. A ring is such a stripe on a circle.

- **Semi-strong regime:** For $D = O(1)$ only $v$ is localized and not $u$. In this regime a spike has a strong interaction (mediated by the global variable $u$) with either the boundary or with other spikes.

- **Weak-interaction regime:** Occurs for $D = O(\varepsilon^2)$ with $D = D_0 \varepsilon^2$. Hence, both $u$ and $v$ are localized. In this regime the interaction of a spike with either the boundary or with neighbouring spikes is exponentially weak.
Mesa Stripe (Mesa is table in Spanish)

A Mesa Stripe has a flat plateau with edges determined by heteroclinic orbits of the 1-D system in the perpendicular direction.

Such a stripe occurs in the GM model under saturation effects:

\[ v_t = \varepsilon^2 \Delta v - v + \frac{v^2}{u(1 + \kappa v^2)} , \quad \tau u_t = D \Delta u - u + v^2 . \]

“The parameter \( \kappa \) has a deep impact on the final pattern” (Koch and Meinhardt 1994), and typically leads to stable stripe patterns.

Mesa stripes also occur under bistable nonlinearities such as for a generalized Fitzhugh-Nagumo model (Tanaguchi, Nishiura 1994, 1996)
Breakup Instabilities

Breakup or varicose instability of a homoclinic ring for the GS model at different values of time in the semi-strong regime. This instability leads to the formation of spots. There is a secondary instability of spot-splitting. The eigenfunction is an even function across the ring cross-section.
Zigzag or transverse instability of a homoclinic stripe for the GS model in the weak interaction regime leading to a labyrinthine pattern. The eigenfunction is an odd function across the stripe cross-section. For this example, there is no breakup instability.
Zigzags, Breakup, and Spot-Splitting

Patterns in the weak interaction regime can depend very sensitively on the parameters. In particular, for the GS model where \( \nu \) concentrates on a triangular-shaped ring, very different patterns occur for slight changes in the feed-rate parameter \( A \) and diffusivity \( D \):

For a combination of zigzags, breakup, and spot-splitting [click here]
Homoclinic Stripes: Semi-Strong Regime

We consider a stripe in a rectangle $\Omega$ for the GM model

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^p}{h^q}, \quad \tau h_t = D \Delta h - h + \frac{a^r}{\varepsilon_0 h^s}, \quad x = (x_1, x_2) \in \Omega,$$

where $(p, q, r, s)$ satisfies

$$p > 1, \quad q > 0, \quad r > 1, \quad s \geq 0, \quad \zeta \equiv \frac{qr}{(p - 1)} - (s + 1) > 0.$$

We assume homogeneous Neumann conditions on $\partial \Omega$ and $\Omega : -1 < x_1 < 1, \quad 0 < x_2 < d_0$.

By re-scaling $a$ and $h$, and from $X = x/l$ where $l = 1/\sqrt{D}$, we can set $D = 1$ above and replace $\Omega$ by $\Omega_l$ where

$$\Omega_l : -l < x_1 < l, \quad 0 < x_2 < d; \quad d \equiv d_0 l; \quad \varepsilon \equiv \varepsilon_0 l; \quad l \equiv 1/\sqrt{D}.$$

We examine the existence and stability of a stripe centered on the line $x_1 = 0$ in the semi-strong regime where $D = O(1)$ and $\varepsilon_0 \ll 1$. 
The Equilibrium Homoclinic Stripe

For $\varepsilon \to 0$, the equilibrium stripe solution $a_e(x_1)$ satisfies

$$a_e(x_1) \sim \mathcal{H} \gamma w(\varepsilon^{-1} x_1), \quad \mathcal{H} \equiv \frac{2}{b_r \coth(D^{-1/2})}, \quad b_r \equiv \int_{-\infty}^\infty [w(y)]^r \, dy.$$

Here $\gamma \equiv q/(p - 1)$, and $w(y)$ is the homoclinic solution to

$$w'' - w + w^p = 0, \quad w \to 0 \text{ as } |y| \to \infty; \quad w'(0) = 0, \quad w(0) > 0,$$

The equilibrium $h_e(x_1)$ can be written in terms of a Green’s function.
Homoclinic Stripe: Stability I

We introduce the perturbation

\[ a = a_e + e^{\lambda t + imx_2} \phi, \quad h = h_e + e^{\lambda t + imx_2} \eta, \quad m = \frac{k \pi}{d}. \]

This leads to the eigenvalue problem on \(|x_1| < l|:

\[ \varepsilon^2 \phi_{x_1x_1} - \phi + \frac{pa_e^{p-1}}{h_e^q} \phi - \frac{qa_e^p}{h_e^{q+1}} \eta = \left( \lambda + \varepsilon^2 m^2 \right) \phi, \quad \phi_{x}(\pm l) = 0, \]

\[ \eta_{x_1x_1} - \left( 1 + \tau \lambda + m^2 \right) \eta = -\frac{ra_e^{r-1}}{\varepsilon h_e^s} \phi + \frac{sa_e^r}{\varepsilon h_e^{s+1}} \eta, \quad \eta_{x}(\pm l) = 0. \]

The \(O(1)\) eigenvalues, which govern breakup instabilities, satisfy

\[ \phi(x_1) \sim \Phi \left( \varepsilon^{-1} x_1 \right), \quad \int_{-\infty}^{\infty} w(y) \Phi(y) \, dy \neq 0. \]

The \(O(\varepsilon^2)\) eigenvalues, which govern zigzag instabilities, satisfy

\[ \phi(x_1) \sim w' \left( \varepsilon^{-1} x_1 \right) + \varepsilon \phi_1 \left( \varepsilon^{-1} x_1 \right) + \cdots. \]
Homoclinic Stripe: Breakup Instability I

A singular perturbation analysis leads to the NLEP problem:

\[ L_0 \Phi - \chi_m w^p \frac{\int_{-\infty}^{\infty} w r^{-1} \Phi \, dy}{\int_{-\infty}^{\infty} w r \, dy} = (\lambda + \varepsilon^2 m^2) \Phi, \quad \Phi \to 0 \text{ as } |y| \to \infty \]

\[ \frac{1}{\chi_m(\lambda)} \equiv \frac{s}{qr} + \frac{\theta_\lambda \tanh(\theta_\lambda l)}{qr \tanh l}, \quad \theta_\lambda \equiv \sqrt{1 + m^2 + \tau \lambda}, \]

\[ L_0 \Phi \equiv \Phi'' - \Phi + pw^{p-1} \Phi. \]

- A rigorous analysis (KSWW, 2005, submitted SIADS 2005) proves stability \( \text{Re}(\lambda) < 0 \) for \( m = 0 \) and \( m \gg 1 \) when \( \tau < \tau_H \) for certain \((p, q, r, s)\).

