The Existence and Stability of Spike Equilibria in the
One-Dimensional Gray-Scott Model on a Finite Domain

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Abstract

Results for the existence, stability, and pulse-splitting behavior of spike patterns in the one-dimensional Gray-Scott model on a finite domain in the semi-strong spike-interaction regime are summarized. Conditions on the parameters for the existence of competition instabilities, synchronous oscillatory instabilities, or pulse-splitting behavior of spike patterns are given.

Keywords: spike patterns, nonlocal eigenvalue problem, oscillatory instability, pulse-splitting.

1 Introduction

In this letter we summarize some results of [1] and [2] for the existence and stability of spike patterns in the one-dimensional Gray-Scott (GS) reaction-diffusion model. The GS model, first introduced in [3], and studied numerically in [4], can be written in dimensionless form as

\[ v_t = \varepsilon^2 v_{xx} - v + Av^2, \quad -1 < x < 1, \quad t > 0; \quad v_x = 0, \quad x = -1, 1, \] (1.1a)
\[ \tau u_t = Du_{xx} + (1 - u) - uv^2 \quad -1 < x < 1, \quad t > 0; \quad u_x = 0, \quad x = -1, 1. \] (1.1b)

Here \( A > 0, D > 0, \tau > 1, \) and \( \varepsilon \ll 1. \) This form of the GS model was first introduced in [5].

We consider (1.1) in the semi-strong spike interaction regime defined by \( D = O(1) \) and \( \varepsilon \ll 1. \) In this limit, singular perturbation techniques can be used to partially decouple \( u \) and \( v. \) From a mathematical viewpoint, our analysis provides a careful case study for the long-term goal of classifying generic mechanisms leading to instabilities of spike-type solutions in reaction-diffusion systems. For general reaction-diffusion systems in the weakly nonlinear regime, a similar classification based on amplitude equations and normal forms has been undertaken over the past several decades. As a step

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towards a classification for instabilities of spike solutions, our results (cf. [1], §2) show that certain instabilities in the GS model have a direct spectral equivalence with those in the Gierer-Meinhardt model of [6] derived in [7] in the semi-strong interaction limit.

For the infinite domain problem, the existence and stability of a one-spike solution for the GS model in the semi-strong interaction regime was studied analytically in [8], [9], [10], and [5]. Periodic solutions were also constructed in [8]. A numerical example of pulse-splitting with a qualitative analysis of the dynamical mechanism of pulse-splitting is given in [11].

The weak spike interaction regime is defined by \( D = O(\varepsilon^2) \), and \( \varepsilon \ll 1 \). In this latter regime, where \( u \) and \( v \) are both localized near certain points in the domain, the pioneering works of [12] and [13] showed that both pulse-splitting behavior and spatio-temporal chaos is possible. In terms of numerically computed bifurcation diagrams, a clear mechanism for the occurrence of pulse-splitting was formulated. General criteria for pulse-splitting are given in [14]. For symmetric initial data, the pulse-splitting that occurs in the weak interaction regime exhibits edge-splitting, whereas in the semi-strong interaction regime pulses split roughly simultaneously.

In the semi-strong regime where \( D = O(1) \), and \( \varepsilon \ll 1 \), there are three main parameter regimes for \( A \) where different behaviors occur. In §2, the results of [1] are summarized for the low-feed rate regime, \( A = O(\varepsilon^{1/2}) \), where both competition and oscillatory instabilities can occur. These results are new. Some new results of [2] for the pulse-splitting regime, \( A = O(1) \), verifying the criteria of [14] and predicting the number of pulse-splitting events, are given in §3. In the intermediate regime \( O(\varepsilon^{1/2}) \ll A \ll 1 \), our results are rather closely related to those in [8], [9], and [10]. For this regime, we refer to §3.4 and §5 of [1] for a detailed comparison to these previous works.

2 The Low-Feed Rate Regime: \( A = O(\varepsilon^{1/2}) \)

In the low feed-rate regime, we define a new \( O(1) \) variable \( A \) by \( A = \varepsilon^{1/2} A \). The first result of [1] for this regime characterizes symmetric \( k \)-spike equilibrium patterns, characterized by spikes in \( v \) that have a common amplitude. For each \( k = 1, 2, \ldots \), and for \( A > A_{ke} \), there are two branches of such solutions for (1.1) that meet at the saddle-node bifurcation value \( A_{ke} \) given by

\[
A_{ke} = \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}, \quad \theta_0 = D^{-1/2}.
\]  

(2.1)

These solution branches are conveniently parameterized in terms of \( s \), with \( 0 < s < \infty \), where

\[
s \equiv \frac{1 - U_\pm}{U_\pm}, \quad U_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1 - \frac{A_{ke}^2}{A^2}} \right].
\]  

(2.2)
We refer to the range $0 < s < 1$ as the *large solution branch* ($U = U_+$ is large), while the range $s > 1$ is called the *small solution branch* ($U = U_-$ is small). The precise result is as follows (see Proposition 2.1 of [1]):

**Proposition 1:** Let $\varepsilon \to 0$, with $A = O(1)$ and $D = O(1)$. Then, when $A > A_{ke}$, the small and large $k$-spike symmetric equilibrium solutions for (1.1), denoted by $u_-, v_-$ and $u_+, v_+$, respectively, are given asymptotically by

$$ v_\pm(x) \sim \frac{1}{\sqrt{\varepsilon A U_\pm}} \sum_{j=1}^{k} w \left[ \varepsilon^{-1} (x - x_j) \right], \quad u_\pm(x) \sim 1 - \frac{(1 - U_\pm)}{a_g} \sum_{j=1}^{k} G(x; x_j). \quad (2.3) $$

Here $x_j = -1 + \left( \frac{2j - 1}{k} \right)$, $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, $w(y) = \frac{3}{2} \text{sech}^2(y/2)$ satisfies $w'' - w + w^2 = 0$, and $G(x; x_j)$ is the Green's function satisfying $DG_{xx} - G = -\delta(x - x_j)$ with $G_x(\pm 1; x_j) = 0$.

