The Stability of Stripes and Spots for Some Reaction-Diffusion Models

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Outline of the Talk

1. Terminology: homoclinic vs. mesa stripes; weak vs. semi-strong interactions; breakup and zigzag instabilities of stripes
2. Homoclinic stripes and rings for the GM and GS models (semi-strong):
   - Rigorous analysis of the spectrum of certain NLEPs.
   - Stripe and pulse-splitting (simultaneous) in the GS Model.
3. Homoclinic stripes for the GM and GS models (weak interaction):
   - Zigzag and breakup instabilities of stripes.
   - Labyrinthine patterns and self-replication (edge-type) is possible.
4. A stripe for the GM model with saturation:
   - “Fat” homoclinic stripes under small saturation.
   - Mesa stripes under larger saturation.
5. Spots for the GM and GS Models: Examples
   - Dynamic bifurcations of spots (semi-strong regime)
   - Equilibrium spot locations correspond to optimal trap locations for a certain Laplacian eigenvalue problem (semi-strong regime)
   - Spot-splitting (weak interaction regime)
6. Open problems

Singularly Perturbed RD Models: Localization

Spatially localized solutions, with strong deviations from the uniform state, occur for singularly perturbed RD models of the form:

$$v_t = \varepsilon^2 \triangle v + g(u,v)\,; \quad \tau u_t = D\triangle u + f(u,v)\,, \quad \partial_n u = \partial_n v = 0 \quad \boldsymbol{x} \in \partial\Omega\,.$$

Since $\varepsilon \ll 1$, then v is the component that can be localized in space as a stripe or spot. The classical Gierer-Meinhardt (GM) and Gray-Scott (GS) models represent particular choices of f, g.

- GM Model:

$$g(u,v) = -v + v^p/u^q \qquad f(u,v) = -u + v^r/u^s\,.$$

  A model for biological morphogenesis and sea-shell patterns (Gierer, Meinhardt 1972; Meinhardt 1982; Koch, Meinhardt 1994; Meinhardt 1995). Simplest in a hierarchy of more complicated models.

- GS Model:

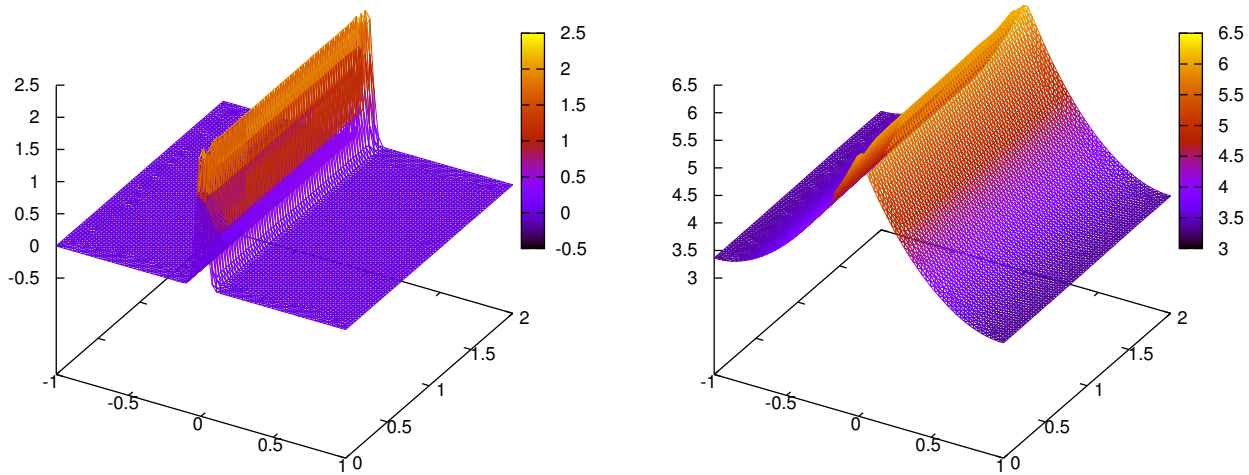
$$g(u,v) = -v + Auv^2\,, \qquad f(u,v) = (1-u) - uv^2\,.$$

  Model for chemical patterns in a continuously fed reactor. Very intricate paterns depending on D and A (Pearson, Science 1993, Swinney et al. Science 1993, Nishiura et al. 1999).

Homoclinic Stripe

Homoclinic Stripe: occurs when v localizes on a planar curve $\mathcal{C}$ (a pulse or spike corresponds to localization on an interval). The stripe cross-section for v is approximated by a homoclinic orbit of the 1-D system in the direction perpendicular to the stripe. A ring is such a stripe on a circle.

- **Semi-strong regime:** For $D = O(1)$ only v is localized and not u. In this regime a spike has a strong interaction (mediated by the global variable u) with either the boundary or with other spikes.

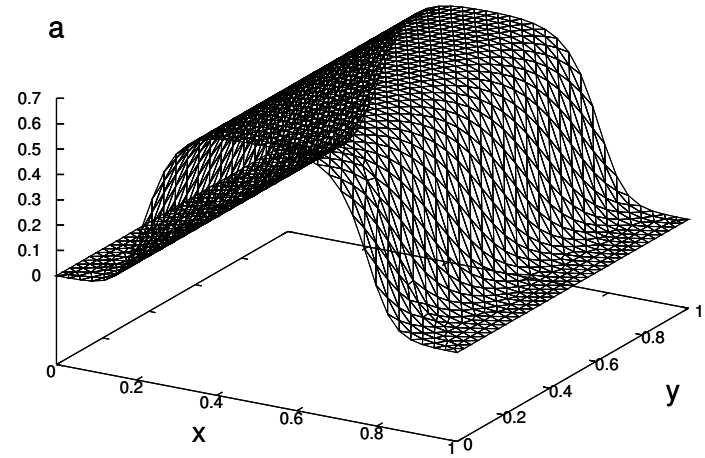


(a) v (localized) (b) u (global)

- **Weak-interaction regime:** Occurs for $D = O(\varepsilon^2)$ with $D = D_0\varepsilon^2$. Hence, both u and v are localized. In this regime the interaction of a spike with either the boundary or with neighbouring spikes is exponentially weak.

Mesa Stripe (Mesa is table in Spanish)

A **Mesa Stripe** has a flat plateau with edges determined by heteroclinic orbits of the 1-D system in the perpedicular direction.



- Such a stripe occurs in the GM model under saturation effects:

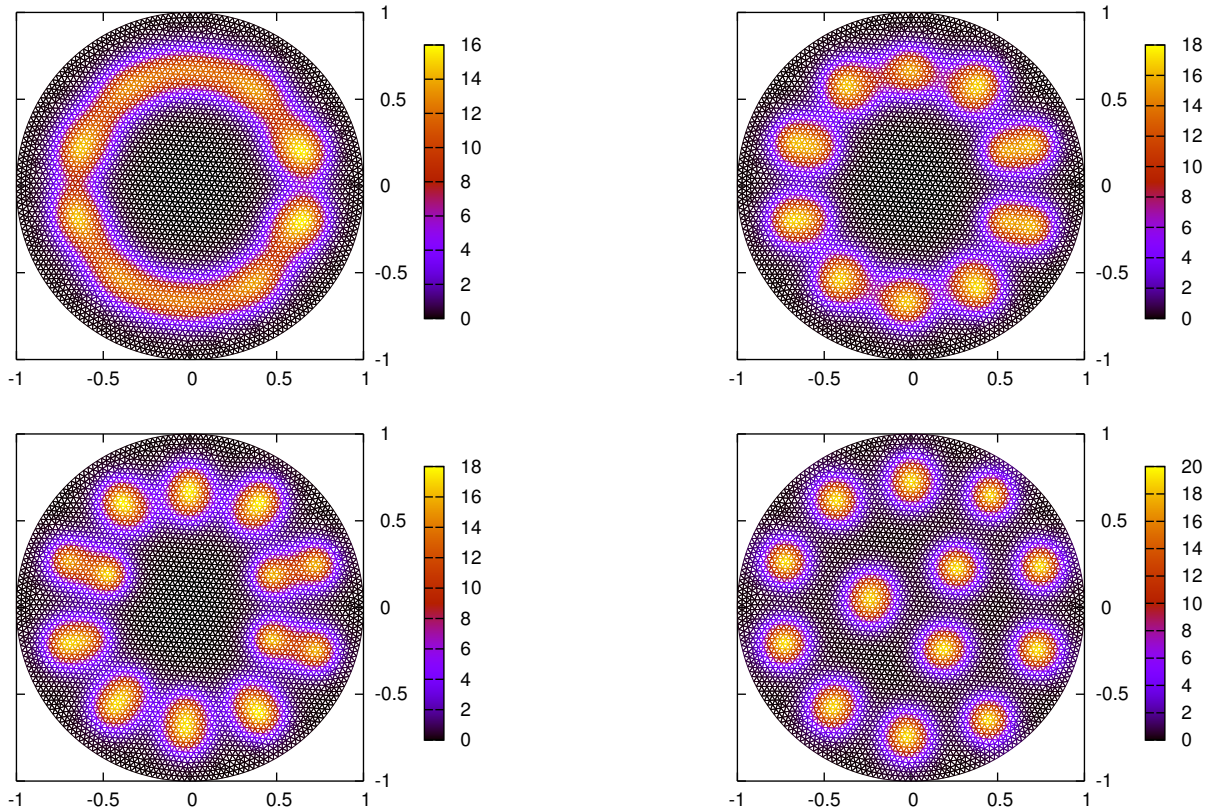
$$v_t = \varepsilon^2 \Delta v - v + \frac{v^2}{u(1+\kappa v^2)}\,, \qquad \tau u_t = D\Delta u - u + v^2\,.$$

  "The parameter κ has a deep impact on the final pattern" (Koch and Meinhardt 1994), and typically leads to stable stripe patterns.

- Mesa stripes also occur under bistable nonlinearities such as for a generalized Fitzhugh-Nagumo model (Tanaguchi, Nishiura 1994, 1996)

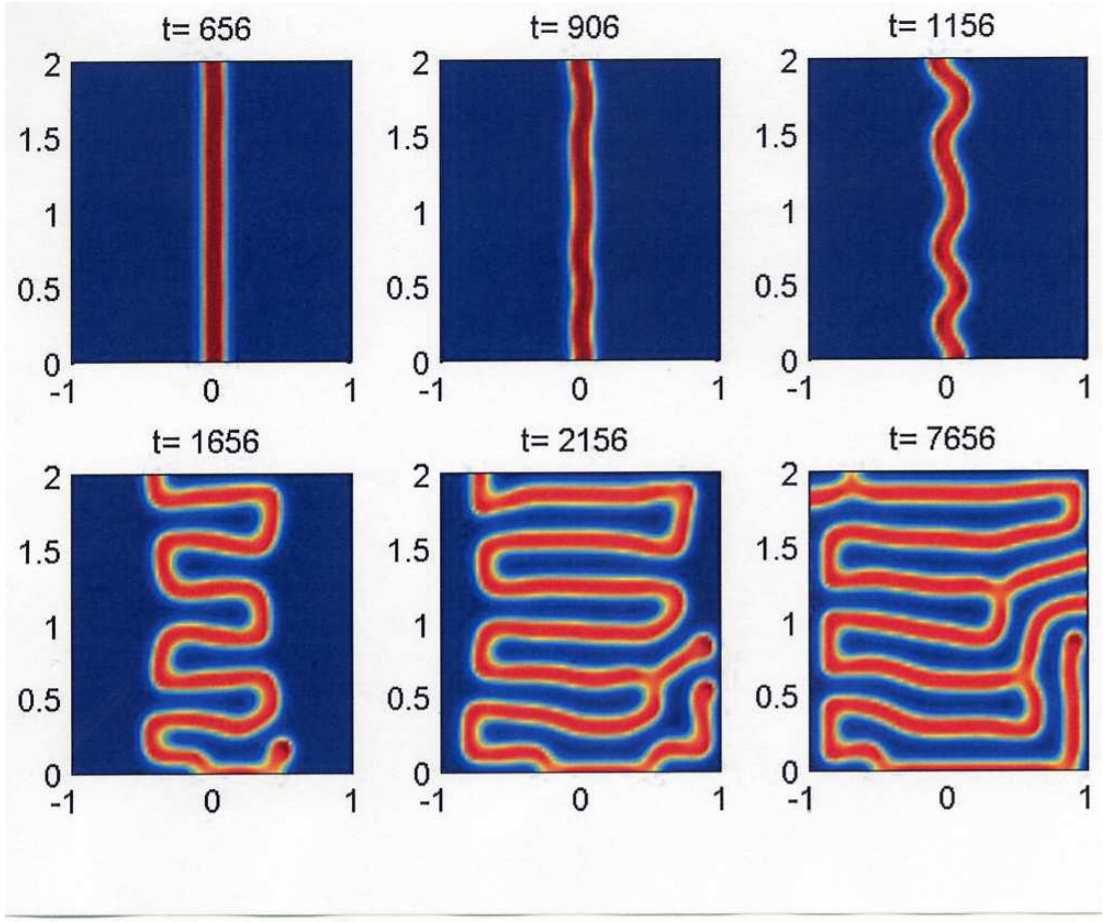
Breakup Instabilities

Breakup or varicose instability of a homoclinic ring for the GS model at different values of time in the semi-strong regime. This instability leads to the formation of spots. There is a secondary instability of spot-splitting. The eigenfunction is an even function across the ring cross-section.



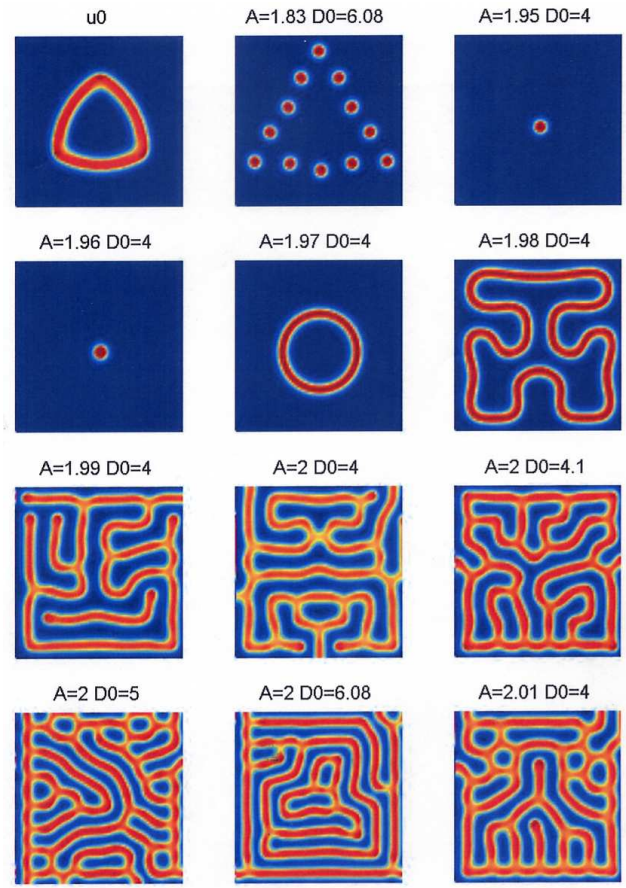
Zigzag Instability of a Stripe

Zigzag or transverse instability of a homoclinic stripe for the GS model in the weak interaction regime leading to a labyrinthine pattern. The eigenfunction is an odd function across the stripe cross-section. For this example, there is no breakup instability.