- For any \( \tau > 0 \), there is an unstable breakup band of the form

\[ m_{b-} < m < m_{b+} = O(\varepsilon^{-1}), \quad m_{b+} \sim \frac{1}{2\varepsilon} \sqrt{(p - 1)(p + 3)} + O(\varepsilon). \]

- Key result: Since \( m_{b+} = O(\varepsilon^{-1}) \), breakup instabilities can only be avoided in asymptotically thin domains where \( d_0 = O(\varepsilon_0) \).

- The analysis extends that of Doelman, Van der Ploeg (SIADS 2002).
From numerical computations of the spectrum of the NLEP we get:

Left figure: Most unstable $\lambda$ versus $m$ for $D = 1$, $\tau = 0$, $(p, q, r, s) = (2, 1, 2, 0)$ but different $\varepsilon$:

Right figure: Most unstable $\lambda$ versus $m$ for $D = 1$, $\varepsilon = 0.025$, $\tau = 0$, but different exponent sets $(p, q, r, s)$:
Homoclinic Stripe: Zigzags

For \( \varepsilon \to 0 \) and \( \tau = O(1) \), a singular perturbation analysis related to the SLEP method (Nishiura, Fujii 1987) shows that the eigenvalue governing a zigzag instability of a stripe of mode \( m \) is

\[
\lambda \sim \varepsilon^2 \left[ 2\gamma \theta \tanh \theta l \tanh(\theta l) - 2\gamma - m^2 \right], \quad \theta \equiv \sqrt{1 + m^2}, \quad \gamma \equiv \frac{q}{p - 1}.
\]

The implications from this formula are:

- There are no unstable zigzag modes when \( \gamma \leq 1 \), which includes the classical GM exponents \((p, q, r, s) = (2, 1, 2, 0)\).

- For \( \gamma > 1 \), there is a band \( m_{z-} < m < m_{z+} \) of unstable zigzag modes only when the domain half-length \( l \equiv 1/\sqrt{D} \) exceeds some critical value \( l_z \), or equivalently when \( D \) is small enough.

- Since \( \lambda = O(\varepsilon^2) \), zigzag instabilities occur for \( t = O(\varepsilon^{-2}) \).

- Hence, in this semi-strong regime a stripe is de-stabilized not by the slow zigzag instabilities but by the faster breakup instabilities.

- Taking the limit \( D \to 0 \), corresponding to the weak interaction regime, we expect that the zigzag and breakup instabilities occur there on the same time-scale.
Homoclinic Stripe: Breakup Instability III

Experiment 1 and 2: Take \((p, q, r, s) = (2, 1, 2, 0), \varepsilon_0 = 0.025, \tau = 0.1\) and \(\Omega = [-1, 1] \times [0, 2]\). The predicted number \(N\) of spots is the number of maxima of the eigenfunction \(\cos(my)\) on \(0 < y < d_0 = 2\), which is \(N = m d_0/(2\pi) = m/\pi\) where \(m\) is most unstable mode.

For \(D = 1.0\), we predict \(N = 8\). Numerically, the initial stripe breaks up into seven spots on an \(O(1)\) time-scale. There is then a competition instability. [The movie.]

For \(D = 0.1\) we predict \(N = 4\). Numerically, the initial stripe breaks up into five spots on an \(O(1)\) time-scale with no competition instability. [The movie.]
Homoclinic Stripes, Rings: Semi-Strong (GS)

Let $l = 1/\sqrt{D}$. Then, in a rectangle or circle, the GS Model is

$$
v_t = \varepsilon^2 \Delta v - v + Auv^2, \quad \tau u_t = \Delta u - u + 1 - uv^2, \quad x \in \Omega.
\Omega : -l < x_1 < l, \quad 0 < x_2 < d \equiv ld_0; \quad \Omega : 0 < x_1^2 + x_2^2 < R^2 \equiv l^2.
$$

The semi-strong regime $l = O(1)$ with $\varepsilon \ll 1$ is sub-divided as follows:

1. **High Feed-Rate (SS) Regime**: $A = O(1)$ as $\varepsilon \to 0$.
   - A homoclinic stripe or ring can undergo a splitting process.
   - In a rectangle, both zigzag and breakup instabilities occur unless the domain with is $O(\varepsilon) \ll 1$ thin.
   - Impossible to have only a zigzag instability with no breakup instability.

2. **Intermediate (SS) regime**: $O(\varepsilon^{1/2}) \ll A \ll 1$. No splitting phenomena and breakup instabilities occur on much faster timescales than zigzags. Breakup instabilities analyzed from a Nonlocal Eigenvalue Problem.

In contrast, the **Weak Interaction Regime**: corresponds to $l = \infty$ and $\varepsilon \to 1$ and has been studied in Nishiura, Ueyema (1999,2001) and Ueyama (2000) in terms of global bifurcation properties.
Homoclinic: High Feed Regime (GS)

An equilibrium stripe (of zero curvature) and ring can be constructed in terms of the solutions $V > 0$, $U > 0$ with $y = \varepsilon^{-1}x_1$ to

$$V'' - V + V^2U = 0, \quad U'' = UV^2, \quad 0 < y < \infty,$$

$$V'(0) = U'(0) = 0; \quad V \to 0, \quad U \sim By, \quad \text{as} \quad y \to \infty.$$

This core problem was identified for pulses in Muratov et al. (2000) (and Doelman et al. (1998)). Inner-outer matching for $u$ determines $B$ as

$$A = B \coth \left(D^{-1/2}\right), \quad \text{(rectangle)}; \quad A = 2BG(\rho_0; \rho_0), \quad \text{(ring)}.$$

Here $G(\rho; \rho_0)$ on $0 < \rho < R$ satisfies

$$G_{\rho\rho} + \frac{1}{\rho}G_{\rho} - G = -\delta(\rho - \rho_0); \quad G_{\rho}(R; \rho_0) = G_{\rho}(0; \rho_0) = 0.$$

