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Outline of the Talk

1. An Eigenvalue Optimization Problem in a Planar Domain
   - Asymptotic expansions, the Neumann Green’s function, and Optimal Trap locations (Kolokolnikov, Titcombe, MJW).
   - Boundary Traps and a Narrow Escape Problem (Kolokolnikov, Pillay, MJW)
   - Analogous problems in a spherical domain (Cheviakov, Kolokolnikov, MJW)

2. Diffusion of Receptor Proteins on a Cylindrical Membrane
   - The reaction rate with one trap (Falcke, Straube, MJW)
   - Steady-state diffusion with many traps (Bressloff, Earnshaw, MJW)

3. Spot Solutions to Reaction-Diffusion Models
   - The stability of spots for the GM model (Kolokolnikov, MJW)
   - Self-Replication of spots for the Schnakenburg model on a growing domain (Kolokolnikov, MJW, Wei)
Eigenvalue Problem with Interior Traps

\[ \Delta u + \lambda u = 0, \quad x \in \Omega \setminus \Omega_p \; ; \quad \int_{\Omega \setminus \Omega_p} u^2 \, dx = 1, \]
\[ \partial_n u = 0 \quad x \in \partial \Omega, \quad u = 0 \quad x \in \partial \Omega_p. \]

Here \( \Omega_p = \bigcup_{i=1}^{N} \Omega_{\varepsilon_i} \) are \( N \) interior non-overlapping holes or traps, each of ‘radius’ \( O(\varepsilon) \ll 1 \). The holes are assumed to be identical up to a translation and rotation.

Also \( \Omega_{\varepsilon_i} \to x_i \) as \( \varepsilon \to 0 \), for \( i = 1, \ldots, N \). The centers \( x_i \) are arbitrary.
The Eigenvalue Optimization Problem

Goal: Let $\lambda_0 > 0$ be the fundamental eigenvalue. For $\varepsilon \to 0$ (small hole radius) find the hole locations $x_i$, for $i = 1, \ldots, N$, that maximize $\lambda_0$. In other words, chose the trap locations to minimize the lifetime of a wandering particle in the domain, i.e. where are the best places to fish?

Specific Questions:

- For $N = 1$ (one hole), is there a unique $x_0$ that maximizes $\lambda_0$? Can one find domains $\Omega$ where there are several values of $x_0$ that locally maximize $\lambda_0$?
- For the unit ball $\Omega = |x| \leq 1$, determine ring-type configurations of holes $x_1, \ldots, x_N$ that maximize $\lambda_0$.

For the Neumann problem, with $N$ circular holes each of radius $\varepsilon \ll 1$, Ozawa (Duke J. 1981) proved that

$$\lambda_0 \sim \frac{2\pi N \nu}{|\Omega|} + O(\nu^2), \quad \nu \equiv \frac{-1}{\log \varepsilon} \ll 1.$$ 

Since this is independent of $x_i$, $i = 1, \ldots, N$, we need the neglected $O(\nu^2)$ term to optimize $\lambda_0$. For the Dirichlet problem, Ozawa (1981) proved

$$\lambda_0 \sim \lambda_{0d} + 2\pi \sum_{i=1}^{N} [u_0(x_i)]^2 \nu + O(\nu^2).$$

To optimize $\lambda_0$, put the hole at a local maxima of $u_0$ (Harrell, (SIMA 2001)). For the Neumann or Dirichlet case, MJW, Henshaw, Keller (SIAP, 1993) showed

$$\lambda_0 \sim \lambda_*(\nu; x_1, \ldots, x_N) + O(\varepsilon/\nu),$$

where $\lambda_*$ (which “sums” all the log terms) satisfies a PDE that must be solved numerically. Highly accurate results for $\lambda_0$, but no analytical insight on how to optimize $\lambda_0$ wrt hole locations.
Eigenvalue Asymptotics I

A singular perturbation analysis shows that all of the logarithmic terms are contained in the solution to

\[ \Delta u^* + \lambda^* u^* = 0, \quad x \in \Omega \setminus \{x_1, \ldots, x_N\}, \]

\[ \int_{\Omega} (u^*)^2 \, dx = 1; \quad \partial_n u^* = 0, \quad x \in \partial \Omega, \]

\[ u^* \sim A_j \nu_j \log |x - x_j| + A_j, \quad x \to x_j, \quad j = 1, \ldots, N. \]

Here \( \nu_j \equiv -1/\log(\varepsilon d_j) \), where \( d_j \) is the logarithmic capacitance of the \( j \)th hole defined by

\[ \Delta_y v = 0, \quad y \notin \Omega_j \equiv \varepsilon^{-1} \Omega \varepsilon_j, \]

\[ v = 0, \quad y \in \partial \Omega_j, \]

\[ v \sim \log |y| - \log d_j + o(1), \quad |y| \to \infty. \]

The highlighted term together with the normalization condition provides \( N + 1 \) constraints for the \( N + 1 \) unknowns \( \lambda^* \) and \( A_j \), for \( j = 1, \ldots, N \).
Define the G-function $G(x, x_0, \lambda^*)$ for the Helmholtz operator as

$$
\Delta G + \lambda^* G = -\delta(x - x_0), \quad x \in \Omega; \quad \partial_n G = 0, \quad x \in \partial\Omega,
$$

$$
G(x, x_0, \lambda^*) = -\frac{1}{2\pi} \log |x - x_0| + R(x; x_0, \lambda^*).
$$

Here $R$ is its “regular part”. Then, $u^* = -2\pi \sum_{k=1}^{N} A_k \nu_k G(x; x_k, \lambda^*)$.

Satisfying the point constraint at each $x_j$ gives the homogeneous system

$$
A_j \left(1 + 2\pi \nu_j R(x_j; x_j, \lambda^*)\right) + 2\pi \sum_{\substack{k=1 \atop k \neq j}}^{N} A_k \nu_k G(x_j; x_k, \lambda^*) = 0, \quad j = 1, \ldots, N.
$$

Consider the first eigenvalue for which $\lambda^* \to 0$ as $\varepsilon \to 0$. Set the determinant to zero and then use for $\lambda^* \ll 1$ that

$$
G(x; x_0, \lambda^*) \sim -\frac{1}{|\Omega|\lambda^*} + G_m(x; x_0), \quad R(x; x_0, \lambda^*) \sim -\frac{1}{|\Omega|\lambda^*} + R_m(x; x_0),
$$

where $G_m$ and $R_m$ are the Neumann G-function and its regular part.