Zigzags, Breakup, and Spot-Splitting

Patterns in the weak interaction regime can depend very sensitively on the parameters. In particular, for the GS model where v concentrates on a triangular-shaped ring, very different patterns occur for slight changes in the feed-rate parameter A and diffusivity D:



For a combination of zigzags, breakup, and spot-splitting click here

Homoclinic Stripes: Semi-Strong Regime

We consider a stripe in a rectangle Ω for the GM model

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^p}{h^q}\,, \quad \tau h_t = D\Delta h - h + \frac{a^r}{\varepsilon_0 h^s}\,, \quad x = (x_1, x_2) \in \Omega\,,$$

where (p, q, r, s) satisfies

$$p > 1\,, \quad q > 0\,, \quad r > 1\,, \quad s \geq 0\,, \quad \zeta \equiv \frac{qr}{(p-1)} - (s+1) > 0\,.$$

We assume homogeneous Neumann conditions on $\partial\Omega$ and

$$\Omega: \ -1 < x_1 < 1\,, \quad 0 < x_2 < d_0\,.$$

By re-scaling a and h, and from $X = x/l$ where $l = 1/\sqrt{D}$, we can set $D = 1$ above and replace Ω by Ω_l where

$$\Omega_l: \ -l < x_1 < l\,, \quad 0 < x_2 < d\,; \quad d \equiv d_0 l\,; \quad \varepsilon \equiv \varepsilon_0 l\,; \quad l \equiv 1/\sqrt{D}\,.$$

We examine the existence and stability of a stripe centered on the line $x_1 = 0$ in the semi-strong regime where $D = O(1)$ and $\varepsilon_0 \ll 1$.

The Equilibrium Homoclinic Stripe

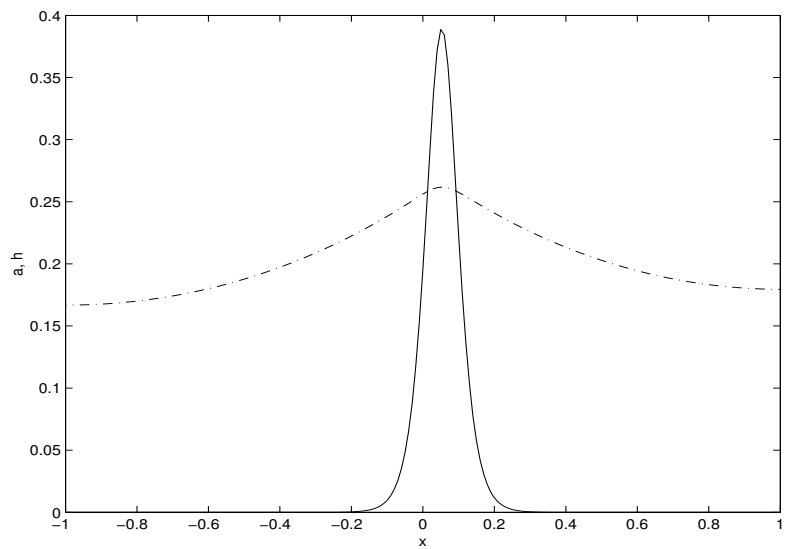
For $\varepsilon \to 0$, the equilibrium stripe solution $a_e(x_1)$ satisfies

$$a_e(x_1) \sim \mathcal{H}^{\gamma} w\left(\varepsilon_0^{-1} x_1\right) , \quad \mathcal{H}^{\zeta} \equiv \frac{2}{b_r \coth\left(D^{-1/2}\right)} , \quad b_r \equiv \int_{-\infty}^{\infty} [w(y)]^r \, dy \, .$$

Here $\gamma \equiv q/(p-1)$, and $w(y)$ is the homoclinic solution to

$$w'' - w + w^p = 0 \, , \quad w \to 0 \quad \text{as} \quad |y| \to \infty \, ; \quad w'(0) = 0 \, , \quad w(0) > 0 \, ,$$

The equilibrium $h_e(x_1)$ can be written in terms of a Green's function.



Homoclinic Stripe: Stability I

We introduce the perturbation

$$a = a_e + e^{\lambda t + imx_2}\phi\,, \quad h = h_e + e^{\lambda t + imx_2}\eta\,, \quad m = \frac{k\pi}{d}\,.$$

This leads to the eigenvalue problem on $|x_1| < l$:

$$\varepsilon^2\phi_{x_1x_1} - \phi + \frac{pa_e^{p-1}}{h_e^q}\phi - \frac{qa_e^p}{h_e^{q+1}}\eta = \left(\lambda + \varepsilon^2 m^2\right)\phi\,, \quad \phi_x(\pm l) = 0\,,$$

$$\eta_{x_1x_1} - \left(1 + \tau\lambda + m^2\right)\eta = -\frac{ra_e^{r-1}}{\varepsilon h_e^s}\phi + \frac{sa_e^r}{\varepsilon h_e^{s+1}}\eta\,, \quad \eta_x(\pm l) = 0\,.$$

The $O(1)$ eigenvalues, which govern breakup instabilities, satisfy

$$\phi(x_1) \sim \Phi\left(\varepsilon^{-1}x_1\right)\,, \quad \int_{-\infty}^{\infty} w(y)\Phi(y)\,dy \neq 0\,.$$

The $O(\varepsilon^2)$ eigenvalues, which govern zigzag instabilities, satisfy

$$\phi(x_1) \sim w'\left(\varepsilon^{-1}x_1\right) + \varepsilon\phi_1\left(\varepsilon^{-1}x_1\right) + \cdots\,.$$

Homoclinic Stripe: Breakup Instability I

A singular perturbation analysis leads to the NLEP problem:

$$L_0\Phi - \chi_m w^p \frac{\int_{-\infty}^{\infty} w^{r-1}\Phi\, dy}{\int_{-\infty}^{\infty} w^r\, dy} = (\lambda + \varepsilon^2 m^2)\Phi\,, \quad \Phi \to 0 \text{ as } |y| \to \infty$$

$$\frac{1}{\chi_m(\lambda)} \equiv \frac{s}{qr} + \frac{\theta_\lambda \tanh(\theta_\lambda l)}{qr \tanh l}\,, \quad \theta_\lambda \equiv \sqrt{1 + m^2 + \tau\lambda}\,,$$

$$L_0\Phi \equiv \Phi'' - \Phi + p w^{p-1}\Phi\,.$$

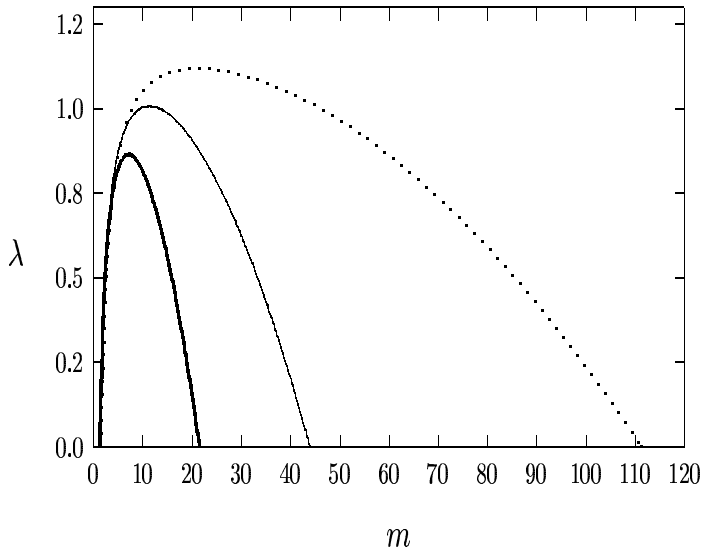
- A rigorous analysis (KSWW, 2005, submitted SIADS 2005) proves stability Re$(\lambda) < 0$ for $m = 0$ and $m \gg 1$ when $\tau < \tau_H$ for certain (p, q, r, s).
- For any $\tau > 0$, there is an unstable breakup band of the form

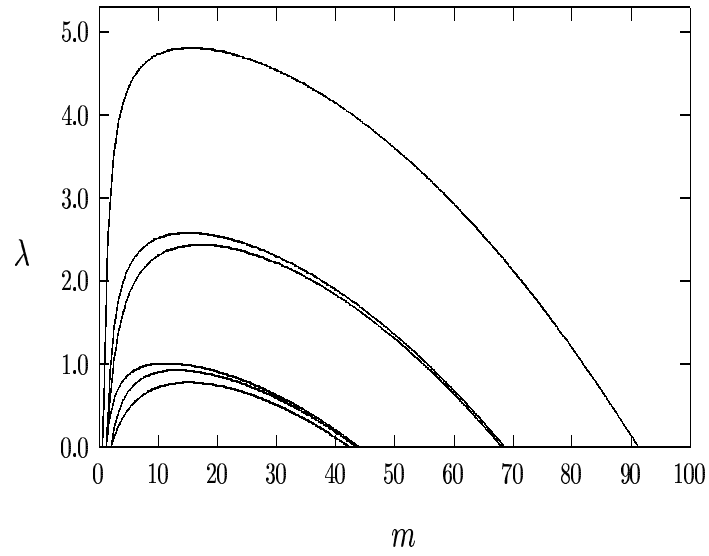
$$m_{b-} < m < m_{b+} = O(\varepsilon^{-1})\,, \quad m_{b+} \sim \frac{1}{2\varepsilon}\sqrt{(p-1)(p+3)} + O(\varepsilon)\,.$$

- Key result: Since $m_{b+} = O(\varepsilon^{-1})$, breakup instabilities can only be avoided in asymptotically thin domains where $d_0 = O(\varepsilon_0)$.
- The analysis extends that of Doelman, Van der Ploeg (SIADS 2002).

Homoclinic Stripe: Breakup Instability II

From numerical computations of the spectrum of the NLEP we get:





Left figure: Most unstable λ versus m for $D = 1$, $\tau = 0$, $(p, q, r, s) = (2, 1, 2, 0)$ but different ε:

Right figure: Most unstable λ versus m for $D = 1$, $\varepsilon = 0.025$, $\tau = 0$, but different exponent sets (p, q, r, s):

Homoclinic Stripe: Zigzags

For $\varepsilon \to 0$ and $\tau = O(1)$, a singular perturbation analysis related to the SLEP method (Nishiura, Fujii 1987) shows that the eigenvalue governing a zigzag instability of a stripe of mode m is

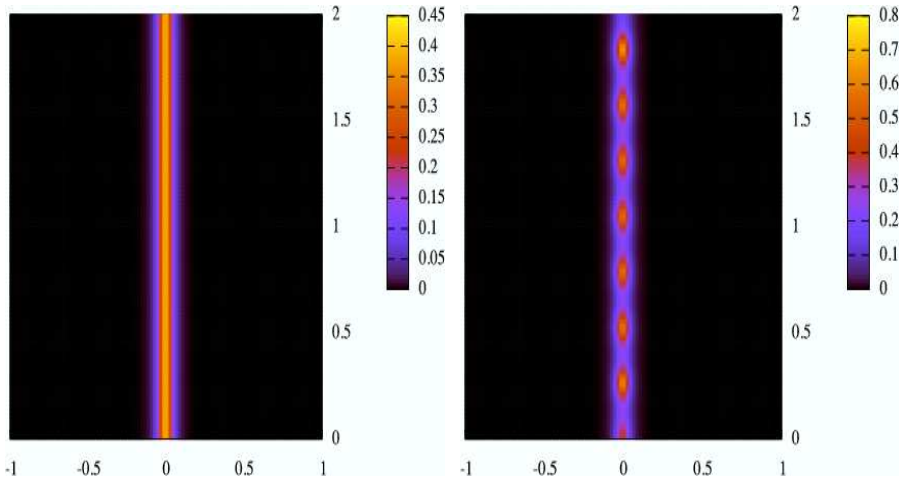
$$\lambda \sim \varepsilon^2 \left[2\gamma\theta \tanh l \tanh(\theta l) - 2\gamma - m^2\right] , \quad \theta \equiv \sqrt{1+m^2} , \quad \gamma \equiv \frac{q}{p-1} .$$

The implications from this formula are:

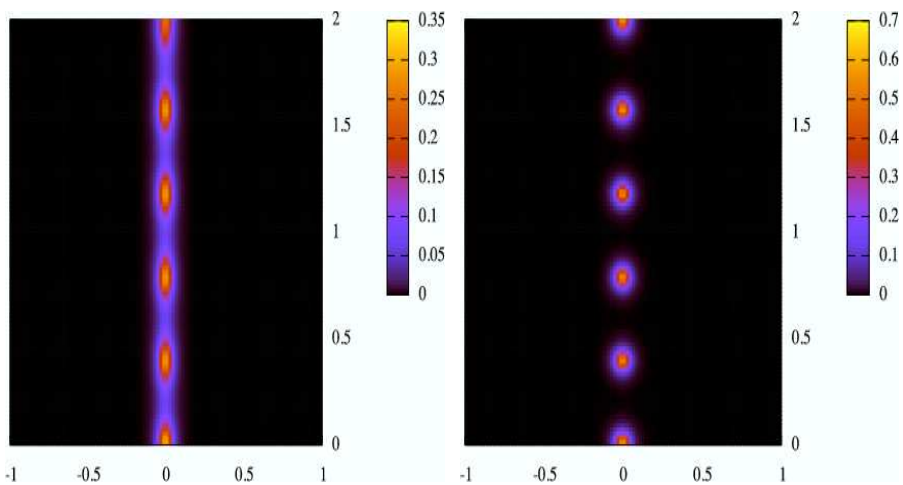
- There are no unstable zigzag modes when $\gamma \leq 1$, which includes the classical GM exponents $(p, q, r, s) = (2, 1, 2, 0)$.
- For $\gamma > 1$, there is a band $m_{z-} < m < m_{z+}$ of unstable zigzag modes only when the domain half-length $l \equiv 1/\sqrt{D}$ exceeds some critical value l_z, or equivalently when D is small enough.
- Since $\lambda = O(\varepsilon^2)$, zigzag instabilities occur for $t = O(\varepsilon^{-2})$.
- Hence, in this semi-strong regime a stripe is de-stabilized not by the slow zigzag instabilities but by the faster breakup instabilities.
- Taking the limit $D \to 0$, corresponding to the weak interaction regime, we expect that the zigzag and breakup instabilities occur there on the same time-scale.

Homoclinic Stripe: Breakup Instability III

Experiment 1 and 2: Take $(p, q, r, s) = (2, 1, 2, 0)$, $\varepsilon_0 = 0.025$, $\tau = 0.1$ and $\Omega = [-1, 1] \times [0, 2]$. The predicted number N of spots is the number of maxima of the eigenfunction $\cos(my)$ on $0 < y < d_0 = 2$, which is $N = md_0/(2\pi) = m/\pi$ where m is most unstable mode.



- For $D = 1.0$, we predict $N = 8$. Numerically, the initial stripe breaks up into seven spots on an $O(1)$ time-scale. There is then a competition instability. The movie.



- For $D = 0.1$ we predict $N = 4$. Numerically, the initial stripe breaks up into five spots on an $O(1)$ time-scale with no competition instability. The movie.