For $R = D^{-1/2}$, the equilibrium ring radius $\rho_0$ is the root of

$$\frac{K_0(\rho_0)}{I_0(\rho_0)} - \frac{K'_0(R)}{I'_0(R)} = \frac{1}{2\rho_0I_0(\rho_0)I'_0(\rho_0)}.$$

When $D \ll 1$ where $R \gg 1$, then $\rho_0 \sim R - \frac{1}{2} \log(2R)$.
Homoclinic: Pulse: High Feed Regime (GS)

The bifurcation plot of $\gamma = U(0)V(0)$ vs. $B$ has a saddle-node behavior:

- saddle node at $B = 1.347$ (Muratov, Osipov 2000). Upper branch is stable for $\tau$ small enough while the lower branch is always unstable.
- The global bifurcation conditions of Ei, Nishiura (JJIAM 2001) can be verified analytically (KWW, Physica D 2005)
- Since $B = \alpha A$, for some $\alpha$, there is no one-pulse solution when $A > A_{\text{split}}$. This leads to pulse-splitting. The pulse-splitting is simultaneous and, for symmetric one-spike initial data, the final number of pulses is determined by the smallest integer $k$ for which

$$A_{p2^k-1} < A < A_{p2^k}, \quad A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right).$$
Homoclinics: Stripe and Ring-Splitting (GS)

Stripe and Ring replication is unstable to breakup instabilities. For stripes we need that the rectangle is $O(\varepsilon)$ thin to prevent these instabilities.

Next, for $A < A_{\text{split}}$, we have the zigzag and breakup instability of a ring.
**Homoclinics: Instabilities of a Ring (GS)**

**Principal Result:** In the intermediate regime $O(\varepsilon^{1/2}) \ll A \ll 1$ all zigzag perturbations of the equilibrium ring solution are unstable in the band

$$m_{zr} < m < A^2 / \left[ 6\varepsilon \rho_0 J^2_{1,0}(\rho_0) J^2_{2,0}(\rho_0) \right],$$

where $m_{zr}$, which depends on $R \equiv 1/\sqrt{D}$ and $\rho_0$, is the root of

$$4\rho_0^2 J_{1,0}(\rho_0) J_{2,0}(\rho_0) J_{1,m}(\rho_0) J_{2,m}(\rho_0) = 1 .$$

The asymptotics of $m_{zr}$ are

$$m_{zr} = 3 , \quad \text{as} \quad R \to 0 ; \quad m_{zr} \sim \sqrt{2R - \ln(2R) + \cdots} , \quad \text{as} \quad R \to \infty .$$

Here $J_{1,m}$, $J_{2,m}$ are related to standard Bessel functions by

$$J_{2,m}(\rho) \equiv K_m(\rho) - K'_m(R)/I'_m(R) I_m(\rho) , \quad J_{1,m}(\rho) \equiv I_m(\rho) ,$$

In the intermediate regime, the breakup instability band is

$$A^2 / \left[ 12\rho_0 J^2_{1,0}(\rho_0) J^2_{2,0}(\rho_0) \right] < \varepsilon m < \sqrt{5} \rho_0 / 2 .$$
Homoclinic Stripes: Weak Interaction Regime

The GM model in a rectangle in the weak interaction regime is

\[ a_t = \varepsilon_0^2 \Delta a - a + \frac{a^p}{h^q}, \quad \tau h_t = \varepsilon_0^2 D_0 \Delta h - h + \frac{a^r}{h^s}, \quad x = (x_1, x_2) \in \Omega. \]

Here \( D = D_0 \varepsilon^2 \) and \( \Omega = [-1, 1] \times [0, d_0] \).

We examine the existence and stability of a stripe centered along the midline \( x_1 = 0 \) of \( \Omega \). Since \( u \) and \( v \) decay exponentially away from \( x_1 = 0 \) for \( \varepsilon_0 \ll 1 \), the sidewalls at \( x_1 = \pm 1 \) can be neglected.

Questions for a Pulse and a Stripe:

- What is the bifurcation behavior of equilibrium homoclinic pulses in terms of \( D_0 \)?
- What are the stability properties for pulses on an interval?
- For a stripe (composed of a pulse trivially extended in the second direction) what are the zigzag and breakup instability bands as a function of \( D_0 \). Can the breakup band disappear?
Homoclinic Pulse: Equilibrium Solution I

For $\varepsilon_0 \to 0$, with $D = \varepsilon_0^2 D_0$, the GM model has two-different one-pulse equilibrium solutions when $D_0 > D_{c0}$ and none when $0 < D_0 < D_{c0}$. When $D_0 > D_{c0}$, the solutions are of the form

$$a_{e\pm}(x_1) \sim u \left[ \varepsilon_0^{-1} x_1 \right], \quad h_{e\pm}(x) \sim v \left[ \varepsilon_0^{-1} x_1 \right].$$

The functions $u(y)$ and $v(y)$ are homoclinic solutions on $0 < y < \infty$ to

$$u'' - u + u^p/v^q = 0, \quad D_0 v'' - v + u^r/v^s = 0,$$

with

$$u(0) = \alpha, \quad u'(0) = v'(0) = 0, \quad u(\infty) = v(\infty).$$

- This problem is easily solved using AUTO for various $(p, q, r, s)$.
- Existence results by geometric theory of singular perturbations (Doelman and Van der Ploeg, SIADS 2002)
- Numerical computations also by Nishiura (AMS Monograph Translations 2002)
Homoclinic Pulse: Equilibrium Solution II

For each exponent set \((p, q, r, s)\) tested:

- There is a saddle-node point at some \(D_0 = D_{c0} > 0\). For \((2, 1, 2, 0)\), \(D_{c0} \approx 7.17\).

- On the dashed portion of the lower branch the homoclinic profile for \(u\) has a double-bump structure so that the maximum of \(u\) occurs on either side of \(y = 0\).

Left figure: \(a(0)\) versus \(D_0\) for \((2, 1, 2, 0)\). The dotted part is where \(a_e\) has a double-bump structure. Right figure: \(a_{e+}\) (solid curve) for \(D_0 = 9.83\) and \(a_{e-}\) (heavy solid curve) for \(D_0 = 10.22\).
Homoclinic Pulse: Stability Properties

- Numerically we find that the upper branch is stable for $\tau < \tau_H$. For $\tau > \tau_H$, where $\tau_H$ depends on $D_0$, the pulse on the upper branch undergoes a Hopf bifurcation. The lower branch is unstable for any $\tau \geq 0$.

- The translation mode $\Phi = u', N = v'$, is always a neutral mode.