**Eigenvalue Expansion: A Two-Term Result**

**Principal Result:** For \(N\) small circular holes centered at \(x_1, \ldots, x_N\) with logarithmic capacitances \(d_1, \ldots, d_N\), then

\[
\lambda_0(\varepsilon) \sim \lambda^*, \quad \lambda^* = \frac{2\pi}{|\Omega|} \sum_{j=1}^{N} \nu_j - \frac{4\pi^2}{|\Omega|} \sum_{j=1}^{N} \sum_{k=1}^{N} \nu_j \nu_k (G)_{jk} + O(\nu^3).
\]

Here \(\nu_j \equiv -1/\log(\varepsilon d_j)\) and \((G)_{jk}\) are the entries of a certain Neumann Green's function matrix \(G\).

For \(N\)-circular holes each of radius \(\varepsilon\) (for which \(d_j = 1\)), then with \(\nu = -1/\log(\varepsilon)\),

\[
\lambda_0(\varepsilon) \sim \lambda^*, \quad \lambda^* = \frac{2\pi N \nu}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|} p(x_1, \ldots, x_N) + O(\nu^3),
\]

where

\[
p(x_1, \ldots, x_N) \equiv \sum_{j=1}^{N} \sum_{k=1}^{N} (G)_{jk}.
\]

Therefore, for \(N\) circular holes and \(\nu \ll 1\), \(\lambda_0\) has a local maximum at a local minimum point of the “Energy-like” function \(p(x_1, \ldots, x_N)\).
The Neumann Green’s Function

The Neumann Green’s function $G_m(x; x_0)$, with regular part $R_m(x; x_0)$, satisfies:

$$\Delta G_m = \frac{1}{|\Omega|} - \delta(x - x_0), \quad x \in \Omega,$$

$$\partial_n G_m = 0, \quad x \in \partial \Omega; \quad \int_{\Omega} G_m \, dx = 0,$$

$$G_m(x, x_0) = -\frac{1}{2\pi} \log |x - x_0| + R_m(x, x_0);$$

The Green’s matrix $\mathcal{G}$ is determined in terms of the hole-interaction term $G_m(x_i; x_j) \equiv G_{mij}$, and the self-interaction $R_m(x_i; x_i) \equiv R_{mi}$ by

$$\mathcal{G} \equiv \begin{pmatrix}
R_{m11} & G_{m12} & \cdots & \cdots & G_{m1N} \\
G_{m21} & R_{m22} & G_{m23} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
G_{mN1} & \cdots & \cdots & G_{mNN-1} & R_{mNN}
\end{pmatrix}.$$
One Hole: Uniqueness of Maximizer?

Corollary: For the case of one circular hole of radius $\varepsilon$, centered at $x_1$, then

$$\lambda_0(\varepsilon) \sim \frac{2\pi \nu}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|} R_m(x_1; x_1) + O(\nu^3), \quad \nu \equiv -1/ \log \varepsilon.$$  

Thus $\lambda_0$ is maximized for a hole location that minimizes $R_m(x_1; x_1)$.

Is there a unique point $x_1$ in $\Omega$ that minimizes $R_m(x_1; x_1)$, and consequently maximizes $\lambda_0$?

- Require properties of $R_m(x; x_1)$ and $\nabla R_{m0} \equiv \nabla R_m(x : x_1)|_{x=x_1}$ (complex analysis).

- In a symmetric dumbbell-shape domain $x_1$ is unique. However, multiple roots of $\nabla R_m = 0$ can occur in non-symmetric dumbbell-shape domains (proof by complex analysis).
Multiple Holes in the Unit Disk

Let $\Omega$ be the unit circle, so that $|\Omega| = \pi$. Then, $G_m$ and $R_m$ are

$$G_m(x; \xi) = -\frac{1}{2\pi} \log |x - \xi| + R_m(x; \xi)$$

$$R_m(x; \xi) = -\frac{1}{2\pi} \log \left| |x| \xi - \frac{\xi}{|\xi|} \right| + \frac{(|x|^2 + |\xi|^2)}{2} - \frac{3}{4}.$$

For the unit disk, the problem of minimizing $p(x_1, \ldots, x_N)$ is equivalent to the problem of minimizing the function $F(x_1, \ldots, x_N)$ defined by

$$F(x_1, \ldots, x_N) = -\sum_{j=1}^{N} \sum_{k=1}^{N} \log |x_j - x_k| - \sum_{j=1}^{N} \sum_{k=1}^{N} \log |1 - x_j \bar{x}_k| + N \sum_{j=1}^{N} |x_j|^2,$$

for $|x_j| < 1$ and $x_j \neq x_k$ when $j \neq k$.

We consider the restricted optimization problem where $F$ is optimized over certain ring-type configurations of holes. We then compare the results with those computed with optimization software from MATLAB.
A Related Concentration Problem: Unit Disk

Our eigenvalue optimization problem is equivalent to minimizing

\[ F(x_1, \ldots, x_N) = -\sum_{j=1}^{N} \sum_{k=1}^{N} \log |x_j - x_k| - \sum_{j=1}^{N} \sum_{k=1}^{N} \log |1 - x_j x_k| + N \sum_{j=1}^{N} |x_j|^2, \]

for \(|x_j| < 1\), and \(x_j \neq x_k\) when \(j \neq k\).

In contrast, by taking a certain limit of a variational formulation of the GL model of superconductivity in the unit disk, Lefter, Radulescu (1996) and Sandier, Soret (2000) showed that equilibrium vortices at \(x_1, \ldots, x_N\) inside the unit disk \(|x_j| < 1\) with a common winding number are located at a minimum point of the renormalized energy \(W\) defined by

\[ W(x_1, \ldots, x_N) = -\sum_{j=1}^{N} \sum_{k=1}^{N} \log |x_j - x_k| - \sum_{j=1}^{N} \sum_{k=1}^{N} \log |1 - x_j x_k|. \]

This problem differs from that of the eigenvalue problem only by the confinement potential \(N \sum_{j=1}^{N} |x_j|^2\).
One-Ring Configurations: Unit Disk

Two Patterns: I (one ring), II (ring with a center hole). Specifically,

\[ x_j = re^{2\pi ij/N}, \quad j = 1, \ldots, N, \quad (P \ I), \]
\[ x_j = re^{2\pi ij/(N-1)}, \quad j = 1, \ldots, N - 1, \quad x_N = 0, \quad (P \ II). \]

More generally, we can construct \( m \) ring patterns with \( m \) rings of radii \( r_1, \ldots, r_m \), with \( r_j < r_{j+1} \), inside the unit disk. Assume that there are \( J_k \) holes on the ring of radius \( r_k \). On the \( k^{th} \) ring, for \( k = 1, \ldots, m \), the centres of the holes are assumed to satisfy

\[ \xi_j^{(k)} = r_k e^{2\pi i j / J_k} e^{i\phi_k}, \quad j = 1, \ldots, J_k. \]

Here \( \phi_k \) is a phase angle with \( \phi_1 = 0 \).