Homoclinic Stripes, Rings: Semi-Strong (GS)

Let $l = 1/\sqrt{D}$. Then, in a rectangle or circle, the GS Model is

$$v_t = \varepsilon^2 \triangle v - v + Auv^2 \,, \quad \tau u_t = \Delta u - u + 1 - uv^2 \,, \quad x \in \Omega \,.$$

$$\Omega : \; -l < x_1 < l, \quad 0 < x_2 < d \equiv l d_0 \,; \quad \Omega : \; 0 < x_1^2 + x_2^2 < R^2 \equiv l^2 \,.$$

The semi-strong regime $l = O(1)$ with $\varepsilon \ll 1$ is sub-divided as follows:

1. **High Feed-Rate (SS) Regime:** $A = O(1)$ as $\varepsilon \to 0$.
   - a homoclinic stripe or ring can undergo a splitting process.
   - in a rectangle, both zigzag and breakup instabilities occur unless the domain with is $O(\varepsilon) \ll 1$ thin.
   - impossible to have only a zigzag instability with no breakup instability.
2. **Intermediate (SS) regime:** $O(\varepsilon^{1/2}) \ll A \ll 1$. No splitting phenomena and breakup instabilities occur on much faster timescales than zigzags. Breakup instabilities analyzed from a **Nonlocal Eigenvalue Problem**.

In contrast, the **Weak Interaction Regime:** corresponds to $l = \infty$ and $\varepsilon \to 1$ and has been studied in Nishiura, Ueyema (1999,2001) and Ueyama (2000) in terms of global bifurcation properties.

Homoclinic: High Feed Regime (GS)

An equilibrium stripe (of zero curvature) and ring can be constructed in terms of the solutions $V > 0$, $U > 0$ with $y = \varepsilon^{-1} x_1$ to

$$V'' - V + V^2 U = 0\,, \quad U'' = UV^2\,, \quad 0 < y < \infty\,,$$

$$V'(0) = U'(0) = 0\,; \quad V \to 0\,, \quad U \sim By\,, \quad \text{as} \quad y \to \infty\,.$$

This core problem was indentified for pulses in Muratov et al. (2000) (and Doelman et al. (1998)). Inner-outer matching for u determines B as

$$A = B \coth\left(D^{-1/2}\right)\,, \quad \text{(rectangle)}\,; \quad A = 2BG(\rho_0;\rho_0)\,, \quad \text{(ring)}$$

Here $G(\rho;\rho_0)$ on $0 < \rho < R$ satisfies

$$G_{\rho\rho} + \frac{1}{\rho} G_\rho - G = -\delta(\rho - \rho_0)\,; \quad G_\rho(R;\rho_0) = G_\rho(0;\rho_0) = 0\,.$$

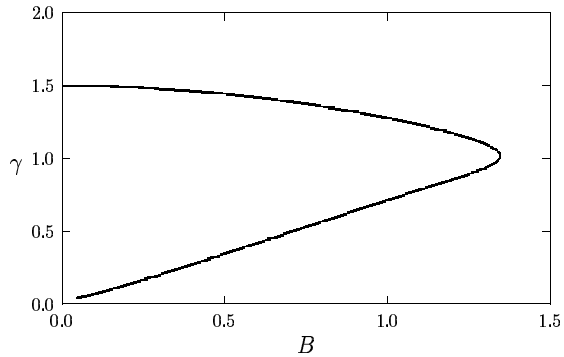
For $R = D^{-1/2}$, the equilibrium ring radius ρ_0 is the root of

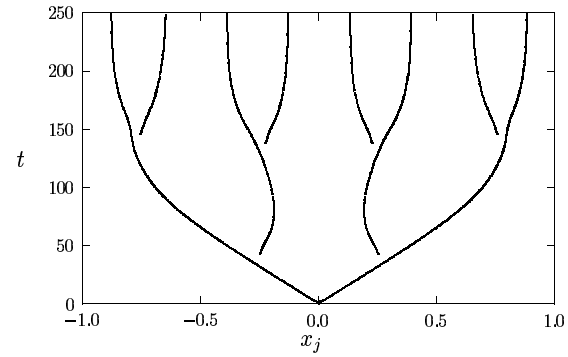
$$\frac{K_0(\rho_0)}{I_0(\rho_0)} - \frac{K_0'(R)}{I_0'(R)} = \frac{1}{2\rho_0 I_0(\rho_0) I_0'(\rho_0)}\,.$$

When $D \ll 1$ where $R \gg 1$, then $\rho_0 \sim R - \frac{1}{2}\log(2R)$.

Homoclinic: Pulse: High Feed Regime (GS)

The bifurcation plot of $\gamma = U(0)V(0)$ vs. B has a saddle-node behavior:



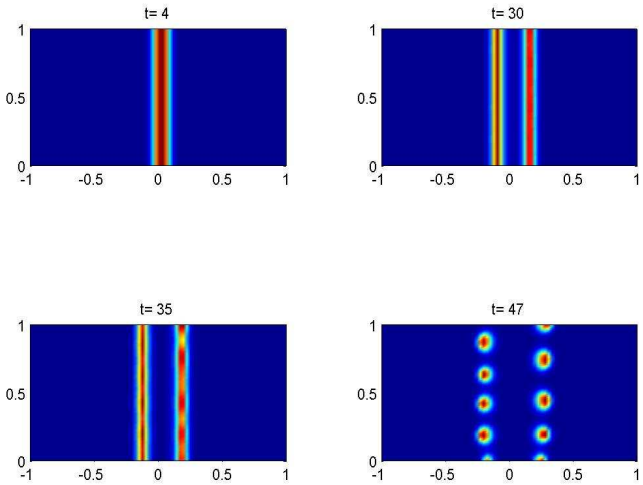


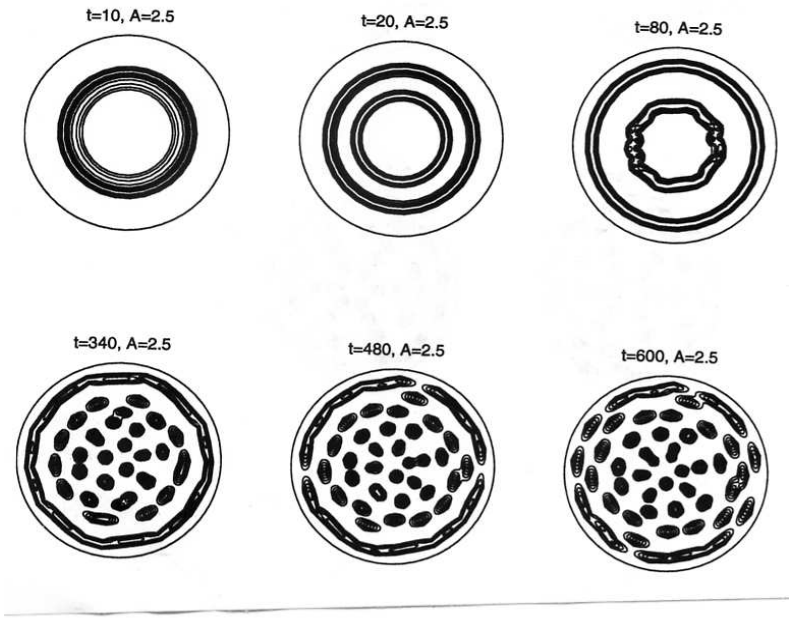
- saddle node at $B = 1.347$ (Muratov, Osipov 2000). Upper branch is stable for τ small enough while the lower branch is always unstable.
- The global bifurcation conditions of Ei, Nishiura (JJIAM 2001) can be verified analytically (KWW, Physica D 2005)
- Since $B = \alpha A$, for some α, there is no one-pulse solution when $A > A_{split}$. This leads to pulse-splitting. The pulse-splitting is simultaneous and, for symmetric one-spike initial data, the final number of pulses is determined by the smallest integer k for which

$$A_{p2^{k-1}} < A < A_{p2^k} \ , \qquad A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right) \ .$$

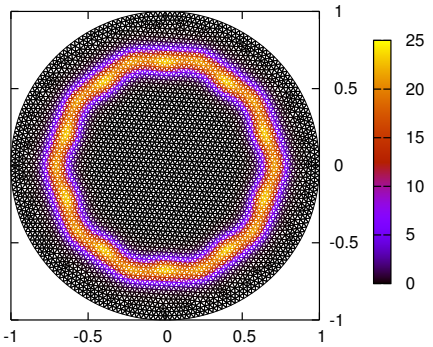
Homoclinics: Stripe and Ring-Splitting (GS)

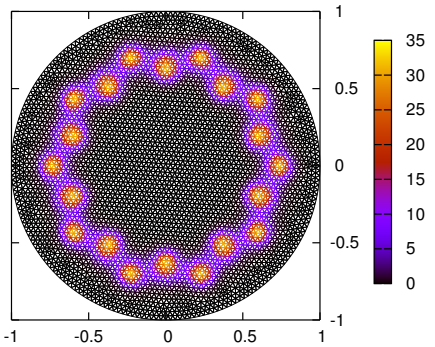
Stripe and Ring replication is unstable to breakup instabilities. For stripes we need that the rectangle is $O(\varepsilon)$ thin to prevent these instabilities.





Next, for $A < A_{\text{split}}$, we have the zigzag and breakup instability of a ring.





Homoclinics: Instabilities of a Ring (GS)

Principal Result: In the intermediate regime $O(\varepsilon^{1/2}) \ll A \ll 1$ all zigzag perturbations of the equilibrium ring solution are unstable in the band

$$m_{zr} < m < A^2 / \left[6\varepsilon\rho_0 J_{1,0}^2(\rho_0) J_{2,0}^2(\rho_0) \right],$$

where m_{zr}, which depends on $R \equiv 1/\sqrt{D}$ and ρ_0, is the root of

$$4\rho_0^2 J_{1,0}(\rho_0) J_{2,0}(\rho_0) J_{1,m}(\rho_0) J_{2,m}(\rho_0) = 1\,.$$

The asymptotics of m_{zr} are

$$m_{zr} = 3\,, \quad \text{as} \quad R \to 0\,; \quad m_{zr} \sim \sqrt{2R - \ln(2R) + \cdots)}\,, \quad \text{as} \quad R \to \infty\,.$$

Here $J_{1,m}$, $J_{2,m}$ are related to standard Bessel functions by

$$J_{2,m}(\rho) \equiv K_m(\rho) - K_m^{'}(R)/I_m^{'}(R) I_m(\rho)\,, \quad J_{1,m}(\rho) \equiv I_m(\rho)\,,$$

In the intermediate regime, the breakup instability band is

$$A^2 / \left[12\rho_0 J_{1,0}^2(\rho_0) J_{2,0}^2(\rho_0) \right] < \varepsilon m < \sqrt{5}\rho_0/2\,.$$

Homoclinic Stripes: Weak Interaction Regime

The GM model in a rectangle in the weak interaction regime is

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^p}{h^q}\,, \quad \tau h_t = \varepsilon_0^2 D_0 \Delta h - h + \frac{a^r}{h^s}\,, \quad x = (x_1, x_2) \in \Omega\,.$$

Here $D = D_0 \varepsilon^2$ and $\Omega = [-1, 1] \times [0, d_0]$.

We examine the existence and stability of a stripe centered along the midline $x_1 = 0$ of Ω. Since u and v decay exponentially away from $x_1 = 0$ for $\varepsilon_0 \ll 1$, the sidewalls at $x_1 = \pm 1$ can be neglected.

Questions for a Pulse and a Stripe:

- What is the bifurcation behavior of equilibrium homoclinic pulses in terms of D_0?
- What are the stability properties for pulses on an interval?
- For a stripe (composed of a pulse trivially extended in the second direction) what are the zigzag and breakup instability bands as a function of D_0. Can the breakup band disappear?

Homoclinic Pulse: Equilibrium Solution I

For $\varepsilon_0 \to 0$, with $D = \varepsilon_0^2 D_0$, the GM model has two-different one-pulse equilibrium solutions when $D_0 > D_{c0}$ and none when $0 < D_0 < D_{c0}$. When $D_0 > D_{c0}$, the solutions are of the form

$$a_{e\pm}(x_1) \sim u\left[\varepsilon_0^{-1} x_1\right] , \quad h_{e\pm}(x) \sim v\left[\varepsilon_0^{-1} x_1\right] .$$

The functions $u(y)$ and $v(y)$ are homoclinic solutions on $0 < y < \infty$ to

$$u^{''} - u + u^p/v^q = 0 , \quad D_0 v^{''} - v + u^r/v^s = 0 ,$$

with

$$u(0) = \alpha , \;\; u^{'}(0) = v^{'}(0) = 0 , \;\; u(\infty) = v(\infty) .$$

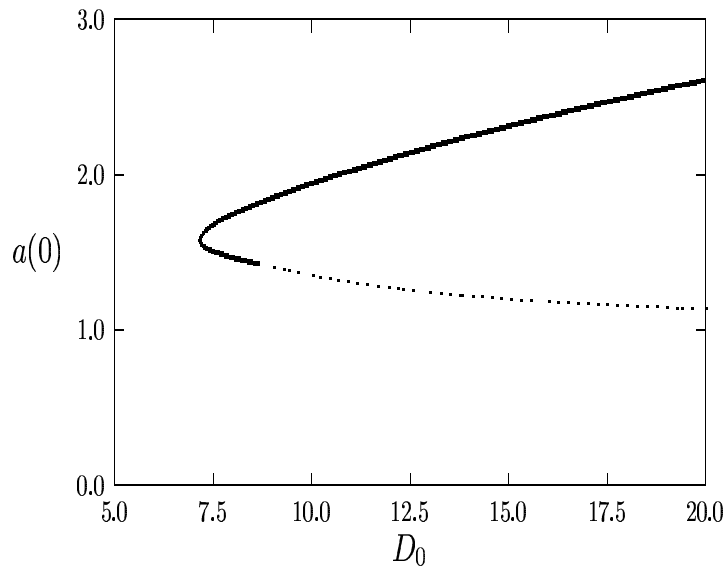
- This problem is easily solved using AUTO for various (p, q, r, s).
- Existence results by geometric theory of singular perturbations (Doelman and Van der Ploeg, SIADS 2002)
- Numerical computations also by Nishiura (AMS Monograph Translations 2002)

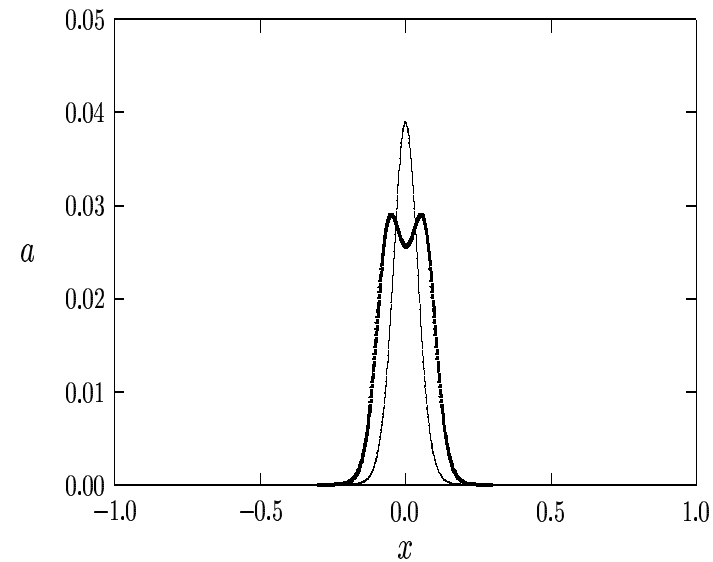
Homoclinic Pulse: Equilibrium Solution II

For each exponent set (p, q, r, s) tested:

- There is a saddle-node point at some $D_0 = D_{c0} > 0$. For $(2, 1, 2, 0)$, $D_{c0} \approx 7.17$.
- On the dashed portion of the lower branch the homoclinic profile for u has a double-bump structure so that the maximum of u occurs on either side of $y = 0$.