- At the fold point, where $D_0 = D_{c0}$, there is also an even dimple-shaped eigenfunction $\Phi = \Phi_d(y)$, associated with a zero eigenvalue, satisfying

\[
\begin{align*}
\Phi_d(0) &< 0; \\
\Phi_d(y) &> 0, \quad y > y_0; \\
\Phi_d(y) &< 0, \quad 0 < y < y_0.
\end{align*}
\]

- Hence, the key global bifurcation conditions of Ei and Nishiura (JJIAM 2001) for edge-splitting pulse-replication behavior are satisfied.
Homoclinic Stripe: Stability Properties I

We extend the pulse trivially in the second direction to obtain a straight stripe. To study the stability of this stripe we let $y = \varepsilon_0^{-1} x$ and introduce

$$a = u(y) + \Phi(y) \, e^{\lambda t} \cos(mx_2), \quad h = v(y) + N(y) \, e^{\lambda t} \cos(mx_2).$$

Defining $\mu \equiv \varepsilon_0^2 m^2$, we get an eigenvalue problem on $0 < y < \infty$:

$$\Phi_{yy} - (1 + \mu) \Phi + \frac{pu^{p-1}}{v^q} \Phi - \frac{qu^p}{v^{q+1}} N = \lambda \Phi,$$

$$D_0 N_{yy} - (1 + D_0 \mu) N + \frac{ru^{r-1}}{v^s} \Phi - \frac{su^r}{v^{s+1}} N = \tau \lambda N.$$

- For **breakup instabilities** we compute the spectrum for even eigenfunctions $\Phi$ and $N$ so that $\Phi_y(0) = N_y(0) = 0$.
- For **zigzag instabilities** we compute the spectrum for odd eigenfunctions $\Phi$ and $N$ so that $\Phi(0) = N(0) = 0$.
- The computations are done by first computing $u$ and $v$. Then, we discretize the eigenvalue problem by finite differences and use a generalized eigenvalue problem solver from Lapack. Instability thresholds are computed using a quasi-Newton method.
Homoclinic Stripe: Stability II (Breakup)

For breakup instabilities we find numerically that:

- For $\tau = 0$ there is a unique real positive eigenvalue in the breakup instability band $\sqrt{\mu_1} < \varepsilon_0 m < \sqrt{\mu_2}$ (provided it exists), and that $\text{Re}(\lambda) < 0$ outside this band. An unstable eigenvalue is $O(1)$.

- As $D_0$ is decreased towards the existence threshold $D_{0c}$, the instability band disappears for certain exponent sets at some critical value $D_{0b} > D_{0c}$ on the upper branch. Thus, for some ($p, q, r, s$) a straight stripe will not break into spots for $D_0$ on the range $D_{0c} < D_0 < D_{0b}$.

Figure 1: Left: Terminating Bands. Right: Non-terminating bands.
**Homoclinic Stripe: Stability III (Zigzag)**

For zigzag instabilities we find numerically that:

- For $\tau = 0$ and $\gamma = q/(p - 1) < 1$ and as $D_0$ is decreased towards the existence threshold $D_{0c}$, an unstable zigzag band first emerges at some critical value $D_{0z}$ with $D_{0z} > D_{0c}$.

- For $\tau = 0$ and $\gamma > 1$, the unstable zigzag band continues into the semi-strong regime corresponding to the limit $D_0 \gg 1$.

- Within the unstable zigzag band there is a unique positive eigenvalue when $\tau = 0$. The time-scale for zigzag instabilities is $O(1)$.

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**Figure 2:** Zigzag bands. Left: Exponent sets where $\gamma < 1$. Right: Exponent sets where $\gamma > 1$.
Homoclinic Stripes: Stability III (Thresholds)

<table>
<thead>
<tr>
<th>$(p, q, m, s)$</th>
<th>$D_{0c}$</th>
<th>$\alpha_c$</th>
<th>$D_{0b}$</th>
<th>$D_{0z}$</th>
<th>$D_{0m}$</th>
<th>$D_{0s}$</th>
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</thead>
<tbody>
<tr>
<td>$(2, 1, 2, 0)$</td>
<td>7.17</td>
<td>1.58</td>
<td>8.06</td>
<td>24.0</td>
<td>8.92</td>
<td>9.82</td>
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<td>$(2, 1, 3, 0)$</td>
<td>10.35</td>
<td>1.42</td>
<td>19.14</td>
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<td>12.36</td>
<td>16.31</td>
</tr>
<tr>
<td>$(3, 2, 2, 0)$</td>
<td>3.91</td>
<td>1.62</td>
<td>——</td>
<td>32.3</td>
<td>5.08</td>
<td>5.23</td>
</tr>
<tr>
<td>$(3, 2, 3, 1)$</td>
<td>4.41</td>
<td>1.53</td>
<td>5.13</td>
<td>28.0</td>
<td>5.36</td>
<td>5.97</td>
</tr>
<tr>
<td>$(2, 2, 3, 3)$</td>
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<td>2.28</td>
<td>——</td>
<td>——</td>
<td>41.80</td>
<td>85.52</td>
</tr>
<tr>
<td>$(4, 2, 2, 0)$</td>
<td>0.89</td>
<td>1.36</td>
<td>1.00</td>
<td>27.9</td>
<td>1.06</td>
<td>0.89</td>
</tr>
</tbody>
</table>

Second and Third columns: the saddle-node bifurcation values $D_{0c}$ and $\alpha_c \equiv a(0)$ for the existence of a stripe. Fourth column: values $D_{0b}$ of $D_0$ for the lower bound of the breakup instability band. A stripe is stable to breakup when $D_{0c} < D_0 < D_{0b}$. Fifth column: values $D_{0z}$ of $D_0$ for the upper bound of the zigzag instability band in the weak-interaction regime. Sixth Column: the smallest values $D_{0m}$ of $D_0$ where $a(y)$ has a double-bump structure on the lower branch of the $a(0)$ versus $D_0$ diagram. Seventh column: the saddle-node values $D_{0s}$ representing the smallest value of $D_0$ where a radially symmetric spot solution exists in $\mathbb{R}^2$. 

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Homoclinic Stripe: Stability Properties IV

- For sets \((p, q, m, s)\) where the breakup instability disappears at some \(D_{0b} > D_{0c}\), the stripe will always be unstable with respect to zigzag instabilities for domain widths \(d_0\) that are \(O(1)\) as \(\varepsilon_0 \to 0\). Zigzags can be suppressed only by taking the domain width \(d_0\) to be \(O(\varepsilon_0)\) thin.