For each pattern we can calculate \( p(x_1, \ldots, x_N) \) explicitly and then optimize over the ring radii.
Pattern I

Principal Result: (Pattern I): Let $N > 1$, then $p = p_*/(2\pi)$ satisfies

$$p_* = -N \log(Nr^{N-1}) - N \log (1 - r^{2N}) + r^2 N^2 - \frac{3N^2}{4}.$$ 

Hence $p(r)$ has a unique minimum at $r = r_c$, where

$$\frac{r^{2N}}{1 - r^{2N}} = \frac{N - 1}{2N} - r^2.$$ 

Left: 4 holes on a ring. Right: $p$ versus $r$ for $N = 2, 3, 4$ holes on a ring.
Pattern II

Principal Result: (Pattern II): Let $N > 1$, then $p = p_*(r)/(2\pi)$ satisfies

\[ p_* = -(N - 1) \log [(N - 1)r^N] + r^2N(N - 1) - \frac{3N^2}{4} \]

\[ - (N - 1) \log \left(1 - r^{2(N-1)}\right). \]

Hence $p(r)$ has a unique minimum at $r = r_c$, where

\[ \frac{r^{2N-2}}{1 - r^{2N-2}} = \frac{N}{N - 1} \left(\frac{1}{2} - r^2\right). \]

Left: $N = 2, 3, 4$ holes on a ring and a center hole. Right: 7 holes on a ring (heavy solid) and 6 holes on a ring with an extra center hole (dotted).
## Restricted Optimization: $m$-ring Patterns

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<th>$N$</th>
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<th>$p_{\text{min}}$</th>
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Table 1: Numerical results for the optimum configuration within the class of two and three-ring patterns with or without a centre hole. The first three columns indicate the optimum configuration, the minimum value of $p$, and the optimum ring radii. The last two columns correspond to the second best pattern. The notation [1](5,12) indicates a two-ring pattern with a centre hole, which has 5 and 12 holes on the inner and outer rings, respectively.
Figure 1: The optimum configurations for $N = 6$ to $N = 25$ holes within the class of two and three-ring patterns, with or without a centre hole.
Figure 2: The optimum configurations for $N = 6$ to $N = 25$ holes computed using the routine \textit{fminunc} of MATLAB. The values of $p$ for each pattern are given in the figure. The dotted circular lines are the optimal ring radii of the $m$-ring configurations of Proposition 5.5.
Comparison: Restricted and Full Optimization

Optimization with respect to radii (dots) is compared with a MATLAB optimization with respect to $2N$ variables.

Open: Is a Hexagonal Lattice the optimal arrangement for $N \gg 1$?

Open: Optimal Configurations in Other Domains such as a Square?
Eigenvalue Problem with Boundary Traps: I

Consider the 2-D problem with boundary traps

\[ \Delta u + \lambda u = 0, \quad x \in \Omega; \quad \int_\Omega u^2 \, dx = 1, \]

\[ \partial_n u = 0 \quad x \in \partial \Omega_r, \quad u = 0, \quad x \in \partial \Omega_a = \bigcup_{j=1}^N \partial \Omega_{\varepsilon_j}. \]

Assume that \( \partial \Omega_{\varepsilon_j} \to x_j \) as \( \varepsilon \to 0 \) and \( |\partial \Omega_{\varepsilon_j}| = 2\varepsilon \) for \( j = 1, \ldots, N \).

Then, with \( \nu \equiv -1/\log[\varepsilon/2] \), the first eigenvalue \( \lambda_1 \) satisfies

\[ \lambda_1 \sim \frac{\pi N \nu}{|\Omega|} - \frac{\pi^2 \nu^2}{|\Omega|} \sum_{j=1}^N \sum_{k=1}^N (G)_{jk} + O(\nu^3), \]

where \( (G)_{jk} \equiv G_m(x_j; x_k) \) for \( j \neq k \) and \( (G)_{jj} \equiv R_m(x_j; x_j) \) where

\[ \Delta G_m = \frac{1}{|\Omega|}, \quad x \in \Omega; \quad \int_\Omega G_m(x; x_0) \, dx = 0, \]

\[ \partial_n G_m = 0, \quad x \in \partial \Omega_r \setminus \{x_0\}, \quad G_m(x, x_0) = -\frac{1}{\pi} \log |x - x_0| + R_m(x, x_0). \]
Some consequences and open issues:

For one patch, $\lambda_1$ and the associated “outer” eigenfunction $u_1$ are

$$
\lambda_1 \sim \frac{\pi \nu}{|\Omega|} - \frac{\pi^2 \nu^2}{|\Omega|} R_m(x_1, x_1) + \cdots ,
$$

$$
u_1 \sim |\Omega|^{-1/2} - \frac{\pi \nu}{|\Omega|^{1/2}} G_m(x; x_1) + O(\nu^2) .
$$

Open: Is the point that minimizes $R_m$, and consequently maximizes $\lambda_1$, related to extrema of the boundary curvature?

For $N$ patches on the boundary of the unit disk, for which $R_m$ is independent of $x_j$, the optimal arrangement is to choose $x_j$ with $|x_j| = 1$ for $j = 1, \ldots, N$ such that $\mathcal{F}(x_1, \ldots, x_N)$ is minimized, where

$$
\mathcal{F}(x_1, \ldots, x_N) = - \sum_{j=1}^{N} \sum_{k=1 \atop k \neq j}^{N} \log |x_j - x_k| .
$$

Clearly, we must choose the roots of unity. Open: what about the optimal arrangement for a general boundary?
Narrow Escape: A Boundary Trap in 2-D

Consider Brownian motion with diffusivity $D$ in a 2-D domain $\Omega$ with a boundary that is insulated except for a small absorbing patch $\partial \Omega_a$ with $|\partial \Omega_a| = 2\varepsilon$. Assume that $\partial \Omega_a \to x_1$ as $\varepsilon \to 0$ and that the initial point for the Brownian motion is $X(0) = x \in \Omega$.

Let $v(x)$ be the mean first passage time (MFPT)

$$v(x) = E[\tau | X(0) = x].$$

It is well-known that $v(x)$ satisfies

$$\Delta v = -\frac{1}{D}, \quad x \in \Omega,$$

$$\partial_n v = 0 \quad x \in \partial \Omega_r; \quad v = 0, \quad x \in \partial \Omega_a.$$

Goal: Calculate $v(x)$ as $\varepsilon \to 0$.

Narrow Escape: A Boundary Trap in 2-D

Alternative More General Derivation: Expand \( v \) in terms of the eigenfunctions \( u_i \) for \( i \geq 1 \). Then,

\[
v = \frac{1}{D \lambda_1} \left( \int_{\Omega} u_1 \, dx \right) u_1 + \sum_{i=2}^{\infty} \frac{1}{D \lambda_i} \left( \int_{\Omega} u_i \, dx \right) u_i ,
\]

\[
v \sim \frac{1}{D \lambda_1} \left( \int_{\Omega} u_1 \, dx \right) u_1 + O(\varepsilon) .
\]

This follows since \( \lambda_1 = O(\nu) \) and \( \lambda_i = O(1) \) for \( i \geq 2 \), with \( \int_{\Omega} u_i \, dx = O(\varepsilon) \) for \( i \geq 2 \) by the Divergence theorem.