Left figure: $a(0)$ versus D_0 for $(2, 1, 2, 0)$. The dotted part is where a_e has a double-bump structure. Right figure: a_{e+} (solid curve) for $D_0 = 9.83$ and a_{e-} (heavy solid curve) for $D_0 = 10.22$.



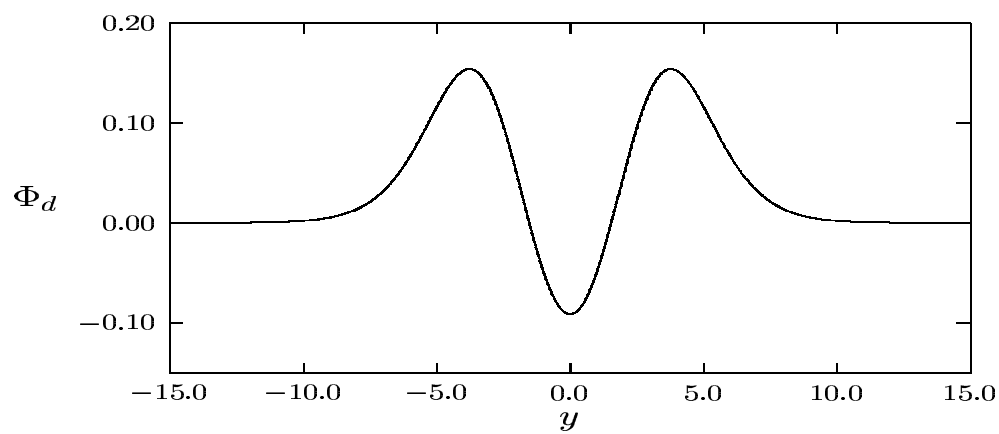


Homoclinic Pulse: Stability Properties

- Numerically we find that the upper branch is stable for $\tau < \tau_H$. For $\tau > \tau_H$, where τ_H depends on D_0, the pulse on the upper branch undergoes a Hopf bifurcation. The lower branch is unstable for any $\tau \geq 0$.
- The translation mode $\Phi = u^{'}$, $N = v^{'}$, is always a neutral mode.
- At the fold point, where $D_0 = D_{c0}$, there is also an even dimple-shaped eigenfunction $\Phi = \Phi_d(y)$, associated with a zero eigenvalue, satisfying

$$\Phi_d(0) < 0\,; \quad \Phi_d(y) > 0\,, \;\; y > y_0\,; \quad \Phi_d(y) < 0\,, \quad 0 < y < y_0\,.$$

- Hence, the key global bifurcation conditions of Ei and Nishiura (JJIAM 2001) for edge-splitting pulse-replication behavior are satisfied.



Homoclinic Stripe: Stability Properties I

We extend the pulse trivially in the second direction to obtain a straight stripe. To study the stability of this stripe we let $y = \varepsilon_0^{-1} x$ and introduce

$$a = u(y) + \Phi(y)\, e^{\lambda t} \cos(m x_2)\,, \quad h = v(y) + N(y)\, e^{\lambda t} \cos(m x_2)\,.$$

Defining $\mu \equiv \varepsilon_0^2 m^2$, we get an eigenvalue problem on $0 < y < \infty$:

$$\Phi_{yy} - (1+\mu)\Phi + \frac{p u^{p-1}}{v^q}\Phi - \frac{q u^p}{v^{q+1}} N = \lambda \Phi\,,$$

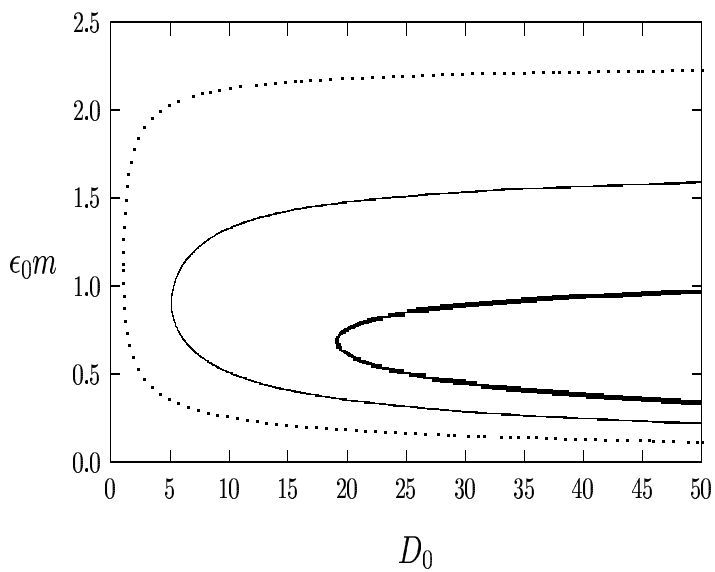
$$D_0 N_{yy} - (1 + D_0 \mu)\, N + \frac{r u^{r-1}}{v^s}\Phi - \frac{s u^r}{v^{s+1}} N = \tau \lambda N\,.$$

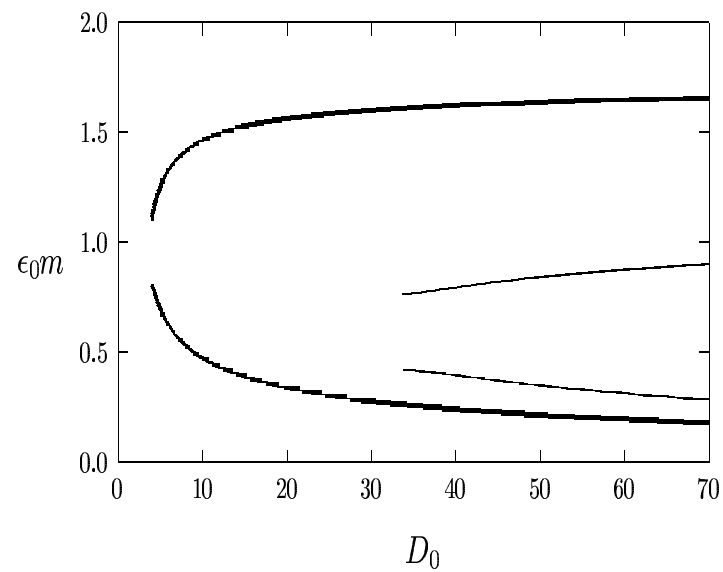
- For breakup instabilities we compute the spectrum for even eigenfunctions Φ and N so that $\Phi_y(0) = N_y(0) = 0$.
- For zigzag instabilities we compute the spectrum for odd eigenfunctions Φ and N so that $\Phi(0) = N(0) = 0$.
- The computations are done by first computing u and v. Then, we discretize the eigenvalue problem by finite differences and use a generalized eigenvalue problem solver from Lapack. Instability thresholds are computed using a quasi-Netwon method.

Homoclinic Stripe: Stability II (Breakup)

For breakup instabilities we find numerically that:

- For $\tau = 0$ there is a unique real positive eigenvalue in the breakup instability band $\sqrt{\mu_1} < \varepsilon_0 m < \sqrt{\mu_2}$ (provided it exists) , and that Re$(\lambda) < 0$ outside this band. An unstable eigenvalue is $O(1)$.
- As D_0 is decreased towards the existence threshold D_{0c}, the instability band disappears for certain exponent sets at some critical value $D_{0b} > D_{0c}$ on the upper branch. Thus, for some (p, q, r, s) a straight stripe will not break into spots for D_0 on the range $D_{0c} < D_0 < D_{0b}$.

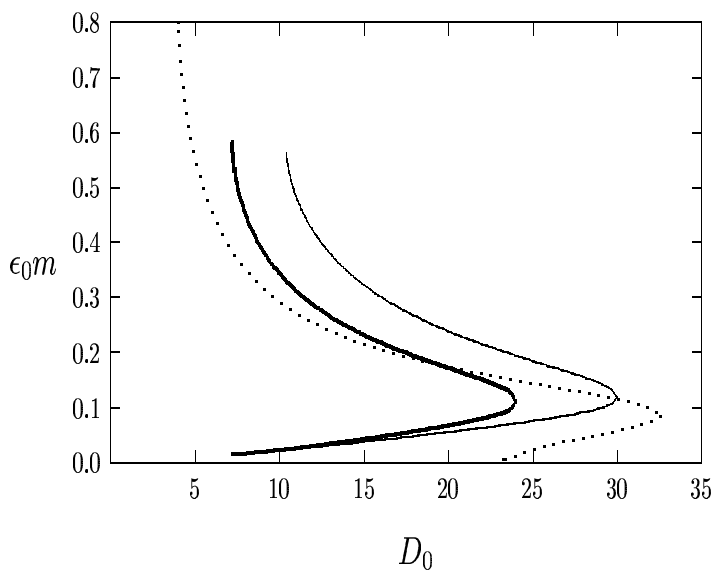


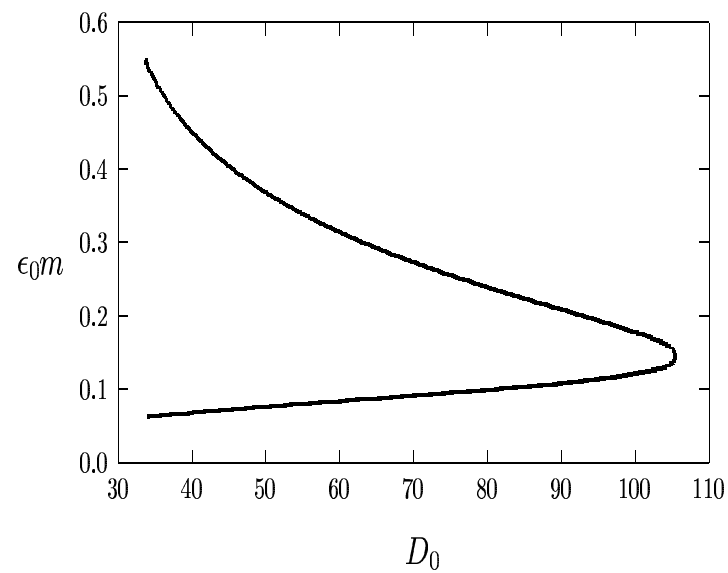


Homoclinic Stripe: Stability III (Zigzag)

For zigzag instabilities we find numerically that:

- For $\tau = 0$ and $\gamma = q/(p-1) < 1$ and as D_0 is decreased towards the existence threshold D_{0c}, an unstable zigzag band first emerges at some critical value D_{0z} with $D_{0z} > D_{0c}$.
- For $\tau = 0$ and $\gamma > 1$, the unstable zigzag band continues into the semi-strong regime corresponding to the limit $D_0 \gg 1$.
- Within the unstable zigzag band there is a unique positive eigenvalue when $\tau = 0$. The time-scale for zigzag instabilities is $O(1)$.





Homoclinic Stripes: Stability III (Thresholds)

(p, q, m, s)	D_{0c}	α_c	D_{0b}	D_{0z}	D_{0m}	D_{0s}
$(2, 1, 2, 0)$	7.17	1.58	8.06	24.0	8.92	9.82
$(2, 1, 3, 0)$	10.35	1.42	19.14	30.0	12.36	16.31
$(3, 2, 2, 0)$	3.91	1.62	$--$	32.3	5.08	5.23
$(3, 2, 3, 1)$	4.41	1.53	5.13	28.0	5.36	5.97
$(2, 2, 3, 3)$	33.7	2.28	$--$	$--$	41.80	85.52
$(4, 2, 2, 0)$	0.89	1.36	1.00	27.9	1.06	0.89

Second and Third columnns: the saddle-node bifurcation values D_{0c} and $\alpha_c \equiv a(0)$ for the existence of a stripe. **Fourth column:** values D_{0b} of D_0 for the lower bound of the breakup instability band. A stripe is stable to breakup when $D_{0c} < D_0 < D_{0b}$. **Fifth column:** values D_{0z} of D_0 for the upper bound of the zigzag instability band in the weak-interaction regime. **Sixth Column:** the smallest values D_{0m} of D_0 where $a(y)$ has a double-bump structure on the lower branch of the $a(0)$ versus D_0 diagram. **Seventh coumn:** the saddle-node values D_{0s} representing the smallest value of D_0 where a radially symmetric spot solution exists in $\mathbb{R}^2$.

Homoclinic Stripe: Stability Properties IV

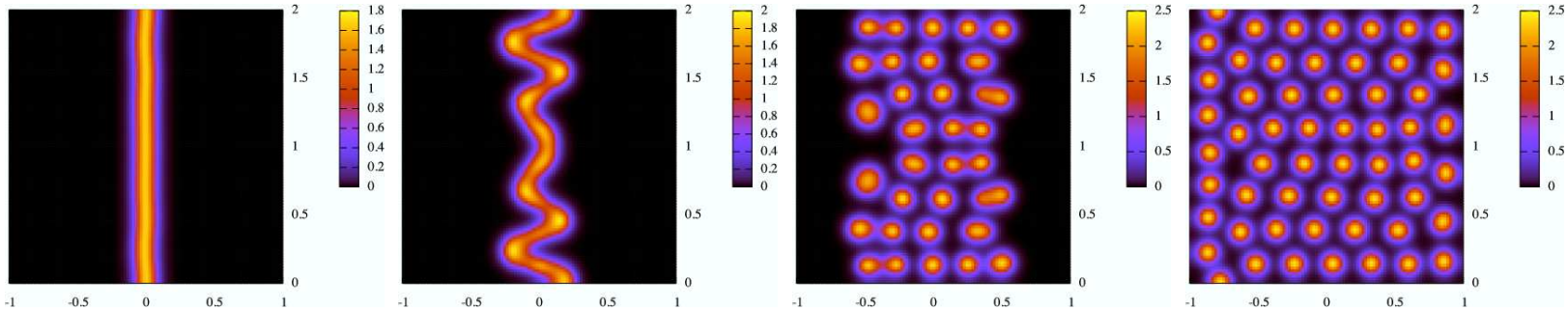
- For sets (p, q, m, s) where the breakup instability disappears at some $D_{0b} > D_{0c}$, the stripe will always be unstable with respect to zigzag instabilities for domain widths d_0 that are $O(1)$ as $\varepsilon_0 \to 0$. Zigzags can be suppressed only by taking the domain width d_0 to be $O(\varepsilon_0)$ thin.
- Whenever a breakup band exists, the breakup and zigzag bands overlap in such a way that there are no domain widths d_0 where a zigzag instability is not accompanied by a breakup instability.
- The time-scale for breakup instabilities is generally faster than for zigzag instabilities. However, both time-scales are independent of ε_0.

Disappearance of the Breakup Band: It appears to be analytically intractable to rigorously study the eigenvalue problem to analytically confirm the coalescence of the breakup instability band and the emergence of the zigzag band near D_{0c}. However, qualitatively, near D_{0c}, the region near the maximum of a becomes wider than for larger values of D. These fattened homoclinic stripes are more reminiscent of mesa-stripes in bistable systems, where breakup instabilities typically do not occur (Taniguchi, Nishiura 1996) and where zigzags are the dominant instabilities.