- Whenever a breakup band exists, the breakup and zigzag bands overlap in such a way that there are no domain widths \(d_0\) where a zigzag instability is not accompanied by a breakup instability.

- The time-scale for breakup instabilities is generally faster than for zigzag instabilities. However, both time-scales are independent of \(\varepsilon_0\).

Disappearance of the Breakup Band: It appears to be analytically intractable to rigorously study the eigenvalue problem to analytically confirm the coalescence of the breakup instability band and the emergence of the zigzag band near \(D_{0c}\). However, qualitatively, near \(D_{0c}\), the region near the maximum of \(a\) becomes wider than for larger values of \(D\). These fattened homoclinic stripes are more reminiscent of mesa-stripes in bistable systems, where breakup instabilities typically do not occur (Taniguchi, Nishiura 1996) and where zigzags are the dominant instabilities.
Homoclinic Stripes: Experiment 1 (GM)

**Experiment 1:** Take \((p, q, r, s) = (2, 1, 2, 0), \varepsilon_0 = 0.025, \text{ and } D_0 = 7.6.\) Since \(D_{0c} < D_0 < D_{0b},\) there is no breakup instability band and we predict that the initially straight stripe will not break up into spots. Theory predicts the number \(N\) of zigzag crests as \(N = \frac{m}{\pi} = \frac{0.315}{\pi \varepsilon_0} \approx 4.\) The wriggled stripe, however, undergoes a breakup instability near the points of its maximum curvature leading to spot formation. Since \(D_0 = 7.6 < D_{0s} = 9.82\) these spots then undergo a self-replication process which fill the entire domain.

**Click here for the movie**

![Movie Frames](https://example.com/movie_frames.png)

**Left:** \(t = 200\) **Left Middle:** \(t = 300\) **Right Middle:** \(t = 400\) **Right:** \(t = 700\)
Homoclinic Stripes: Experiment 2 (GM)

Experiment 2: Take \((p, q, r, s) = (2, 1, 2, 0), \varepsilon_0 = 0.025,\) and increase \(D_0\) to \(D_0 = 15.0.\) Since \(D_0 > D_{0b}\) we predict that the initially straight stripe will break up into spots. From theory, the predicted number \(N\) of spots is \(N = \frac{m}{\pi} = \frac{0.623}{\pi \varepsilon_0} \approx 8.\) The zigzag instability breaks the vertical symmetry. Since \(D_0 = 15\) is well above the spot-existence threshold \(D_{0s} = 9.82,\) there is no self-replication behavior of spots. Instead there is an exponentially slow, or metastable, drift of the spots towards a stable equilibrium configuration that has a hexagonal structure.

Left: \(t = 500\) Left Middle: \(t = 800\) Right Middle: \(t = 3000\) Right: \(t = 30000.\)
Homoclinic Stripes: Experiment 3 (GM)

Experiment 3: Take \((p, q, r, s) = (2, 1, 2, 0), \varepsilon_0 = 0.025, \) and \(D_0 = 6.8.\) Since \(D_0 < D_{0c}\), there is no equilibrium stripe solution. The stripe then splits into two and develops a zigzag instability. The wriggled stripes undergo a breakup instability near the points of maximum curvature of the stripe. Since \(D_0 < D_{0s}\) the emerging spots then undergo a spot-splitting process.

Click here for the movie

Left: \(t = 75\) Left Middle: \(t = 175\) Right Middle: \(t = 225\) Right: \(t = 250.\)

Click here for another movie for a smaller \(D.\)
Homoclinic Stripes: Experiment 4 (GM)

**Experiment 4**: Take \((p, q, r, s) = (3, 2, 2, 0), \varepsilon_0 = 0.025, \text{ and } D_0 = 4.5\). In addition to the zigzag instability, a breakup instability is guaranteed since the breakup band does not terminate in the weak interaction regime. From theory the most unstable **breakup mode** predicts \(N = \frac{m}{\pi} = \frac{0.96}{\pi \varepsilon_0} \approx 12\) spots. The most unstable **zigzag mode** predicts \(N = \frac{m}{\pi} = \frac{0.43}{\pi \varepsilon_0} \approx 5\) crests. The numerical results show that the initially straight stripe breaks up into fourteen spots and then develops a zigzag instability with six crests. Since \(D_0 < D_{0s} = 5.23\) the resulting spots then undergo a spot-splitting process.

Left: \(t = 100\) Left Middle: \(t = 200\) Right Middle: \(t = 250\) Right: \(t = 1000\).
Homoclinic Stripes: Experiment 5 (GM)

Experiment 5: Take \((p, q, r, s) = (2, 1, 3, 0), \epsilon_0 = 0.01, \text{ and } D_0 = 14.0\). For this set there are no breakup instabilities when \(10.35 < D_0 < 19.14\). Therefore, we expect no breakup instability. From theory the expected number of zigzag crests is \(N = \frac{m}{\pi} = \frac{0.23}{\pi \epsilon_0} \approx 7\). The numerical results show a zigzag instability with seven crests. The zigzag instability leads to a curve of increasing length.

Click here for the movie

Left: \(t = 1400\) Left Middle: \(t = 1600\) Right Middle: \(t = 1800\) Right: \(t = 4300\).
For a stripe $C$ of non-constant curvature $\kappa$ in the weak interaction regime, the stripe profile and the stability analysis for the stripe profile are, to leading order in $\varepsilon_0$, the same as that of a straight stripe. The key assumption that we need is $\kappa \ll O(\varepsilon_0^{-1})$.

For a curved stripe, the local $y$ is replaced by the local distance $\eta$ perpendicular to the stripe cross-section.

For the stability of this curved stripe we let $a = a_e (\eta) + e^{\lambda t + im\sigma} \Phi (\eta)$, where $\sigma$ is arclength along $C$. If $C$ is closed, and $L$ is the length of $C$, we have $m = 2\pi k/L$. Thus, the most unstable zigzag or breakup mode is

$$k = \frac{(\varepsilon_0 m)L}{2\pi \varepsilon_0}.$$  

Here $\sqrt{\mu} = \varepsilon_0 m$ is computed from the eigenvalue problem for a straight stripe.