Use the asymptotics for \( \lambda_1 \) and the “outer” form for \( u_1 \). In 3-lines we get

\[
v(x) = E [\tau|X(0) = x] \sim \frac{|\Omega|}{\pi D} \left[ -\log \varepsilon + \log 2 + \pi (R_m(x_1; x_1) - G_m(x; x_1)) \right],
\]

where \( R_m \) and \( G_m \) are the surface Neumann G-functions.

For the unit disk, \( R_m = (8\pi)^{-1} \) and \( G_m \) is known/ For example, if \( x_1 = 1 \) and \( x = 0 \), then we readily recover a result of [SSH]:

\[
v(0) = E [\tau|X(0) = 0] \sim \frac{|\Omega|}{\pi D} \left[ -\log \varepsilon + \log 2 + \frac{1}{4} \right].
\]
Eigenvalues in 3-D Domains: Interior Traps I

In a 3-D bounded domain $\Omega$ consider

\[ \Delta u + \lambda u = 0, \quad x \in \Omega \setminus \Omega_p; \quad \int_{\Omega \setminus \Omega_p} u^2 \, dx = 1, \]

\[ \partial_n u = 0 \quad x \in \partial \Omega, \quad u = 0, \quad x \in \partial \Omega_p. \]

Here $\Omega_p = \bigcup_{i=1}^N \Omega_{\varepsilon_i}$, with $\Omega_{\varepsilon_i} \to x_i$ as $\varepsilon \to 0$ and non-overlapping.

The first eigenvalue has the asymptotics

\[ \lambda_1 \sim \frac{4\pi \varepsilon}{|\Omega|} \sum_{j=1}^N C_j - \frac{16\pi^2 \varepsilon^2}{|\Omega|} \sum_{j=1}^N \sum_{k=1}^N C_j C_k (\mathcal{G})_{jk} + O(\varepsilon^3). \]

Here $(\mathcal{G})_{jk} \equiv G_m(x_j; x_k)$ for $j \neq k$ and $(\mathcal{G})_{jj} \equiv R_m(x_j; x_j)$ where $G_m(x; \xi)$ and $R_m(x; \xi)$ are now the 3-D Neumann G-function.

Also $C_j$ is the electrostatic capacitance of the $j^{th}$ hole defined by

\[ \Delta v = 0, \quad y \notin \Omega_j = \varepsilon^{-1} \Omega_{\varepsilon_j}, \]

\[ v = 1, \quad y \in \partial \Omega_j; \quad v \sim -\frac{C_j}{|y|}, \quad |y| \to \infty. \]
Eigenvalues in 3-D Domains: Interior Traps II

The matrix $G$ can be found explicitly when $\Omega$ is the unit sphere. By summing series related to Legendre polynomials

$$G_m(x; \xi) = \frac{1}{4\pi |x - \xi|} + \frac{1}{4\pi |x|^2} + \frac{1}{4\pi} \ln \left[ \frac{2}{1 - |x||\xi| \cos \theta + |x|^2} \right]$$

$$+ \frac{1}{8\pi} (|x|^2 + |\xi|^2) - \frac{13}{20\pi}.$$

Here $r' = |x' - \xi|$, where $x' = x/|x|^2$ is the image point and $\theta$ is the angle between $x$ and $\xi$. The regular part $R_m(\xi, \xi)$ is

$$R_m(\xi, \xi) = \frac{1}{4\pi (1 - |\xi|^2)} - \frac{1}{4\pi} \log (1 - |\xi|^2) + \frac{|\xi|^2}{4\pi} - \frac{13}{20\pi}.$$

Open: Where are the optimal trap locations $x_j$ for $j = 1, \ldots, N$ inside the unit sphere that maximize the first eigenvalue? For identical traps we need to minimize the explicitly known function $p(x_1, \ldots, x_N) = \sum \sum G_{jk}$.

Open: What about more general domains such as $\Omega$ a cube. Here we need Ewald summation techniques to build the matrix $G$. 
**Eigenvalues in 3-D Domains: Boundary Traps**

Consider a spherical domain $\Omega$ with $N$-small non-overlapping absorbing boundary patches on an otherwise reflecting boundary.

Open: Where are the optimal locations to put $N$ small patches to maximize the first eigenvalue $\lambda_1$?

One might guess that the trap locations that maximize $\lambda_1$ are given by the minimum of the discrete energy $\mathcal{F}(x_1, \ldots, x_N)$ defined by

$$
\mathcal{F}(x_1, \ldots, x_N) = \sum_{j=1}^{N} \sum_{\substack{k=1 \atop k \neq j}}^{N} \frac{1}{|x_j - x_k|}, \quad |x_j| = 1.
$$

If so, this is a famous discrete optimization problem of finding the minimal discrete energy of “electrons” confined to the boundary of a sphere. This is related to the discovery of Carbon-60 molecules. Long list of references; E. Saff, N. Sloane, A. Kuijlaars etc..

The difficulty with this problem is that the number of local minima grow exponentially with $N$, and so finding the global minimum is not trivial computationally.
Biomembrane Surface Diffusion With Trap I

Consider the diffusion of proteins on the cylindrical surface (a biomembrane) of length $2L$ and radius $R$ having a small circular trap $\Omega_\delta = |x| \leq \delta$. The concentration with $x = (x, y)$, where $|x| < L$ and $|y| < \pi R$, satisfies

$$c_t = D \Delta c, \quad x \in \Omega \setminus \Omega_\delta,$$

$$\partial_x c = 0, \quad x = \pm L; \quad c, \partial_y c, \quad 2\pi R \text{ periodic in } y,$$

$$c = 0, \quad x \in \partial \Omega_\delta.$$

Initially, $c(x, 0) = c_0$. We want to calculate the reaction rate $k(t)$,

$$k(t) = D \int_{\partial \Omega_\delta} \nabla c|_{|x| = \delta} \cdot \hat{n} \, dS.$$
For $t \gg 1$, then $c(x, t) \sim d_0 \phi_0 e^{-\lambda_0 D t}$, where $d_0 = c_0 \int_{\Omega \setminus \Omega_\delta} \phi_0 \, dx$. Here $\lambda_0$ and $\phi_0$ are the principal eigenpair of

$$\Delta \phi + \lambda \phi = 0, \quad x \in \Omega; \quad \int_\Omega \phi^2 \, dx = 1,$$

$$\partial_x \phi = 0, \quad x = \pm L; \quad \phi, \partial_y \phi, \quad 2\pi R \text{ periodic in } y,$$

$$\phi = 0, \quad x \in \partial \Omega_\delta.$$

The principal eigenvalue has the following asymptotics for $\delta \ll 1$:

$$\lambda_0 \sim \frac{2\pi \nu}{|\Omega|} - \frac{4\pi^2 \nu^2}{|\Omega|} R_m(0; 0) + \frac{8\pi^3 \nu^3}{|\Omega|} \left( [R_m(0; 0)]^2 - \frac{G_{m2}(0; 0)}{|\Omega_0|} \right).$$