Homoclinic Stripes: Experiment 1 (GM)

Experiment 1: Take $(p, q, r, s) = (2, 1, 2, 0)$, $\varepsilon_0 = 0.025$, and $D_0 = 7.6$. Since $D_{0c} < D_0 < D_{0b}$, there is no breakup instability band and we predict that the initially straight stripe will not break up into spots. Theory predicts the number N of zigzag crests as $N = \frac{m}{\pi} = \frac{0.315}{\pi\varepsilon_0} \approx 4$. The wriggled stripe, however, undergoes a breakup instability near the points of its maximum curvature leading to spot formation. Since $D_0 = 7.6 < D_{0s} = 9.82$ these spots then undergo a self-replication process which fill the entire domain.

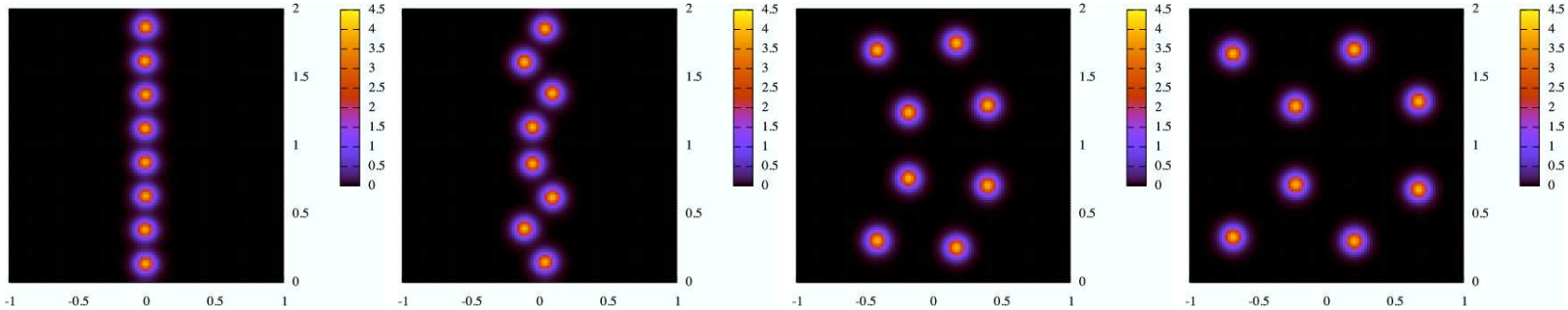
Click here for the movie



Left: $t = 200$ Left Middle: $t = 300$ Right Middle: $t = 400$ Right: $t = 700$

Homoclinic Stripes: Experiment 2 (GM)

Experiment 2: Take $(p, q, r, s) = (2, 1, 2, 0)$, $\varepsilon_0 = 0.025$, and increase D_0 to $D_0 = 15.0$. Since $D_0 > D_{0b}$ we predict that the initially straight stripe will break up into spots. From theory, the predicted number N of spots is $N = \frac{m}{\pi} = \frac{0.623}{\pi \varepsilon_0} \approx 8$. The zigzag instability breaks the vertical symmetry. Since $D_0 = 15$ is well above the spot-existence threshold $D_{0s} = 9.82$, there is no self-replication behavior of spots. Instead there is an exponentially slow, or metastable, drift of the spots towards a stable equilibrium configuration that has a hexagonal structure.

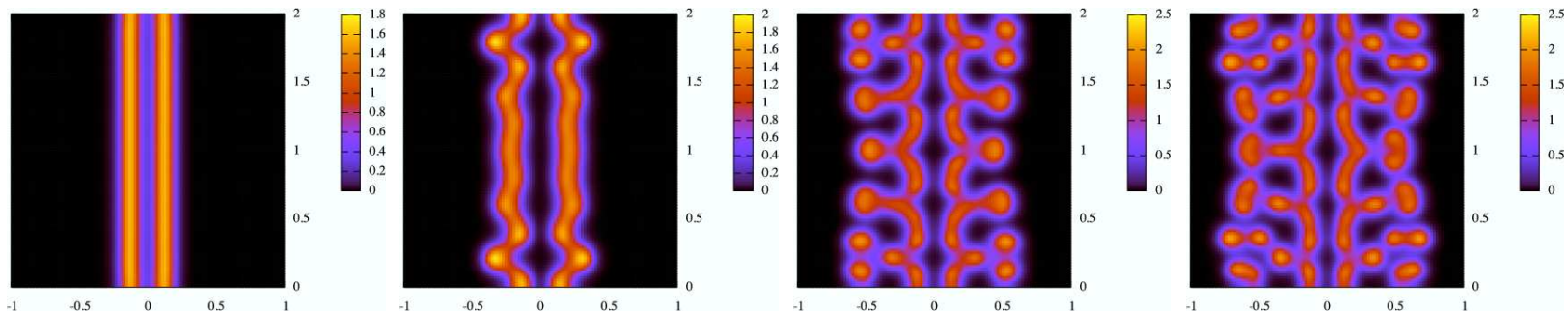


Left: $t = 500$ Left Middle: $t = 800$ Right Middle: $t = 3000$ Right: $t = 30000$.

Homoclinic Stripes: Experiment 3 (GM)

Experiment 3: Take $(p, q, r, s) = (2, 1, 2, 0)$, $\varepsilon_0 = 0.025$, and $D_0 = 6.8$. Since $D_0 < D_{0c}$, there is no equilibrium stripe solution. The stripe then splits into two and develops a zigzag instability. The wriggled stripes undergo a breakup instability near the points of maximum curvature of the stripe. Since $D_0 < D_{0s}$ the emerging spots then undergo a spot-splitting process.

Click here for the movie

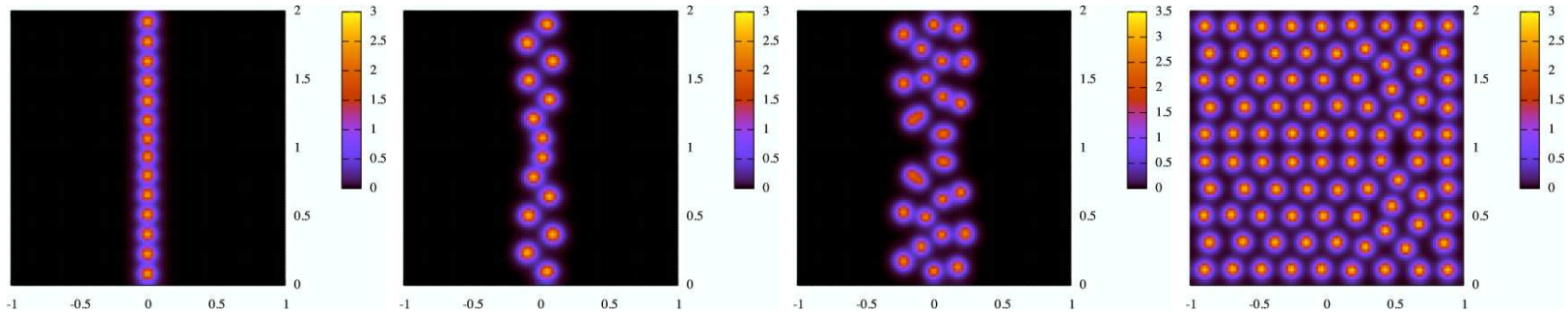


Left: $t = 75$ Left Middle: $t = 175$ Right Middle: $t = 225$ Right: $t = 250$.

Click here for another movie for a smaller D.

Homoclinic Stripes: Experiment 4 (GM)

Experiment 4: Take $(p, q, r, s) = (3, 2, 2, 0)$, $\varepsilon_0 = 0.025$, and $D_0 = 4.5$. In addition to the zigzag instability, a breakup instability is guaranteed since the breakup band does not terminate in the weak interaction regime. From theory the most unstable breakup mode predicts $N = \frac{m}{\pi} = \frac{0.96}{\pi\varepsilon_0} \approx 12$ spots. The most unstable zigzag mode predicts $N = \frac{m}{\pi} = \frac{0.43}{\pi\varepsilon_0} \approx 5$ crests. The numerical results show that the initially straight stripe breaks up into fourteen spots and then develops a zigzag instability with six crests. Since $D_0 < D_{0s} = 5.23$ the resulting spots then undergo a spot-splitting process.

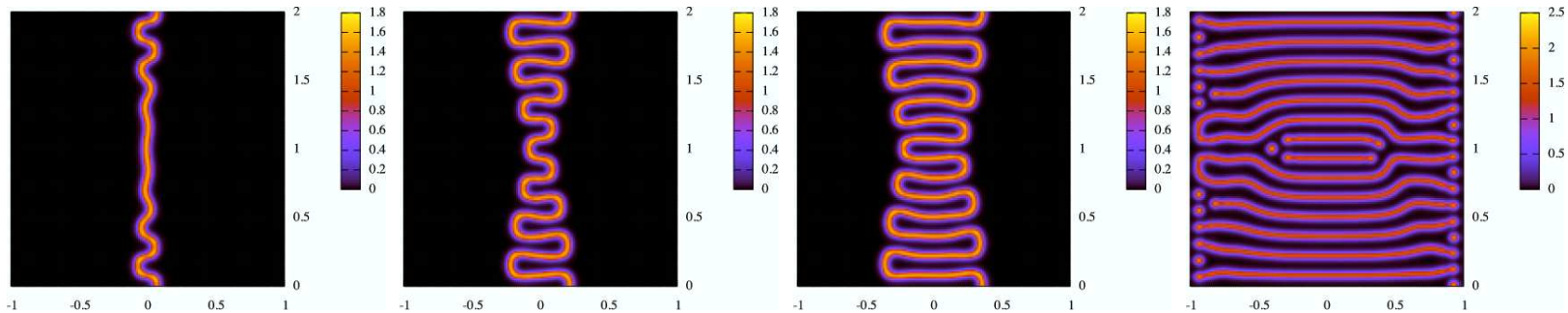


Left: $t = 100$ Left Middle: $t = 200$ Right Middle: $t = 250$ Right: $t = 1000$.

Homoclinic Stripes: Experiment 5 (GM)

Experiment 5: Take $(p, q, r, s) = (2, 1, 3, 0)$, $\varepsilon_0 = 0.01$, and $D_0 = 14.0$. For this set there are no breakup instabilities when $10.35 < D_0 < 19.14$. Therefore, we expect no breakup instability. From theory the expected number of zigzag crests is $N = \frac{m}{\pi} = \frac{0.23}{\pi\varepsilon_0} \approx 7$. The numerical results show a zigzag instability with seven crests. The zigzag instability leads to a curve of increasing length.

Click here for the movie



Left: $t = 1400$ Left Middle: $t = 1600$ Right Middle: $t = 1800$ Right: $t = 4300$.

Homoclinic Stripes: Non-Zero Curvature I

For a stripe $\mathcal{C}$ of non-constant curvature κ in the weak interaction regime, the stripe profile and the stability analysis for the stripe profile are, to leading order in ε_0, the same as that of a straight stripe. The key assumption that we need is $\kappa \ll O(\varepsilon_0^{-1})$.

For a curved stripe, the local y is replaced by the local distance η perpendicular to the stripe cross-section.

For the stability of this curved stripe we let $a = a_e(\eta) + e^{\lambda t + im\sigma}\Phi(\eta)$, where σ is arclength along $\mathcal{C}$. If $\mathcal{C}$ is closed, and L is the length of $\mathcal{C}$, we have $m = 2\pi k/L$. Thus, the most unstable zigzag or breakup mode is

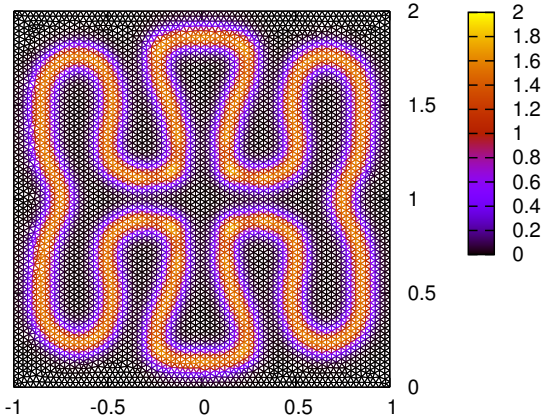
$$k = \frac{(\varepsilon_0 m)L}{2\pi\varepsilon_0}\,.$$

Here $\sqrt{\mu} = \varepsilon_0 m$ is computed from the eigenvalue problem for a straight stripe.

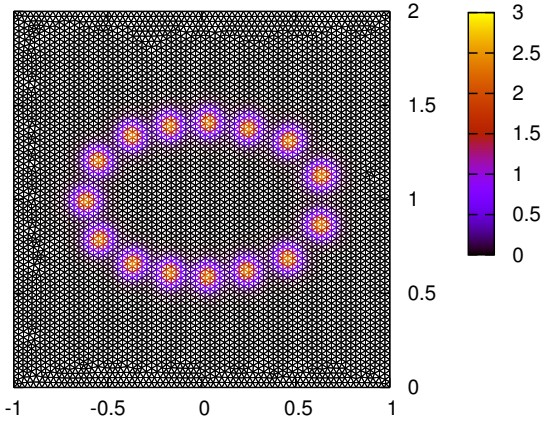
Conjecture: A zigzag instability in the absence of a breakup instability can lead to some type of space-filling labyrinthine pattern.

Homoclinic Stripes: Non-Zero Curvature II

Experiment 6: Consider the GM model with exponent set $(2, 1, 3, 0)$, $\tau = 0.1$ and $\varepsilon_0 = 0.02$ in $\Omega = [-1, 1] \times [0, 2]$. Suppose that the initial data concentrates on the ellipse $\mathcal{C} : x_1^2 + 2(x_2 - 1)^2 = 1/4$ in Ω. Then, since $L = 2.71$ for $\mathcal{C}$, we use the eigenvalue computation for a straight stripe to predict the following:



- For $D_0 = 14$ we predict only zigzags with most unstable mode $\varepsilon_0 m \approx 0.227$ and $\lambda = 0.0131$. Thus, we predict $k = 5$ zigzag crests. Plot at $t = 625$. The movie.



- For $D_0 = 20$ we predict a breakup with with most unstable mode $\varepsilon_0 m \approx 0.682$ and $\lambda = 0.0145$. Thus, we predict $k = 15$ spots. Plot at $t = 187$. The movie.

Homoclinic Stripes: Weak Interaction I (GS)

Qualitatively similar results can be obtained for the GS model in the weak interaction regime in that a "fattened" stripe can be stable wrt respect to breakup instabilities but unstable wrt zigzags. The homoclinic problem is to seek even solutions with $v \to 0$ and $u \to 1$ as $|y| \to \pm\infty$ to

$$v_{yy} - v + Auv^2 = 0\,, \quad D_0 u_{yy} - (u-1) - uv^2 = 0\,.$$

The existence problem involves A and D_0 and is very complicated (AUTO: Rademacher, MJW). Instead we take a path $A = A(\beta)$, $D_0 = D_0(\beta)$ with β a parameter, which passes through $(A, D_0) = (2.02, 4.667)$, where Pearson (1993) observed wriggled stripes.