Conjecture: A zigzag instability in the absence of a breakup instability can lead to some type of space-filling labyrinthine pattern.
Homoclinic Stripes: Non-Zero Curvature II

Experiment 6: Consider the GM model with exponent set \((2, 1, 3, 0)\), \(\tau = 0.1\) and \(\varepsilon_0 = 0.02\) in \(\Omega = [-1, 1] \times [0, 2]\). Suppose that the initial data concentrates on the ellipse \(C: x_1^2 + 2(x_2 - 1)^2 = 1/4\) in \(\Omega\). Then, since \(L = 2.71\) for \(C\), we use the eigenvalue computation for a straight stripe to predict the following:

For \(D_0 = 14\) we predict only zigzags with most unstable mode \(\varepsilon_0 m \approx 0.227\) and \(\lambda = 0.0131\). Thus, we predict \(k = 5\) zigzag crests. Plot at \(t = 625\). [The movie]

For \(D_0 = 20\) we predict a breakup with with most unstable mode \(\varepsilon_0 m \approx 0.682\) and \(\lambda = 0.0145\). Thus, we predict \(k = 15\) spots. Plot at \(t = 187\). [The movie]
Homoclinic Stripes: Weak Interaction I (GS)

Qualitatively similar results can be obtained for the GS model in the weak interaction regime in that a “fattened” stripe can be stable wrt respect to breakup instabilities but unstable wrt zigzags. The homoclinic problem is to seek even solutions with \( v \to 0 \) and \( u \to 1 \) as \( |y| \to \pm \infty \) to

\[
v_{yy} - v + Auv^2 = 0, \quad D_0 u_{yy} - (u - 1) - uv^2 = 0.
\]

The existence problem involves \( A \) and \( D_0 \) and is very complicated (AUTO: Rademacher, MJW). Instead we take a path \( A = A(\beta) \), \( D_0 = D_0(\beta) \) with \( \beta \) a parameter, which passes through \( (A, D_0) = (2.02, 4.667) \), where Pearson (1993) observed wriggled stripes.

The eigenvalue problem for a straight or a curved stripe is

\[
\Phi_{yy} - (1 + \mu)\Phi + 2Auv\Phi + Av^2 N = \lambda \Phi, \\
D_0 N_{yy} - (1 + D_0 \mu) N - 2uv\Phi - v^2 N = \lambda N\tau.
\]

Here \( \mu \equiv \varepsilon_0^2 m^2 \). An unstable integer mode \( k \) is related to the length \( L \) of a closed concentration curve \( C \) and an unstable \( m \) by

\[
k = \frac{(\varepsilon_0 m) L}{2\pi \varepsilon_0}.
\]
By solving the homoclinic and eigenvalue problems numerically we get:

Left figure: $u$ and $v$ for $(D_0, A) = (19.91, 1.57)$ (dashed curve), $(D_0, A) = (5.94, 1.84)$, (solid curve), and $(D_0, A) = (4.0, 2.0)$ (heavy solid curve). The pulse “fattens” as $D_0$ decreases and $A$ tends to 2 due to a nearby heteroclinic connection.

Right figure: the breakup instability thresholds (heavy solid curves) and the upper zigzag instability threshold (dashed curve). The top solid curve is the path for $A = A(D_0)$. The breakup band disappears when $D_0 = 5.46$ and $A = 1.87$. Thus, for $D_0 < 5.46$ on this path $A = A(D_0)$, there is a zigzag instability in the absence of a breakup instability.
**Homoclinic Stripes: Weak Interaction III (GS)**

**Experiment 1:** Let $\epsilon_0 = 0.02$, $D = 6.08\epsilon_0^2$ (i.e. $D_0 = 6.08$), $A = 1.83$, $\tau = 1$ in $\Omega = [-1, 1] \times [0, 2]$. The initial condition concentrates on a triangular curve with $L \approx 3.14$. The unstable breakup band for $D_0 = 6.08$ is $0.316 < \epsilon_0 m < 0.703$, with the most unstable mode roughly in the center. With $m = 2\pi k/L$ we get $2\pi\epsilon_0 k/L \approx 0.51$. Since $\epsilon_0 = 0.02$ and $L \approx 3.14$, we predict $k \approx 12$ spots (as observed).
Homoclinic Stripes: Weak Interaction IV (GS)

Experiment 2: Take $\varepsilon_0 = 0.02$, $A = 2$, $\tau = 1$, and $D = 4\varepsilon_0^2$, with the same initial data. For these $A$ and $D$ there is only a zigzag instability band and no breakup band. The most unstable zigzag mode is roughly $\varepsilon_0 m \approx 0.2$. With $m = 2\pi k/L$, and $L \approx 3.14$, we predict $k \approx 5$ crests. The full numerical results give a $k = 6$ zigzag instability, and the subsequent temporal development of a large-scale labyrinthian pattern.
Mesa Stripes: GM with Small Saturation I

Consider the GM model with saturation $k > 0$ on $\Omega = [-1, 1] \times [0, d_0]$:

$$a_t = \varepsilon^2 \Delta a - a + \frac{a^2}{h(1 + ka^2)}, \quad \tau h_t = D\Delta h - h + \frac{a^2}{\varepsilon}.$$ 

For $\varepsilon_0 \ll 1$, there is a homoclinic stripe solution

$$a \sim \mathcal{H} w \left[ \varepsilon^{-1} x_1 \right], \quad h \sim \mathcal{H} \frac{\cosh[\theta_0(1 - |x_1|)]}{\cosh \theta_0}, \quad \theta_0 \equiv \frac{1}{\sqrt{D}},$$

$$\mathcal{H} \equiv \frac{2 \tanh \theta_0}{\beta \theta_0}, \quad \beta \equiv \int_{-\infty}^{\infty} w^2 \, dy, \quad b\beta^2 = 4kD \tanh^2(\theta_0).$$

For $0 < b < b_0 \approx 0.21138$, $w$ is the fat homoclinic solution to

$$w'' - w + w^2/(1 + bw^2) = 0, \quad w(\pm \infty) = 0.$$
Mesa Stripes: GM with Small Saturation II