Here $\nu = -1/\log \delta$ and $|\Omega| = 4\pi LR$ is the area of the cylindrical surface. The reaction rate is given by

$$k(t) \sim c_0 D |\Omega| \lambda_0 e^{-\lambda_0 D t} \left( 1 - \frac{4\pi^2 \nu^2}{|\Omega|} G_{m2}(0; 0) \right).$$
Biomembrane Surface Diffusion With Trap III

The Neumann Green’s function $G_m(x; 0)$ with regular part $R_m(0; 0)$ satisfy

$$\Delta G_m = \frac{1}{|\Omega|} - \delta(x); \quad \int_{\Omega} G_m \, dx = 0,$$

$$\partial_x G_m = 0, \quad x = \pm L; \quad G_m, \partial_y G_m, \quad 2\pi R \text{ periodic in } y,$$

$$G_m(x; 0) \sim -\frac{1}{2\pi} \log |x| + R_m(0; 0) \quad \text{as } x \to 0.$$
By using Ewald-type summation techniques to extract $R_m$ from the slowly converging doubly-infinite series we calculate

$$R_m(0; 0) = \frac{1}{2\pi} \left( \frac{L}{6R} - \log \left( \frac{L}{R} \right) - 2 \sum_{n=1}^{\infty} \log \left( 1 - e^{-2nL/R} \right) \right).$$

The solution $G_{m2}$ has the Fourier series representation

$$G_{m2}(x; 0) = \frac{2}{|\Omega_0^s|} \left( \sum_{m=1}^{\infty} \cos \left( \frac{m\pi x}{R} \right) \left( \frac{\pi m}{4} \right)^4 + \sum_{n=1}^{\infty} \cos \left( \frac{nL y}{R} \right) \left( \frac{nL}{R} \right)^4 \right) + \frac{2}{|\Omega_0^s|} \left( \sum_{m,n=1}^{\infty} \frac{2 \cos \left( \frac{m\pi x}{R} \right) \cos \left( \frac{nL y}{R} \right)}{\left( \frac{m\pi}{R} \right)^2 + \left( \frac{nL}{R} \right)^2} \right).$$

Now, $G_2(0, 0)$ is readily evaluated by interchanging the infinite summations with the limiting procedure $x \to 0$, since the resulting infinite series are absolutely convergent. This gives

$$G_{m2}(0; 0) = \frac{1}{4\pi} \left( \frac{1}{45 \ R} \frac{L}{R} + \frac{R}{L} \sum_{n=1}^{\infty} \frac{1}{n^2 \sinh^2 \left( \frac{L}{R} n \right)} + \frac{R^2}{L^2} \sum_{n=1}^{\infty} \frac{\coth \left( \frac{L}{R} n \right)}{n^3} \right).$$
Diffusion of Protein Receptors I

The problem for the diffusion of protein receptors on a cylindrical dendritic membrane $\Omega = \{ |x| < L, |y| < 2\pi l \}$, with partially absorbing traps is

$$U_t = \Delta U, \quad x \in \Omega \setminus \Omega_p, \quad \Omega_p = \bigcup_{j=1}^N \Omega_{\varepsilon_j},$$

$$\partial_x U(-L, y) = -\sigma, \quad \partial_x U(L, y) = 0; \quad U, \partial_y U, \quad 2\pi l \text{ periodic in } y,$$

$$\varepsilon \partial_n U = -\kappa_j (U - T_j), \quad x \in \partial \Omega_{\varepsilon_j}, \quad j = 1, \ldots, N.$$

Here $\sigma > 0$ models the influx of protein receptors from the soma.

Here $\sigma$ models the influx of protein receptors from the soma.
Diffusion of Protein Receptors II

Define the average concentration $U_j$ on the boundary of the $j^{th}$ spine

$$U_j = \frac{1}{2\pi\varepsilon} \int_{\partial\Omega_{\varepsilon_j}} U \, dx.$$  

Within each spine $T_j(t)$ and $S_j(t)$ for $j = 1, \ldots, N$ satisfy coupled ODE’s of the form

$$T_j' = \mathcal{F}_j(T_j, S_j, U_j), \quad S_j' = \mathcal{H}_j(T_j, S_j).$$

This model is due to Bressloff and Earnshaw (Phys. Rev. E. (2007), J. Neuroscience (2006)). The 1-D problem was studied by them.

Calculation of the steady-state solution in terms of $\sigma$ and the locations of the dendritic spines.

Stability analysis: couples the stability of ODE’s within each spine to the stability problem for the “outer” diffusion equation.

Time dependent computations?
The 2-D steady-state problem fits in the same framework as the other problems. Define $U$ by

$$U = U - U_c(x), \quad U_c(x) = \frac{\sigma}{2L}(x - L)^2.$$ 

With $\nu = -1/\log \varepsilon$. By using inner-outer matching, $U$ satisfies

$$\begin{align*}
\Delta U &= -\frac{\sigma}{L}, \quad x \in \Omega \setminus \Omega_p, \\
\partial_x U(\pm L, y) &= 0; \quad U, \ \partial_y U, \quad 2\pi l \text{ periodic in } y, \\
U &\sim \nu A_j \log |x - x_j| + A_j + U_j - U_c(x_j), \quad \text{as } x \to x_j.
\end{align*}$$

The solution in terms of $G_m$ is $U = -2\pi \nu \sum_{j=1}^{N} A_j G_m(x; x_j) + \chi$. Then, we obtain $2N + 1$ equations for the unknowns $A_j, U_j$ and $\chi$:

$$2\pi \nu \sum_{j=1}^{N} A_j = \frac{\sigma}{L} |\Omega|, \quad \text{(Divergence Theorem)};$$

$$(1 + 2\pi \nu R_{m,j,j}) A_j + 2\pi \nu \sum_{i \neq j} A_i G_{m,ji} = U_c(x_j) - U_j + \chi, \quad \text{(Point constraint)};$$

$$2\pi \nu A_j = \kappa_j(U_j - T_j), \quad \text{(BC on each trap)}.$$
Spot Stability for the GM Model