The eigenvalue problem for a straight or a curved stripe is

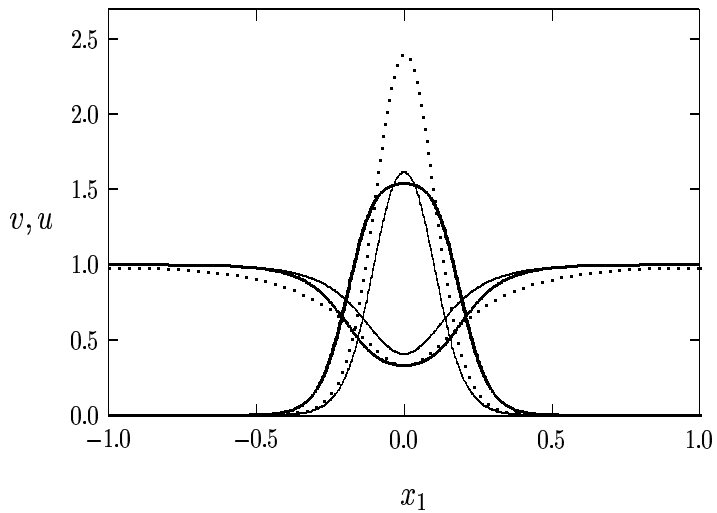
$$\Phi_{yy} - (1+\mu)\Phi + 2Auv\Phi + Av^2 N = \lambda\Phi\,,$$
$$D_0 N_{yy} - (1 + D_0\mu)\,N - 2uv\Phi - v^2 N = \lambda N\tau\,.$$

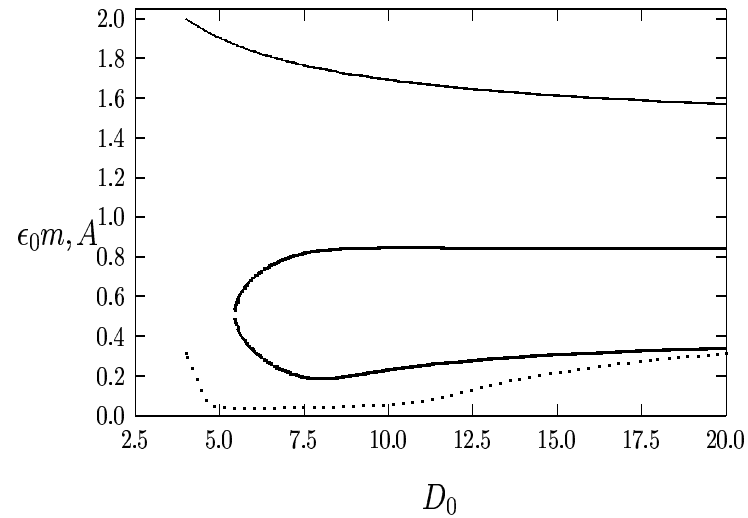
Here $\mu \equiv \varepsilon_0^2 m^2$. An unstable integer mode k is related to the length L of a closed concentration curve $\mathcal{C}$ and an unstable m by

$$k = \frac{(\varepsilon_0 m)L}{2\pi\varepsilon_0}\,.$$

Homoclinic Stripes: Weak Interaction II (GS)

By solving the homoclinic and eigenvalue problems numerically we get:



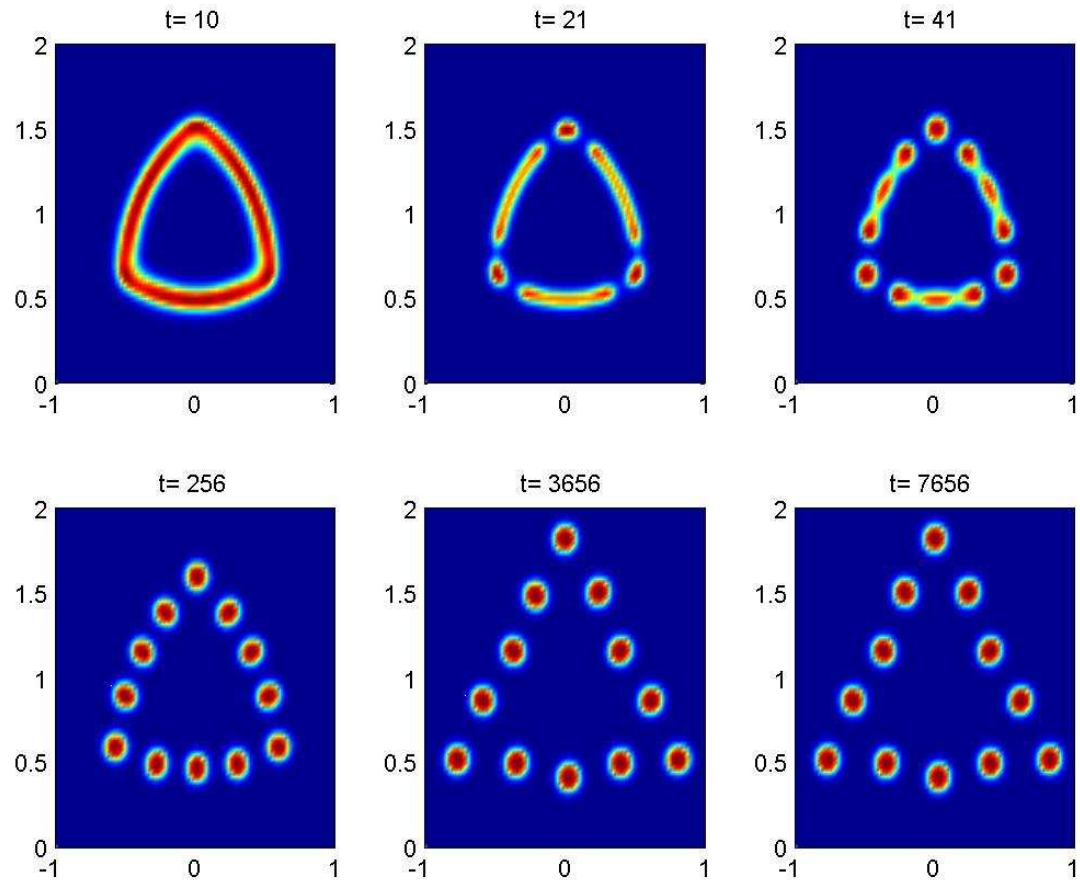


Left figure: u and v for $(D_0, A) = (19.91, 1.57)$ (dashed curve), $(D_0, A) = (5.94, 1.84)$, (solid curve), and $(D_0, A) = (4.0, 2.0)$ (heavy solid curve). The pulse "fattens" as D_0 decreases and A tends to 2 due to a nearby heteroclinic connection.

Right figure: the breakup instability thresholds (heavy solid curves) and the upper zigzag instability threshold (dashed curve). The top solid curve is the path for $A = A(D_0)$. The breakup band disappears when $D_0 = 5.46$ and $A = 1.87$. Thus, for $D_0 < 5.46$ on this path $A = A(D_0)$, there is a zigzag instability in the absence of a breakup instability.

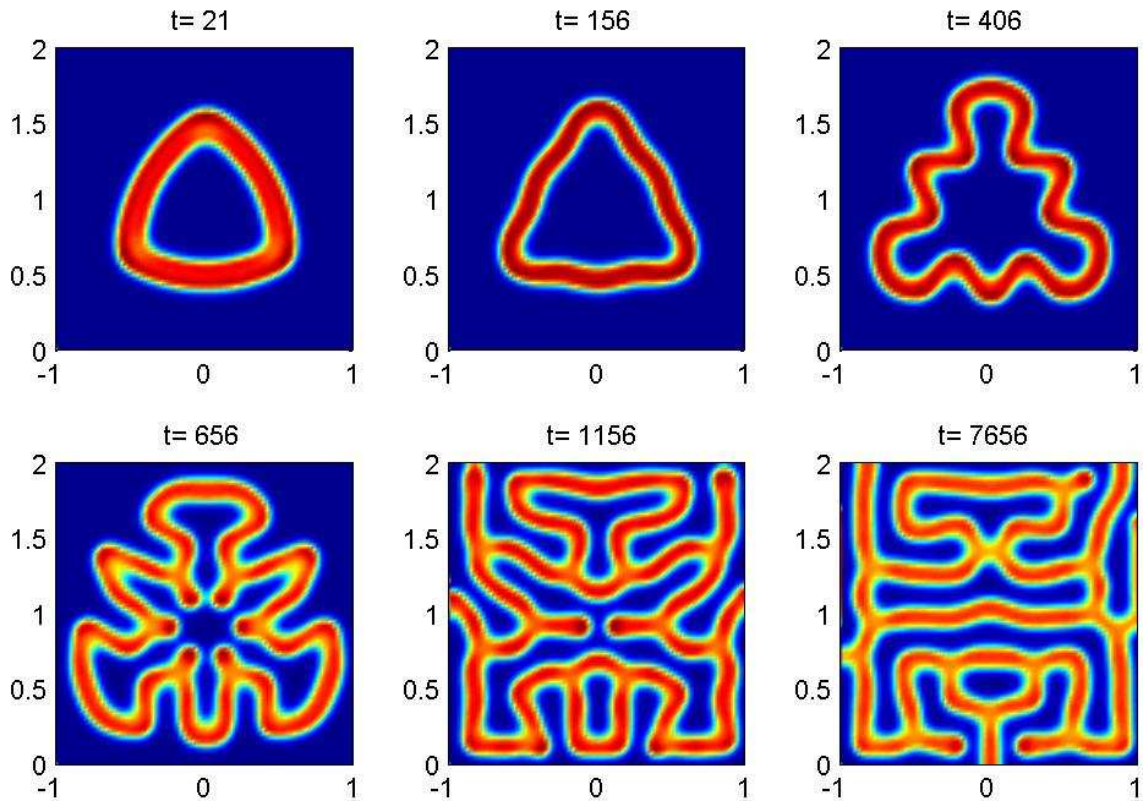
Homoclinic Stripes: Weak Interaction III (GS)

Experiment 1: Let $\varepsilon_0 = 0.02$, $D = 6.08\varepsilon_0^2$ (i.e. $D_0 = 6.08$), $A = 1.83$, $\tau = 1$ in $\Omega = [-1, 1] \times [0, 2]$. The initial condition concentrates on a triangular curve with $L \approx 3.14$. The unstable breakup band for $D_0 = 6.08$ is $0.316 < \varepsilon_0 m < 0.703$, with the most unstable mode roughly in the center. With $m = 2\pi k/L$ we get $2\pi\varepsilon_0 k/L \approx 0.51$. Since $\varepsilon_0 = 0.02$ and $L \approx 3.14$, we predict $k \approx 12$ spots (as observed).



Homoclinic Stripes: Weak Interaction IV (GS)

Experiment 2: Take $\varepsilon_0 = 0.02$, $A = 2$, $\tau = 1$, and $D = 4\varepsilon_0^2$, with the same initial data. For these A and D there is only a zigzag instability band and no breakup band. The most unstable zigzag mode is roughly $\varepsilon_0 m \approx 0.2$. With $m = 2\pi k/L$, and $L \approx 3.14$, we predict $k \approx 5$ crests. The full numerical results give a $k = 6$ zigzag instability, and the subsequent temporal development of a large-scale labyrithinian pattern.



Mesa Stripes: GM with Small Saturation I

Consider the GM model with saturation $k > 0$ on $\Omega = [-1, 1] \times [0, d_0]$:

$$a_t = \varepsilon^2 \Delta a - a + \frac{a^2}{h\,(1 + ka^2)}\,, \quad \tau h_t = D\Delta h - h + \frac{a^2}{\varepsilon}\,.$$

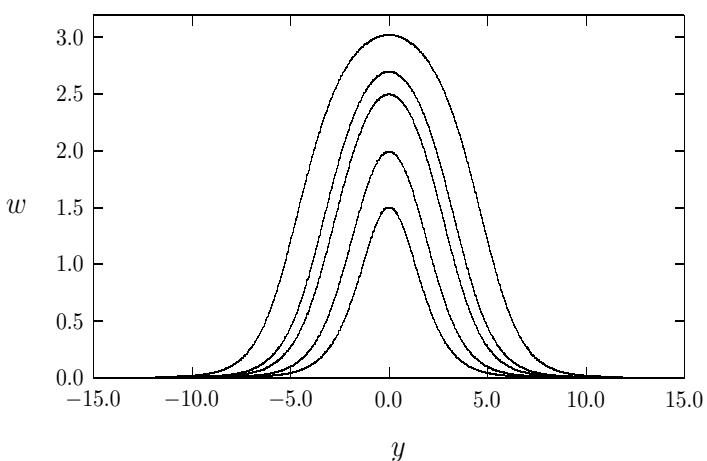
For $\varepsilon_0 \ll 1$, there is a homoclinic stripe solution

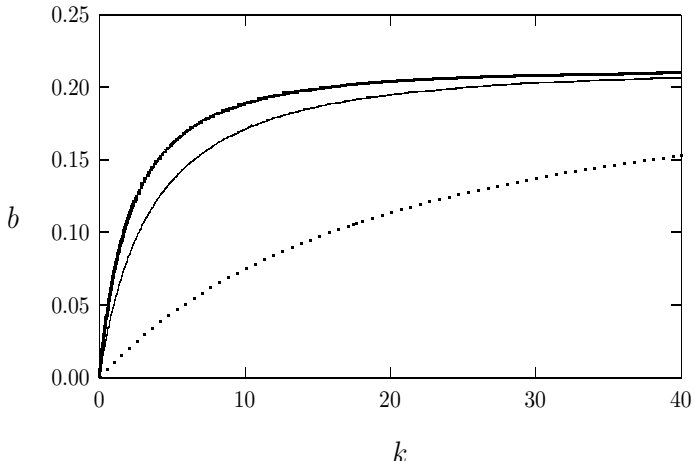
$$a \sim \mathcal{H} w\left[\varepsilon^{-1} x_1\right]\,, \quad h \sim \mathcal{H}\frac{\cosh\left[\theta_0(1 - |x_1|)\right]}{\cosh\theta_0}\,, \quad \theta_0 \equiv \frac{1}{\sqrt{D}}\,,$$

$$\mathcal{H} \equiv \frac{2\tanh\theta_0}{\beta\theta_0}\,, \quad \beta \equiv \int_{-\infty}^{\infty} w^2\,dy\,, \quad b\beta^2 = 4kD\tanh^2(\theta_0)\,.$$

For $0 < b < b_0 \approx 0.21138$, w is the fat homoclinic solution to

$$w'' - w + w^2/(1 + bw^2) = 0\,, \quad w(\pm\infty) = 0\,.$$





Mesa Stripes: GM with Small Saturation II

A singular perturbation analysis leads to the NLEP governing breakup instabilities:

$$L_0\Phi - \frac{\chi_m w^2}{1+bw^2}\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w^2\, dy} = (\lambda + \varepsilon^2 m^2)\Phi\,, \quad \Phi \to 0\,, \text{ as } |y| \to \infty$$

$$\chi_m(\lambda) \equiv \frac{2\theta_0 \tanh\theta_0}{\theta_\lambda \tanh\theta_\lambda}\,, \quad \theta_\lambda \equiv \sqrt{m^2 + \frac{(1+\tau\lambda)}{D}}\,, \quad \theta_0 \equiv \frac{1}{\sqrt{D}}\,,$$

$$L_0\Phi \equiv \Phi'' - \Phi + \frac{2w}{(1+bw^2)^2}\Phi\,.$$

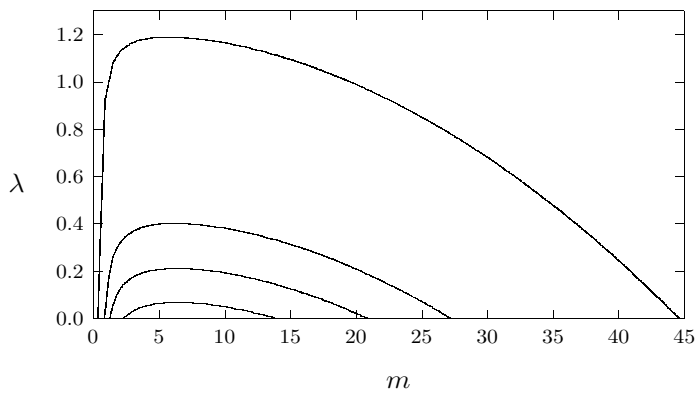
Since the pulse is stable when $m = 0$, and that $\chi_m \to 0$ as $m \to \infty$, the breakup instability band has the form