A singular perturbation analysis leads to the NLEP governing breakup instabilities:

\[
L_0 \Phi - \frac{\chi_m w^2}{1 + bw^2} \int_{-\infty}^{\infty} w \Phi \, dy \int_{-\infty}^{\infty} w^2 \, dy = (\lambda + \varepsilon^2 m^2) \Phi, \quad \Phi \to 0, \text{ as } |y| \to \infty
\]

\[
\chi_m(\lambda) \equiv \frac{2\theta_0 \tanh \theta_0}{\theta_\lambda \tanh \theta_\lambda}, \quad \theta_\lambda \equiv \sqrt{m^2 + \frac{(1 + \tau \lambda)}{D}}, \quad \theta_0 \equiv \frac{1}{\sqrt{D}},
\]

\[
L_0 \Phi \equiv \Phi^{\prime\prime} - \Phi + \frac{2w}{(1 + bw^2)^2} \Phi.
\]

Since the pulse is stable when \( m = 0 \), and that \( \chi_m \to 0 \) as \( m \to \infty \), the breakup instability band has the form

\[
m_{b-} < m < \sqrt{\nu_0} / \varepsilon.
\]

Here \( \nu_0 = \nu_0(b) \) is the principal eigenvalue of the local operator \( L_0 \). Since \( \nu_0 \to 0 \) as \( b \to b_0 = 0.211 \) corresponding to a fat homoclinic, it suggests that the breakup band disappears for some \( b_0 \) slightly below \( b_0 \).
Mesa Stripes: GM with Small Saturation III

From numerical computations of the spectrum of the NLEP we get:

Left: Most unstable $\lambda$ versus $m$ for $D = 10.0$, $\varepsilon = 0.025$ with $k = 0$, $k = 3.6$, $k = 6.8$, and $k = 12.5$. The band disappears near $k = 12.5$. Right: principal eigenvalue $\nu_0$ of $L_0$ versus $b$.

Left: $m_{b\pm}$ versus $k$ for $D = 10$ and different $\varepsilon$. Right: $m_{b\pm}$ versus $k$ for $\varepsilon = 0.025$ and different $D$. 
Mesa Stripes: GM with Large Saturation

On a rectangular domain $0 < x_1 < 1, 0 < x_2 < d_0$, we consider

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^2}{h(1 + \kappa a^2)}, \quad \tau h_t = D \Delta h - h + a^2.$$ 

If we set $\kappa = \varepsilon_0^2 k$ and enlarge $a$ and $h$ by a factor of $1/\varepsilon_0$ we recover the small saturation formulation.

The constant $\kappa > 0$ is the saturation parameter. For $\kappa > 0$, with $\kappa = O(1)$, the numerical simulations of Koch, Meinhardt (1994), Meinhardt (1995) suggest that a stripe will not disintegrate into spots. This model seems to robustly support stable stripe patterns. Theoretical Analysis?

For $\kappa = O(1) > 0$ we will construct a mesa stripe centered about the mid-line $x_1 = 0$ when the inhibitor diffusivity is asymptotically large with $D = O(\varepsilon_0^{-1})$. We write

$$D = \frac{D}{\varepsilon_0}.$$
The Equilibrium Mesa Stripe

For $\epsilon_0 \ll 1$, the mesa stripe is given by

$$a \sim \mathcal{H} [w_l + w_r - w_+] , \quad h \sim \mathcal{H} = \frac{1}{w_+ L} ,$$

$$w_l (y_l) \equiv w \left[ \epsilon_0^{-1} (x_1 - \xi_l) \right] , \quad w_r (y_r) \equiv w \left[ \epsilon_0^{-1} (\xi_r - x_1) \right] ,$$

$$L \equiv \xi_r - \xi_l \sim \frac{\sqrt{\kappa}}{\sqrt{b_0 w_+^2}} < 1 .$$

Here $L$ is the length of the mesa plateau. With $b \equiv \kappa \mathcal{H}^2$, there is a heteroclinic solution $w(y)$ for $b_0 \approx 0.21138$ to

$$w'' - w + \frac{w^2}{1 + b_0 w^2} = 0 , \quad w(\infty) = w_+ \approx 3.295 , \quad w(-\infty) = 0 .$$
Let $a_e$, $h_e$ be the equilibrium mesa stripe. We linearize as

$$a = a_e + e^{\lambda t + imx_2} \phi, \quad h = h_e + e^{\lambda t + imx_2} \psi, \quad m = \frac{k\pi}{d_0}, \quad k = 1, 2, \ldots,$$

where $\phi = \phi(x_1) \ll 1$ and $\psi = \psi(x_1) \ll 1$. We will assume that $\tau = O(1)$. For $\varepsilon_0 \ll 1$ the eigenfunction for $\phi$ has the form

$$\phi \sim \left\{ \begin{array}{ll}
  c_l \left( w'(y_l) + O(\varepsilon_0) + \cdots \right), & y_l \equiv \varepsilon^{-1}_0(x_1 - \xi_l) = 0(1), \\
  \phi_i \equiv \mu \psi, & \xi_l < x_1 < \xi_r, \\
  c_r \left( w'(y_r) + O(\varepsilon_0) + \cdots \right), & y_r \equiv \varepsilon^{-1}_0(\xi_r - x_1) = 0(1),
\end{array} \right.$$

with $\mu \equiv \frac{w_+^2}{2-w_+}$. The eigenfunction for $\psi$ on $0 < x_1 < 1$ has the form

$$\psi_{x_1x_1} - \theta^2 \psi = -\frac{\varepsilon_0^2 \mathcal{H} w_+^2}{D} \left[ c_r \delta(x_1 - \xi_r) + c_l \delta(x_1 - \xi_l) \right],$$

$$\psi_{x_1}(0) = \psi_{x_1}(1) = 0.$$
The Stability of the Mesa Stripe II

From a singular perturbation analysis related to the SLEP method of Fujii and Nishiura, we find for $\varepsilon_0 \ll 1$ that there are no $O(1)$ positive eigenvalues and that the two $O(\varepsilon_0^2) \ll 1$ eigenvalues satisfy the matrix eigenvalue problem

$$\alpha(\lambda + \varepsilon_0^2 m^2) c \sim \varepsilon_0^2 \left[ \frac{1}{2} L(1 - L) I - \mathcal{G} \right] c, \quad \alpha \equiv \frac{2\beta LD}{w_+^2}.$$

Here $I$ is the identity matrix. Also $c$ and $\mathcal{G}$ are

$$\mathcal{G} \equiv \frac{1}{d^2 - e^2} \begin{pmatrix} d & e \\ e & d \end{pmatrix}, \quad c \equiv \begin{pmatrix} c_l \\ c_r \end{pmatrix},$$

$$d \equiv \theta_+ \coth(\theta_+ L) + \theta_- \tanh \left( \frac{\theta_-(1 - L)}{2} \right), \quad e \equiv \theta_+ \operatorname{csch}(\theta_+ L).$$