The activator $a$ and inhibitor $h$ in a 2-D domain $\Omega$, with $\varepsilon \ll 1$ satisfy

$$a_t = \varepsilon^2 \Delta a - a + \frac{a^2}{h}, \quad \partial_n a = 0, \quad x \in \partial \Omega$$

$$\tau h_t = D \Delta h - h + \varepsilon^{-2} a^2, \quad \partial_n h = 0, \quad x \in \partial \Omega.$$

The problem has no variational structure. There are particle-like solutions for $a$, called spots, when $\varepsilon \ll 1$. Since $a$ is localized, $\varepsilon^{-1} a^2 \to \sum_{j=1}^{N} S_j \delta(x - x_j)$ in the “outer” region. Hence, the “outer” equilibrium problem for $h$ is solvable by Green’s functions.
**Spot Stability for the GM model: I**

By analyzing a leading order nonlocal eigenvalue problem (NLEP):

**Theorem:** [Winter, Wei, JNLS 2001] For $\tau = 0$, $\varepsilon \to 0$, and $D \geq O(-\ln \varepsilon)$, an $N$-spot equilibrium solution is stable on an $O(1)$ time-scale iff

$$D < D_N \sim \frac{|\Omega|}{2\pi \nu N}, \quad \nu \equiv -1/ \ln \varepsilon.$$  

- This leading-order term in a logarithmic expansion predicts that $D_N$ is independent of the spot locations $x_j$, $j = 1, \ldots, N$.
- We need higher order terms in the logarithmic series for $D_N$. As for the Neumann eigenvalue problem with traps, we anticipate

$$D_N \sim \frac{|\Omega|}{2\pi \nu N} + F(x_1, \ldots, x_N) + O(\nu), \quad \nu \equiv -1/ \ln \varepsilon.$$  

For a movie showing a spike collapse due to overcrowding [click here](http://example.com).

\[ \text{Graph of } u(x,y,t) \text{ showing spike collapse} \]
Spot Stability for the GM model: II

Upon including the next term in the logarithmic series for the stability analysis:

**Principal Result** [KW, 2006] Let $\tau = 0$, $\varepsilon \to 0$, $D \geq O(\nu^{-1})$ where $\nu \equiv -1/\ln \varepsilon$. Then, an $N$-spot quasi-equilibrium solution is stable on an $O(1)$ time-scale iff

$$D < D_N \sim \frac{|\Omega|}{2\pi\nu N} + |\Omega| \left( -p(x_1, \ldots, x_N) + \frac{2}{N} \min_{j=1,\ldots,N-1} c_j^t G c_j \right) + O(\nu).$$

Here $e^t = (1, \ldots, 1)$ and the $c_j$ correspond to an $N-1$ dimensional subspace perpendicular to $e$: i.e. $c_j^t e = 0$ for with $c_j^t c_j = 1$.

**Sketch:** Let $w$ be the radially symmetric ground state solution for the spatial profile of the activator. The NLEP problems for $\tau = 0$ are

$$\Delta \Phi - \Phi + 2w \Phi - \chi_j w^2 \int_{\mathbb{R}^2} w \Phi \, dy = \lambda \Phi, \quad j = 1, \ldots, N,$$

$$\chi_j \equiv \frac{2N \mu_j}{e^t G e}, \quad C c_j = \mu_j c_j, \quad C \equiv I + \frac{2\pi\nu D}{|\Omega|} e e^t + 2\pi\nu G.$$

To calculate the stability threshold set $\min \chi_j = 1$ and solve for $D = D_N$. 

Oxford – p.36
Spot Replication for the Gray-Scott Model: I

\[ v_t = \varepsilon^2 \Delta v - v + Auv^2, \quad \tau u_t = D \Delta u - (1 - u) - uv^2. \]

Spot splitting: 2-D GS Model: \( A = 3.87, D = 1, \varepsilon = 0.04: \) (Movie)

The Core Solution: Near the $j^{th}$ spot we introduce $U$, $V$, and $y$ by
$$u = \frac{\varepsilon}{A \sqrt{D}} U, \quad v = \frac{\sqrt{D}}{\varepsilon} V, \quad \text{and} \quad y = \varepsilon^{-1}(x - x_j).$$

The quasi-steady spatial profile for the $j^{th}$ spot, referred to as the core problem, is to look for radially symmetric solutions in $\mathbb{R}^2$ to
$$\Delta_y U - UV^2 = 0, \quad \Delta_y V - V + UV^2 = 0,$$
$$V \to 0, \quad U \sim S_j \log \rho + \chi(S_j) \quad \text{as} \quad \rho = |y| \to \infty.$$ 

Here $S_j = \int_0^\infty \rho UV^2 \, d\rho$ is a parameter, and $\chi(S_j)$ is to be computed. Notice the volcano pattern for $V$ when $S_j > s_v \approx 4.78$. 

![Graphs showing the solution profiles](image-url)
Spot Replication for the Gray-Scott Model: III

The Outer Problem: In the “outer” region away from the spots, $uv^2$ is approximated by delta functions. By including all logarithmic terms:

$$\Delta u + \frac{1}{D} (1 - u) = \frac{2\pi \nu}{A} \sum_{j=1}^{N} S_j \delta(x - x_j), \ x \in \Omega; \quad \partial_n u = 0, \ x \in \partial\Omega$$

$$u \sim \frac{S_j \nu}{A} \log |x - x_j| + \frac{1}{A} [S_j + \chi(S_j) \nu], \ \text{as} \ x \to x_j, \ j = 1, \ldots, N.$$  

Here $\nu = -1/\log \varepsilon$ and $A$ is related to $A$ by $A \equiv D^{-1/2} \varepsilon (-\log \varepsilon) A$.

The point constraint gives $N$ nonlinear algebraic equations for $S_j$:

$$A = S_j (1 + 2\pi \nu R_{jj}) + \nu \chi(S_j) + 2\pi \nu \sum_{k \neq j}^{N} S_k G(x_j; x_k), \ j = 1, \ldots, N.$$  

Here $R(x; \xi)$ and $G(x; \xi)$ correspond to the Reduced-Wave G-function:

$$\Delta G - \frac{1}{D} G = -\delta(x - \xi), \ x \in \Omega; \quad \partial_n G = 0, \ x \in \partial\Omega$$

$$G(x, \xi) = -\frac{1}{2\pi} \log |x - x_0| + R(x, \xi).$$
Spot Replication for the Gray-Scott Model: IV

Stability to Angular Perturbations: The stability of the core solution to $e^{im\theta}$ perturbations with $m \geq 2$ is determined by the eigenvalue problem

$$
\mathcal{L}_m \Phi - \Phi + 2UV\Phi + V^2N = \lambda \Phi, \quad \Phi \to 0, \quad \text{as } \rho \to \infty,
$$

$$
\mathcal{L}_m N - 2UV\Phi - V^2N = 0, \quad N \to 0, \quad \text{as } \rho \to \infty,
$$

$$
\mathcal{L}_m \zeta \equiv \zeta'' + \frac{1}{\rho} \zeta' - \frac{m^2}{\rho^2} \zeta.
$$

Goal: determine critical values $s_m$ of $S_j$ for which we have stability wrt mode $m \geq 2$ iff $S_j < s_m$. Note $N = N(\rho)$ and $\Phi = \Phi(\rho)$.