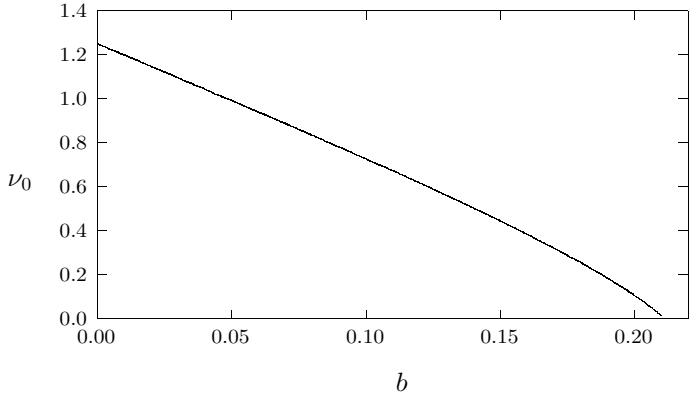
$$m_{b-} < m < \sqrt{\nu_0}/\varepsilon\,.$$

Here $\nu_0 = \nu_0(b)$ is the principal eigenvalue of the local operator L_0. Since $\nu_0 \to 0$ as $b \to b_0 = 0.211$ corresponding to a fat homoclinic, it suggests that the breakup band disappears for some b_0 slightly below b_0.

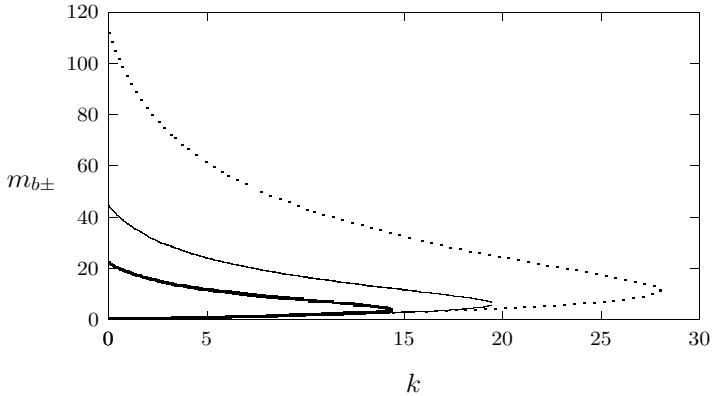
Mesa Stripes: GM with Small Saturation III

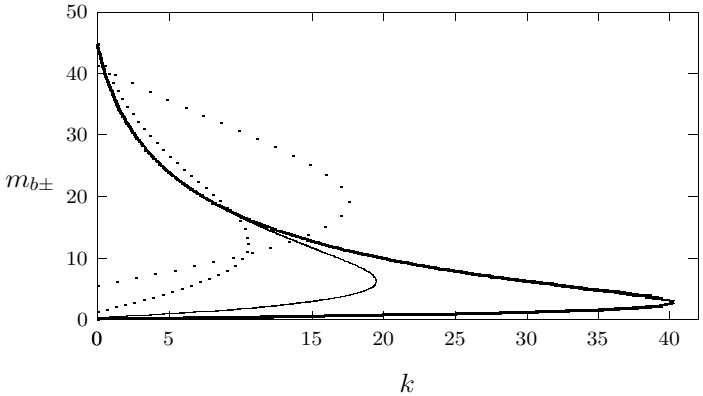
From numerical computations of the spectrum of the NLEP we get:





Left: Most unstable λ versus m for $D = 10.0$, $\varepsilon = 0.025$ with $k = 0$, $k = 3.6$, $k = 6.8$, and $k = 12.5$. The band disappears near $k = 12.5$. **Right:** principal eigenvalue ν_0 of L_0 vesus b.





Left: $m_{b\pm}$ versus k for $D = 10$ and different ε. **Right:** $m_{b\pm}$ versus k for $\varepsilon = 0.025$ and different D.

Mesa Stripes: GM with Large Saturation

On a rectangular domain $0 < x_1 < 1, 0 < x_2 < d_0$, we consider

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^2}{h\,(1 + \kappa a^2)}\,, \quad \tau h_t = D\Delta h - h + a^2\,.$$

- If we set $\kappa = \varepsilon_0^2 k$ and enlarge a and h by a factor of $1/\varepsilon_0$ we recover the small saturation formulation.
- The constant $\kappa > 0$ is the saturation parameter. For $\kappa > 0$, with $\kappa = O(1)$, the numerical simulations of Koch, Meinhardt (1994), Meinhardt (1995) suggest that a stripe will not disintegrate into spots. This model seems to robustly support stable stripe patterns. Theoretical Analysis?
- For $\kappa = O(1) > 0$ we will construct a mesa stripe centered about the mid-line $x_1 = 0$ when the inhibitor diffusivity is asymptotically large with $D = O(\varepsilon_0^{-1})$. We write

$$D = \frac{\mathcal{D}}{\varepsilon_0}\,.$$

The Equilibrium Mesa Stripe

For $\varepsilon_0 \ll 1$, the mesa stripe is given by

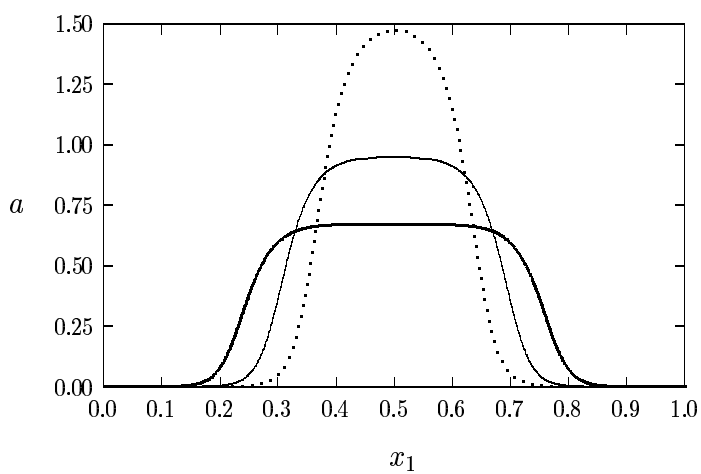
$$a \sim \mathcal{H}\left[w_l + w_r - w_+\right] , \quad h \sim \mathcal{H} = \frac{1}{w_+^2 L} ,$$

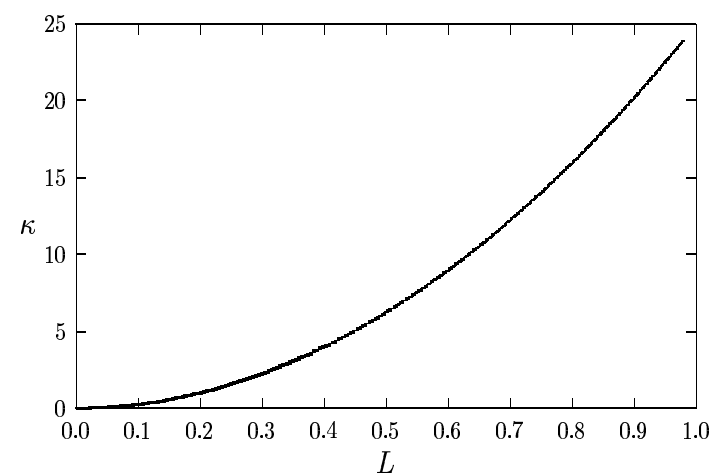
$$w_l(y_l) \equiv w\left[\varepsilon_0^{-1}(x_1 - \xi_l)\right] , \quad w_r(y_r) \equiv w\left[\varepsilon_0^{-1}(\xi_r - x_1)\right] ,$$

$$L \equiv \xi_r - \xi_l \sim \frac{\sqrt{\kappa}}{\sqrt{b_0} w_+^2} < 1 .$$

Here L is the length of the mesa plateau. With $b \equiv \kappa \mathcal{H}^2$, there is a heteroclinic solution $w(y)$ for $b_0 \approx 0.21138$ to

$$w'' - w + \frac{w^2}{1 + b_0 w^2} = 0 , \quad w(\infty) = w_+ \approx 3.295 , \quad w(-\infty) = 0 .$$





The Stability of the Mesa Stripe I

Let a_e, h_e be the equilibrium mesa stripe. We linearize as

$$a = a_e + e^{\lambda t + imx_2}\phi\,, \quad h = h_e + e^{\lambda t + imx_2}\psi\,, \quad m = \frac{k\pi}{d_0}\,, \quad k = 1, 2, \ldots\,,$$

where $\phi = \phi(x_1) \ll 1$ and $\psi = \psi(x_1) \ll 1$. We will assume that $\tau = O(1)$. For $\varepsilon_0 \ll 1$ the eigenfunction for ϕ has the form

$$\phi \sim \begin{cases} c_l\left(w'(y_l) + O(\varepsilon_0) + \cdots\right)\,, & y_l \equiv \varepsilon_0^{-1}(x_1 - \xi_l) = 0(1)\,, \\ \phi_i \equiv \mu\psi\,, & \xi_l < x_1 < \xi_r\,, \\ c_r\left(w'(y_r) + O(\varepsilon_0) + \cdots\right)\,, & y_r \equiv \varepsilon_0^{-1}(\xi_r - x_1) = 0(1)\,, \end{cases}$$

with $\mu \equiv \frac{w_+^2}{2-w_+}$. The eigenfunction for ψ on $0 < x_1 < 1$ has the form

$$\psi_{x_1x_1} - \theta^2\psi = -\frac{\varepsilon_0^2\mathcal{H}w_+^2}{\mathcal{D}}\left[c_r\delta(x_1 - \xi_r) + c_l\delta(x_1 - \xi_l)\right]\,,$$
$$\psi_{x_1}(0) = \psi_{x_1}(1) = 0\,.$$

The Stability of the Mesa Stripe II

From a singular perturbation analysis related to the SLEP method of Fujii and Nishiura, we find for $\varepsilon_0 \ll 1$ that there are no $O(1)$ positive eigenvalues and that the two $O(\varepsilon_0^2) \ll 1$ eigenvalues satisfy the matrix eigenvalue problem

$$\alpha(\lambda + \varepsilon_0^2 m^2)\boldsymbol{c} \sim \varepsilon_0^2 \left[\frac{1}{2}L(1-L)I - \mathcal{G}\right]\boldsymbol{c}\,, \qquad \alpha \equiv \frac{2\beta L\mathcal{D}}{w_+^2}\,.$$

Here I is the identity matrix. Also $\boldsymbol{c}$ and $\mathcal{G}$ are

$$\mathcal{G} \equiv \frac{1}{d^2 - e^2}\begin{pmatrix} d & e \\ e & d \end{pmatrix}\,, \quad \boldsymbol{c} \equiv \begin{pmatrix} c_l \\ c_r \end{pmatrix}\,,$$

$$d \equiv \theta_+ \coth(\theta_+ L) + \theta_- \tanh\left(\frac{\theta_-(1-L)}{2}\right)\,, \quad e \equiv \theta_+ \mathrm{csch}(\theta_+ L)\,.$$

Here $\theta_\pm$ are given by

$$\theta \equiv \begin{cases} \theta_- \equiv \left[m^2 + \frac{\varepsilon_0}{\mathcal{D}}\right]^{1/2}\,, & 0 < x_1 < \xi_l\,,\ \ \xi_r < x_1 < 1\,, \\ \theta_+ \equiv \left[m^2 + \frac{\varepsilon_0}{\mathcal{D}}\left(1 + \frac{2w_+}{L(w_+ - 2)}\right)\right]^{1/2}\,, & \xi_l < x_1 < \xi_r\,. \end{cases}$$

The Stability of the Mesa Stripe III

By calculating the spectrum of $\mathcal{G}$ we obtain

$$\lambda_\pm \sim \frac{\varepsilon_0^2}{\alpha}\left[-\alpha m^2 + \frac{L}{2}(1-L) - \sigma_\pm\right], \quad \boldsymbol{c}_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

Note that $\boldsymbol{c}_+$ is a breather mode while $\boldsymbol{c}_-$ is a zigzag mode. Here

$$\sigma_+ = \left[\theta_+ \tanh\left(\frac{\theta_+ L}{2}\right) + \theta_- \tanh\left(\frac{\theta_-(1-L)}{2}\right)\right]^{-1},$$

$$\sigma_- = \left[\theta_+ \coth\left(\frac{\theta_+ L}{2}\right) + \theta_- \tanh\left(\frac{\theta - (1-L)}{2}\right)\right]^{-1}.$$

The Stability of the Mesa Pulse

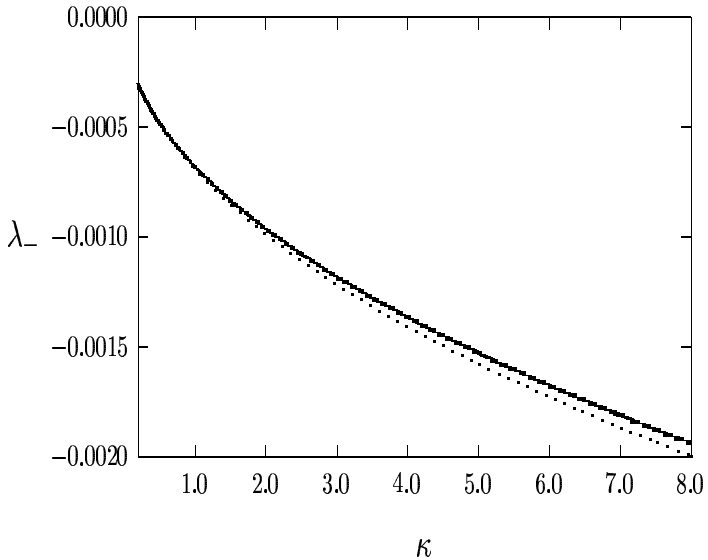
For $m = 0$ and $\varepsilon_0 \ll 1$, we calculate

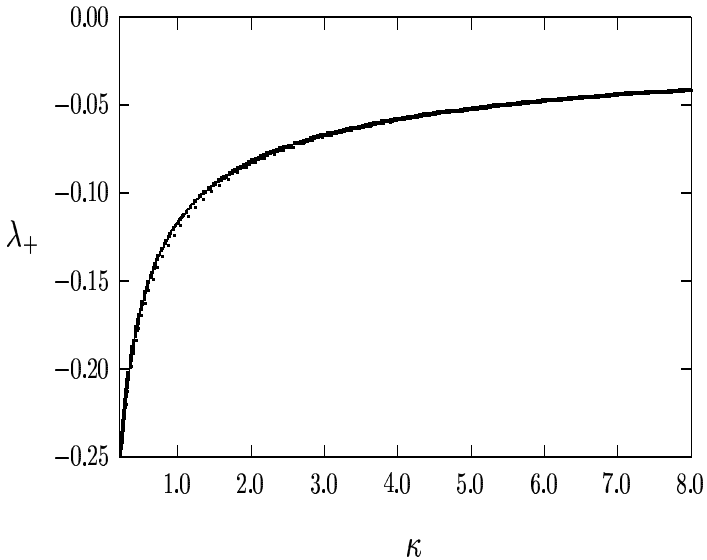
$$\lambda_- \sim -1.8112 \frac{\varepsilon_0^2 L}{\mathcal{D}} < 0\,, \quad \lambda_+ \sim -1.1899 \frac{\varepsilon_0}{L} < 0\,.$$

Thus $\lambda_\pm < 0$ for $m = 0$ and $\varepsilon_0 \ll 1$. Recall that $L = L(\kappa)$.