Here $\theta_\pm$ are given by

$$\theta \equiv \begin{cases} 
\theta_- \equiv \left[ m^2 + \frac{\varepsilon_0}{D} \right]^{1/2}, & 0 < x_1 < \xi_l, \quad \xi_r < x_1 < 1, \\
\theta_+ \equiv \left[ m^2 + \frac{\varepsilon_0}{D} \left( 1 + \frac{2w_+}{L(w_+ - 2)} \right) \right]^{1/2}, & \xi_l < x_1 < \xi_r.
\end{cases}$$
The Stability of the Mesa Stripe III

By calculating the spectrum of $\mathcal{G}$ we obtain

$$\lambda_\pm \sim \frac{\varepsilon_0^2}{\alpha} \left[ -\alpha m^2 + \frac{L}{2} (1 - L) - \sigma_\pm \right], \quad c_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$ 

Note that $c_+$ is a breather mode while $c_-$ is a zigzag mode. Here

$$\sigma_+ = \left[ \theta_+ \tanh \left( \frac{\theta_+ L}{2} \right) + \theta_- \tanh \left( \frac{\theta_-(1 - L)}{2} \right) \right]^{-1},$$

$$\sigma_- = \left[ \theta_+ \coth \left( \frac{\theta_+ L}{2} \right) + \theta_- \tanh \left( \frac{\theta_-(1 - L)}{2} \right) \right]^{-1}.$$
The Stability of the Mesa Pulse

For $m = 0$ and $\varepsilon_0 \ll 1$, we calculate

$$
\lambda_- \sim -1.8112 \frac{\varepsilon_0^2 L}{D} < 0, \quad \lambda_+ \sim -1.1899 \frac{\varepsilon_0}{L} < 0.
$$

Thus $\lambda_\pm < 0$ for $m = 0$ and $\varepsilon_0 \ll 1$. Recall that $L = L(\kappa)$.

Hence, a one-dimensional Mesa pulse for $\tau = O(1)$, $\varepsilon_0 \ll 1$ and $D = O(\varepsilon_0^{-1})$ is always stable under the effect of activator saturation.

We plot $\lambda_\pm$ versus $\kappa$ for $m = 0$, $D = 10$, and $\varepsilon_0 = 0.02$. 

\[ 
\begin{array}{c}
\text{(a)} \\
\text{(b)} 
\end{array} 
\]
The Stability of the Mesa Stripe I

We can show $\lambda_\pm < 0$ for $m = 0$ (pulse) and for $m \gg 1$. However, for $D$ below certain thresholds, there are unstable bands of zigzag and breather modes for $m = O(1)$.

Left figure: $\lambda_-$ versus $m$ for $\kappa = 2.0$ and $\varepsilon_0 = 0.0025$ when $D = 10.0$ (heavy solid curve), $D = 6.0$ (solid curve), and $D = 4.0$ (dashed curve).

Right figure: $\lambda_+$ versus $m$ for $\kappa = 4.0$ and $\varepsilon = 0.0025$ when $D = 10.0$ (heavy solid curve), $D = 3.5$ (solid curve), and $D = 2.8$ (dashed curve).
The Stability of the Mesa Stripe II

The critical values $\mathcal{D}_z$ and $\mathcal{D}_b$ where a zigzag and breather instability emerge at some $m = m_z$ and $m = m_b$ are roots of

$$
\frac{m}{2} \frac{d\sigma_-}{dm} = \sigma_- - \frac{L}{2}(1 - L), \quad \mathcal{D}_z = \frac{w_+^2}{2\beta L m_z^2} \left[ \frac{L}{2} (1 - L) - \sigma_- \right].
$$

$$
\frac{m}{2} \frac{d\sigma_+}{dm} = \sigma_+ - \frac{L}{2}(1 - L), \quad \mathcal{D}_b = \frac{w_+^2}{2\beta L m_z^2} \left[ \frac{L}{2} (1 - L) - \sigma_+ \right].
$$

Since $\mathcal{D}_z > \mathcal{D}_b$ a zigzag instability always occurs before a breather instability as $\mathcal{D}$ is decreased.
Numerical Realization of Zigzag Instability

Experiment: The numerical solution to the saturated GM model in $[0, 1] \times [0, 1]$ for $\varepsilon_0 = 0.01$ and $\kappa = 1.92$. A zigzag instability occurs in each case, except when $D = 1.4$ which is above the zigzag instability threshold of $D = 1.24$. 
**Mesa Stripe-Splitting: Growing Domains**


- Previous theory for $D \gg 1$. For the saturated GM model we can get new behavior: mesa-splitting for some $D = O(1)$ depending on $\kappa$.
- Decreasing $D$ and $\varepsilon^2$ exponentially slowly in time on a fixed size domain is equivalent to the problem of fixed $\varepsilon$ and $D$ on a slowly exponentially growing domain.
- From a single stripe, can one robustly obtain further stripes through mesa-split events without triggering zigzag instabilities?

**Experiment:** Take $\kappa = 1.9$, $D(t) = 0.2e^{-0.0002t}$ and $\varepsilon(t) = 0.15 \times D(t)$. First mesa-split at $t = 350$ when $D < 0.17$. Further splits occur and slow zigzag instabilities do not have time to develop. [The movie.]
Summary and Open Problems

Many semi-rigorous results are available for stripes, spots, and pulses in the GS and GM Models in the semi-strong regime in both $\mathbb{R}^1$ and in $\mathbb{R}^2$ where breakup instabilities are the dominant mechanism. Nonlocal Eigenvalue Problems are key here. Complicated patterns of zigzags, breakup, and spot-splitting occur near the existence threshold of a stripe in the weak interaction regime. Breakup instability bands can disappear for mesa stripes and for fat homoclinics.

Key Open problems:

- provide a rigorous analysis of stripes, spots, and pulses, for general classes of RD models in the semi-strong regime.
- provide some type of rigorous analysis in the weak interaction regime focusing on the disappearance of breakup bands.
- show analytically that the breakup band for the GM model with small saturation and fat homoclinics can disappear. Similar phenomena elsewhere?
- to study mesa-splitting analytically.
- to study breakup and zigzag instabilities of stripes and rings for the hybrid RD-chemotaxis systems of Painter et al. and Woodward et al.
- study stripe replication in slowly growing domains for RD systems.
References


Related papers are available on my website at UBC.