- We compute numerically that $s_2 \approx 4.31$, $s_3 \approx 5.44$, $s_4 \approx 6.14$. Thus a peanut-splitting instability of the $j^{th}$ spot will occur when its $S_j$ value satisfies $s_2 < S_j < s_3$. This leads to spot self-replication.

- The spot locations evolve dynamically on a slow (in $\epsilon$) time-scale. Thus, each $S_j$ depends not only on the fixed quantities $\Omega$, $D$, $A$, but also on all the drifting spot locations $x_j$.

- Thus, a (local) spot-splitting instability can be triggered at some time during the evolution of the collection of spots.
Spot Replication for the Gray-Scott Model: V

Let $\Omega$ be the unit circle and $\tau = 1$. For $N = 1$ and $x_1 = 0$, we get

$$R_{11} \equiv R(0; 0) = \frac{1}{2\pi} \left[ \frac{1}{2} \log D + \log 2 - \gamma - \frac{K'_0(\theta_0)}{I'_0(\theta_0)} \right], \quad \theta_0 = 1/\sqrt{D}.$$

Recall that

$$A = S \left( 1 + 2\pi \nu R_{11} \right) + \nu \chi(S).$$

Let $D = 1$ and $\varepsilon = 0.05$. Since $s_2 = 4.31$ and $\chi(s_2) = -1.79$ the spot-splitting threshold is $A_2 = 5.41$. Similarly $A_v = 5.79$, and $A_3 = 6.29$. Full numerics yields a threshold between $A = 5.6 \sim 5.7$. (Movie) for $A = 5.8$. (Movie) for $A = 7.2$. 

![Movie](Image1)

![Movie](Image2)
Spot Replication for the Gray-Scott Model: VI

- **NLEP Overcrowding Instability** when $A = O \left[ \varepsilon \left( - \log \varepsilon \right)^{1/2} \right]$. A positive real eigenvalue leads to spot-annihilation when there are too many.

- **Spot-Splitting Instability** when $A = O \left[ \varepsilon(- \log \varepsilon) \right]$ due to an instability of the core solution the angular mode $m = 2$ (peanut-splitting).

- **Annihilation–Creation Attractors** should be possible since the self-replication and NLEP thresholds are so close in 2-D. Nishiura has observed these numerically in a different parameter regime of the GS model. Imagine that for some spots, $S_j$ exceeds splitting threshold for some $j$. Then spots are created and all the $S_j$ decrease. This gives a smaller effective value of $A$ and we enter NLEP regime, where the spot over-crowding instability occurs. Some spots are destroyed, and the values of $S_j$ increase again.
Spot Replication: Schnakenburg Model I

The Schnakenburg model in $\Omega$ with no flux BC on $\partial \Omega$ is

$$v_t = \varepsilon^2 \triangle v + b - v + \mu v^2 , \quad \mu_t = D_u \triangle \mu + a - \mu v^2 .$$

Let $\nu = \varepsilon^{-2} v$, $\mu = \varepsilon^2 u$, and $D = \varepsilon^2 D_u$. Then,

$$v_t = \varepsilon^2 \triangle v + b \varepsilon^2 - v + \nu v^2 , \quad \varepsilon^2 u_t = D \triangle u + a - \varepsilon^{-2} \nu v^2 .$$

We neglect the $b \varepsilon^2$ term.

- The Schnakenburg model has been used as a prototype RD model to exhibit the effect of pattern generation by domain growth.


- Adiabatically slow domain growth of a square or a circle is equivalent to fixing $\Omega$ and decreasing $D$ and $\varepsilon^2$ at the same rate (neglect dilution term).
Spot Replication: Schnakenburg Model II

Goal: Explain Mode-Doubling in 2-D starting from \( N \) Spots:

The Core Solution: Near the \( j^{th} \) spot we let \( u = \frac{1}{\sqrt{D}} U, \ v = \sqrt{D} V, \) and \( y = \varepsilon^{-1}(x - x_j). \) We obtain the radially symmetric GS core problem in \( \mathbb{R}^2: \)

\[
\Delta_y U - UV^2 = 0, \quad \Delta_y V - V + UV^2 = 0, \\
V \to 0, \quad U \sim S_j \log \rho + \chi(S_j) \quad \text{as} \quad \rho = |y| \to \infty.
\]

The Outer Problem: Let \( \nu = -1/\log \varepsilon. \) The outer problem that accounts for all logarithmic terms is

\[
\Delta u = -\frac{a}{D} + \frac{2\pi}{\sqrt{D}} \sum_{j=1}^{N} S_j \delta(x - x_j), \quad x \in \Omega; \quad \partial_n u = 0, \quad x \in \partial\Omega \\
u \sim \frac{S_j}{\sqrt{D}} \log |x - x_j| + \frac{1}{\sqrt{D}} \left[ \frac{S_j}{\nu} + \chi(S_j) \right], \quad \text{as} \quad x \to x_j, \quad j = 1, \ldots, N.
\]

The solution in terms of the Neumann G-function is

\[
u = -\frac{2\pi}{\sqrt{D}} \sum_{j=1}^{N} S_j G_m(x; x_j) + \frac{u_c}{\sqrt{D}}.
\]
**Spot Replication: Schnakenburg Model III**

The point constraints and the div. theorem: \(N + 1\) equations for \(S_j\) and \(u_c\)

\[
n u_c = S_j (1 + 2\pi \nu R_{m,j}) + \nu \chi(S_j) + 2\pi \nu \sum_{k \neq j}^N S_k G_m(x_j; x_k), \quad j = 1, \ldots, N,
\]

\[
2\pi \sum_{j=1}^N S_j = \frac{a}{\sqrt{D}} |\Omega|.
\]

- For \(N = 1\), \(S_1 = \frac{a|\Omega|}{2\pi \sqrt{D}}\) is independent of \(G_m, R_m,\) and \(x_1\). When \(S_1 > s_2 = 4.31\), spot-splitting (i.e. mode-doubling) occurs (independent of the spot location \(x_1\)) when \(|\Omega|\) is sufficiently large.