Hence, a one-dimensional Mesa pulse for $\tau = O(1)$, $\varepsilon_0 \ll 1$ and $D = O(\varepsilon_0^{-1})$ is always stable under the effect of activator saturation.

We plot $\lambda_\pm$ versus κ for $m = 0$, $D = 10$, and $\varepsilon_0 = 0.02$.



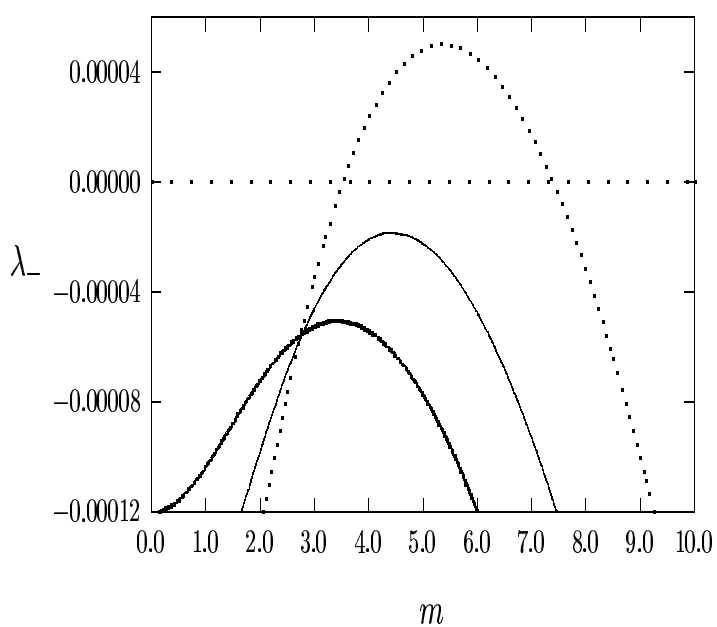


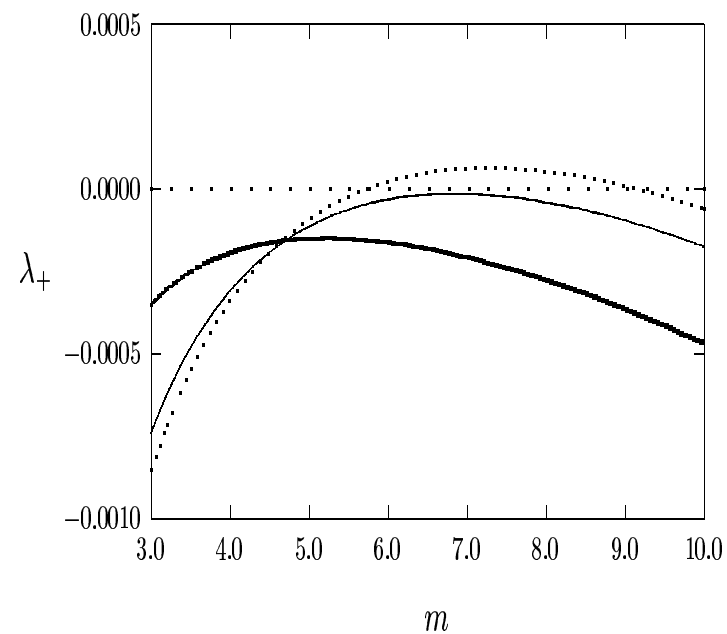
The Stability of the Mesa Stripe I

We can show $\lambda_\pm < 0$ for $m = 0$ (pulse) and for $m \gg 1$. However, for $\mathcal{D}$ below certain thresholds, there are unstable bands of zigzag and breather modes for $m = O(1)$.

Left figure: λ_- versus m for $\kappa = 2.0$ and $\varepsilon_0 = 0.0025$ when $D = 10.0$ (heavy solid curve), $D = 6.0$ (solid curve), and $D = 4.0$ (dashed curve).

Right figure: λ_+ versus m for $\kappa = 4.0$ and $\varepsilon = 0.0025$ when $D = 10.0$ (heavy solid curve), $D = 3.5$ (solid curve), and $D = 2.8$ (dashed curve).





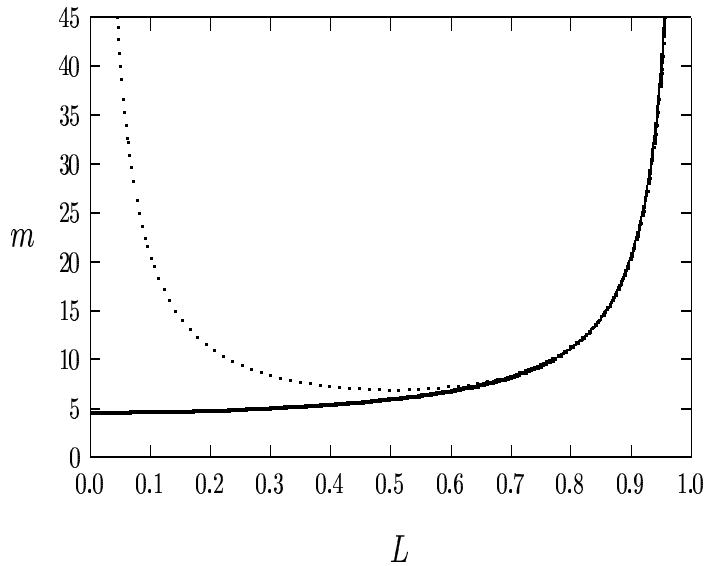
The Stability of the Mesa Stripe II

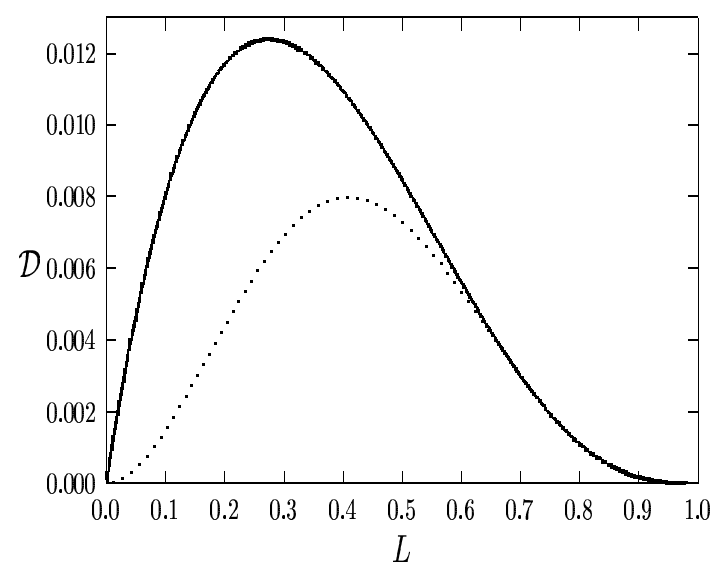
The critical values $\mathcal{D}_z$ and $\mathcal{D}_b$ where a zigzag and breather instability emerge at some $m = m_z$ and $m = m_b$ are roots of

$$\frac{m}{2}\frac{d\sigma_-}{dm} = \sigma_- - \frac{L}{2}(1-L)\,, \quad \mathcal{D}_z = \frac{w_+^2}{2\beta L m_z^2}\left[\frac{L}{2}(1-L) - \sigma_-\right]\,.$$

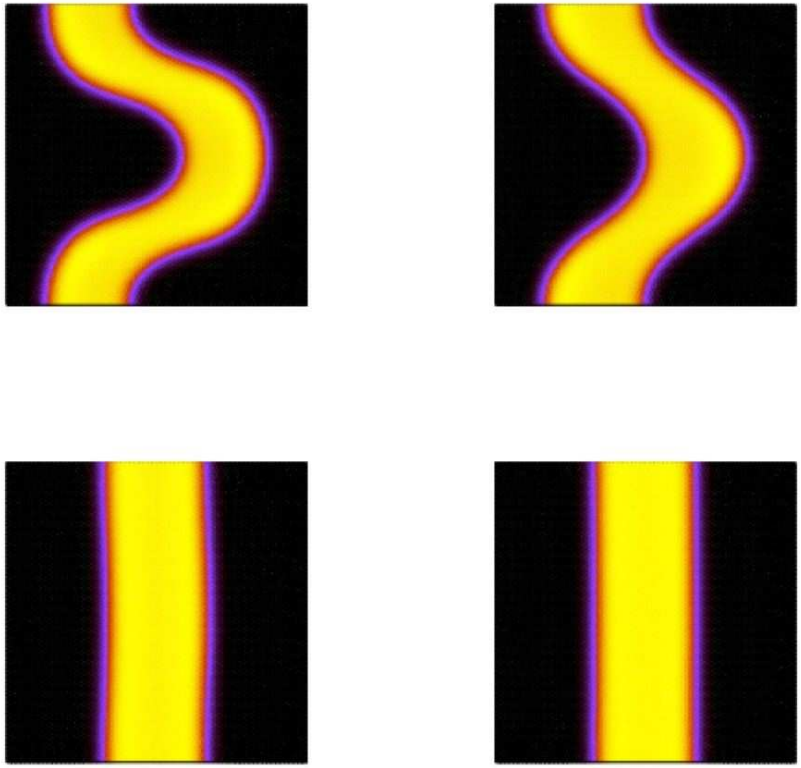
$$\frac{m}{2}\frac{d\sigma_+}{dm} = \sigma_+ - \frac{L}{2}(1-L)\,, \quad \mathcal{D}_b = \frac{w_+^2}{2\beta L m_z^2}\left[\frac{L}{2}(1-L) - \sigma_+\right]\,.$$

Since $\mathcal{D}_z > \mathcal{D}_b$ a zigzag instability always occurs before a breather instability as $\mathcal{D}$ is decreased.





Numerical Realization of Zigzag Instability



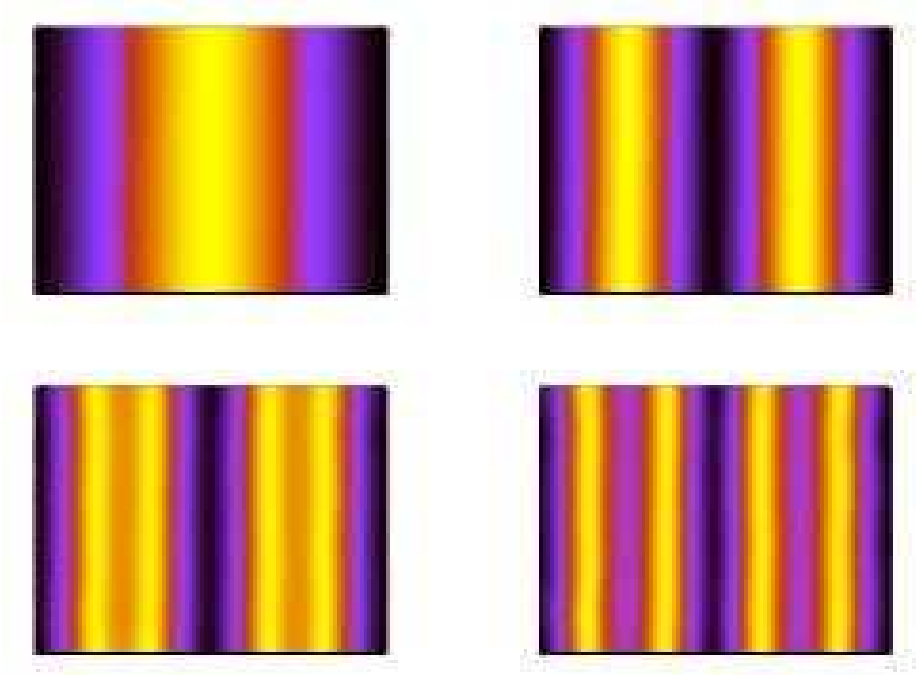
- Top left: $D = 0.6$ at $t = 10,000$.
- Top right: $D = 0.8$ at $t = 10,000$.
- Bottom left: $D = 1.0$ at $t = 20,000$.
- Bottom right: $D = 1.4$ at $t = 20,000$.

Experiment: The numerical solution to the saturated GM model in $[0,1] \times [0,1]$ for $\varepsilon_0 = 0.01$ and $\kappa = 1.92$. A zigzag instability occurs in each case, except when $D = 1.4$ which is above the zigzag instability threshold of $D = 1.24$.

Mesa Stripe-Splitting: Growing Domains

Pattern formation on growing domains: Kondo, Asai (Nature, 1995); Crampin, Maini (J. Math Bio. 2000-2002); Painter et al. (PNAS, 2001).

- Previous theory for $D \gg 1$. For the saturated GM model we can get new behavior: mesa-splitting for some $D = O(1)$ depending on κ
- Decreasing D and ε^2 exponentially slowly in time on a fixed size domain is equivalent to the problem of fixed ε and D on a slowly exponentially growing.domain.
- From a single stripe, can one robustly obtain further stripes through mesa-split events without triggering zigzag instabilities?



Experiment: Take $\kappa = 1.9$, $D(t) = 0.2e^{-0.0002t}$ and $\varepsilon(t) = 0.15 * D(t)$. First mesa-split at $t = 350$ when $D < 0.17$. Further splits occur and slow zigzag instabilities do not have time to develop. The movie.

Summary and Open Problems

Many semi-rigorous results are available for stripes, spots, and pulses in the GS and GM Models in the semi-strong regime in both $\mathbb{R}^1$ and in $\mathbb{R}^2$ where breakup instabilities are the dominant mechanism. Nonlocal Eigenvalue Problems are key here. Complicated patterns of zigzags, breakup, and spot-splitting occur near the existence threshold of a stripe in the weak interaction regime. Breakup instability bands can disappear for mesa stripes and for fat homoclinics.

Key Open problems:

- provide a rigorous analysis of stripes, spots, and pulses, for general classes of RD models in the semi-strong regime.
- provide some type of rigorous analysis in the weak interaction regime focusing on the disappearance of breakup bands.
- show analytically that the breakup band for the GM model with small saturation and fat homoclinics can disappear. Similar phenomena elsewhere?
- to study mesa-splitting analytically.
- to study breakup and zigzag instabilities of stripes and rings for the hybrid RD-chemotaxis systems of Painter et al. and Woodward et al.
- study stripe replication in slowly growing domains for RD systems.

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