- For \(N > 1\), we get \(u_c = O(\nu^{-1})\), and so \(S_j \sim S + O(\nu)\). Therefore, to leading order in \(\nu\), mode-doubling transitions of \(N\) spots will occur (approximately) simultaneously when

\[
|\Omega| > \frac{2\pi N s_2 \sqrt{D}}{a}, \quad s_2 = 4.31.
\]
Spot Replication: Schnakenburg Model IV

**A Simple Example**

Let $\Omega$ be the unit circle and $a = 4$. Then, starting with one initial spot, a spot-replication event will occur when $D$ crosses below $D_2$ where

$$D_2 = \left( \frac{a|\Omega|}{2\pi s^2} \right)^2 = \left( \frac{a}{2s^2} \right)^2 = \frac{4}{(4.31)^2} \approx 0.215.$$ 

**Full Numerics:** Left: $D = 0.19$ (split). Right: $D_2 = 0.22$ (split).
References


- C. Wan, MJW, *Spot-Replication and Annihilation for the 2-D Gray-Scott Model*, in progress.

- T. Kolokolnikov, MJW, J. Wei *Mode-Doubling of Spot Patterns for the 2-D Schnakenburg Model*, in progress.
Dirichlet Green’s Function: Regular Part

Consider the Dirichlet Green’s function $G_d$, with regular part $R_d$:

$$\nabla G_d = -\delta(x - x_0) \quad x \in \Omega; \quad G_d = 0, \quad x \in \partial\Omega,$$

$$R_d(x, x_0) = G_d(x, x_0) + \frac{1}{2\pi} \log |x - x_0|, \quad \nabla R_{d0} \equiv \nabla R_d(x, x_0)|_{x=x_0}.$$

For a strictly convex domain $\Omega$, $-R_{d0}$ is strictly convex, and thus there is a unique root to $\nabla R_{d0} = 0$. (B. Gustafsson, Duke J. (1990), Caffarelli and Friedman, Duke J. (1985)).

$\nabla R_{d0}$ can be found for certain mappings $f(z)$ of the unit disk as

$$f'(z_0) \nabla R_{d0} = -\frac{1}{2\pi} \left( \frac{z_0}{1 - |z_0|^2} + \frac{f''(\overline{z}_0)}{2f'(\overline{z}_0)} \right).$$

Let $B$ be the unit disk and $f(z) = \frac{(1-a^2)z}{z^2-a^2}$. Then $f(B)$ is a symmetric nonconvex dumbbell domain for $1 < a < 1 + \sqrt{2}$. Gustafson (1990) proved that $\nabla R_{d0} = 0$ has three roots when $1 < a < \sqrt{3}$.

Can one derive a similar result for the Neumann Green’s function?
**An Explicit Formula for $\nabla \mathbf{R}_{m0}$**

**Theorem:** (KW) Let $f(z)$ map the unit disk $B$ onto $\Omega$ satisfying:

(i) $f$ is analytic and is invertible on $\overline{B}$, with $\overline{f(z)} = f(\overline{z})$.

(ii) $f$ has only simple poles at $z_1, \ldots, z_k$, and $f$ is bounded at $\infty$.

(iii) $f = g/h$, with $g(z_i) \neq 0$, where $g$ and $h$ are analytic everywhere.

On the image $\Omega = f(B)$, let $R_m$ be the regular part of $G_m$. Then, at $x_0$, with $z_0 \in B$ satisfying $x_0 = f(z_0)$,

$$\nabla R_{m0} = \frac{\nabla s(z_0)}{f'(z_0)} , \quad \nabla s(z_0) = \frac{1}{2\pi} \left( \frac{z_0}{1 - |z_0|^2} + \frac{f''(\overline{z}_0)}{2f'(\overline{z}_0)} \right)$$

$$f'(\overline{z}_0) \left( f(z_0) - f\left(\frac{1}{\overline{z}_0}\right) \right) + \sum_j g(z_j)f'(\frac{1}{z_j}) \frac{z_j^2 h'(z_j)}{z_j^2 h'(z_j)} \chi$$

$$- \frac{g(z_j)f'(\frac{1}{z_j})}{z_j^2 h'(z_j)} , \quad \chi = \left( \frac{1}{z_j - \overline{z}_0} + \frac{z_j}{1 - z_j \overline{z}_0} \right)$$

The Zeroes of $\nabla R_{m0}$

Example 1: Let $f(z; a) = \frac{(1-a^2)z}{z^2-a^2}$. Then $f(B)$ is nonconvex for $1 < a < 1 + \sqrt{3}$. For any $a > 1$, the complex variable formula can be used to show that $\nabla R_{m0} = 0$ has exactly one root at $z = 0$, which maximizes $\lambda_0$ for $\nu \ll 1$. This is qualitatively different than for the Dirichlet problem.

Example 2: A boundary integral computes $\nabla R_{m0}$ for other nonconvex symmetric domains. The numerical results give only one root to $\nabla R_{m0} = 0$. The boundary of the domain shown is $(x, y) = (\sin^2 2t + \frac{1}{4} \sin t)(\cos(t), \sin(t)), t \in [0, \pi]$. The vector field $\nabla R_{m0}$ has a unique equilibrium at approximately $(0, 0.2)$. 
A Non-Uniqueness Result I

Is there a unique root of $\nabla R_{m_0} = 0$ in any simply-connected nonconvex domain? Not necessarily. Let $B$ be the unit ball and $\Omega = f(B)$ where

$$f(z) = -\frac{\kappa z}{(z - a)(z + b)}, \quad a = 1 + \varepsilon, \quad b = 1 + \varepsilon \gamma,$$

with $\kappa = (a - 1)(b + 1)$ and $f(1) = 1$. Then, for $\varepsilon \to 0$, the area of $\Omega$ is

$$|\Omega| \sim \frac{\pi(1 + \gamma^2)}{4\gamma^2}; \quad \gamma^2 = \text{ratio of area of big lobe to small lobe}$$

Let $\gamma > 1$. For $\varepsilon \to 0$, $\Omega = f(B)$ approaches the union of two circles; a larger circle centred at $(1/2, 0)$ of radius $1/2$, and a smaller circle centred at $(-1/(2\gamma), 0)$ of radius $1/(2\gamma)$. This is an asymmetric dumbbell.
A Non-Uniqueness Result II

- For $\varepsilon \to 0$, $\nabla R_m = 0$ has a unique root except on $1.5966 < \gamma < \sqrt{3}$.
- For a slightly asymmetric dumbbell, where $1 < \gamma < 1.5966$, the optimum place to maximize $\lambda_0$ is to put the trap in the channel region of the dumbbell, but shifted slightly towards the largest (right) lobe.
- For $\gamma \gg 1$, where the left lobe of the dumbbell is very small the optimum place to insert the trap is near the centre of the right lobe.
- A saddle-node bifurcation structure for $1.5966 < \gamma < \sqrt{3}$ where $\lambda_0$ has two local maxima and a local minimum.

$\gamma^2$ vs. $x_0$ for $\varepsilon = 0.01$ (solid), $\varepsilon = 0.03$ (dotted), and $\varepsilon = 0.05$ (dashed).