To analyze the stability of these equilibrium solutions on an $O(1)$ time-scale, we let $u(x,t) = u_\pm(x) + e^{\lambda t} \eta(x)$, and $v(x,t) = v_\pm(x) + e^{\lambda t} \phi(x)$, where $\phi$ is a localized eigenfunction of the form

$$ \phi(x) \sim \sum_{j=1}^{k} c_j \Phi \left[ \varepsilon^{-1} (x - x_j) \right], \quad (2.4) $$

for some $c_j$ to be found. We consider eigenfunctions with $\int_{-\infty}^{\infty} w(y) \Phi(y) dy \neq 0$. In §3.1 of [1], the following spectral problem for $\Phi(y)$ is derived (see Proposition 3.2 of [1]):

**Proposition 2:** Assume that $0 < \varepsilon \ll 1$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues determining the stability of $k$-spike equilibria satisfy the nonlocal eigenvalue problem

$$ L_0 \Phi - \chi_{gs} w^2 \left( \int_{-\infty}^{\infty} w \Phi dy \int_{-\infty}^{\infty} w dy \right) = \lambda \Phi, \quad -\infty < y < \infty; \quad \Phi \to 0, \quad \text{as } |y| \to \infty. \quad (2.5) $$

Here the operator $L_0$ is $L_0 \Phi \equiv \Phi'' - \Phi + 2w\Phi$, and the multiplier $\chi_{gs} = \chi_{gs}(z;j)$ is defined by

$$ \chi_{gs} \equiv 2s \left( s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)} \left[ \tanh(\theta_\lambda/k) + \frac{1 - \cos[\pi(j-1)/k]}{\sinh(2\theta_\lambda/k)} \right] \right)^{-1}, \quad (2.6) $$

where $z \equiv \tau \lambda, \theta_\lambda \equiv \theta_0 \sqrt{1+z}$, and $\theta_0 \equiv D^{-1/2}$. Here $s$ is given in (2.2). The coefficients $c^t = (c_1, \ldots, c_k)$ in (2.4) are given by

$$ c_1^t = \frac{1}{\sqrt{k}} (1, \ldots, 1); \quad c_{t,j} = \sqrt{\frac{2}{k}} \cos \left( \frac{\pi(j-1)}{k} (l-1/2) \right), \quad j = 2, \ldots, k. \quad (2.7) $$
There is an equivalent formulation of (2.5) which states that the eigenvalues of (2.5) with \( \int_{-\infty}^{\infty} w \Phi \, dy \neq 0 \) are the union of the zeros of the functions \( g_j(\lambda) = 0 \) for \( j = 1, \ldots, k \), where

\[
g_j(\lambda) = C_j(\lambda) - f(\lambda), \quad f(\lambda) = \frac{\int_{-\infty}^{\infty} w (L_0 - \lambda)^{-1} w^2 \, dy}{\int_{-\infty}^{\infty} w^2 \, dy}, \quad C_j(\lambda) = [\chi_{gs}]^{-1}. \tag{2.8}
\]

By analyzing the zeroes of \( g_j(\lambda) \) in the right half-plane \( \text{Re}(\lambda) \geq 0 \), the main stability result for multi-spike solutions is as follows (cf. Proposition 3.10 and 3.13 of [1]):

**Proposition 3:** The large solution \( u_+, v_+ \) is unstable for any \( 0 < s < 1, \ k \geq 1, \) and \( D > 0 \). Next, let \( k > 1 \), and consider the multi-spike small solution \( u_-, v_- \), where \( s > 1 \). For \( A > A_{kl} \), the solution will be stable on an \( O(1) \) time-scale when \( 0 < \tau < \tau_{hl} \). Alternatively, on the range \( A_{ke} < A < A_{kl} \), the small solution is unstable for any \( \tau > 0 \). The threshold \( A_{kl} \) is given analytically by

\[
A_{kl} = \frac{A_{ke} (\gamma_{k}/2 + 2 \sinh^2 (\theta_0/k))}{\left([\gamma_{k}/2 + 2 \sinh^2 (\theta_0/k)]^2 - (\gamma_{k}/2)^2\right)^{1/2}}, \quad \gamma_k = 1 + \cos \left(\frac{\pi}{k}\right). \tag{2.9}
\]

Let \( A \) satisfy \( A > A_{kl} \). Then, as \( \tau \) increases beyond \( \tau_{hl} \), a Hopf bifurcation in the spike amplitudes was computed numerically in [1]. The threshold \( \tau_{hl} \) is given by the minimum value of the set \( \tau_j \), \( j = 1, \ldots, k \), for which \( g_j(\lambda) = 0 \), \( j = 1, \ldots, k \), has complex conjugate roots on the imaginary axis. Let \( \lambda = \pm i \lambda_k \) be the corresponding value of \( \lambda \). Then, as was shown in [1], the unstable eigenfunction generically has the form of a **synchronous oscillatory instability** with

\[
v = v_- + \delta e^{i \lambda_k t} \phi + \text{c.c}, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi [e^{-l}(x-x_n)], \quad c_n = 1, \quad n = 1, \ldots, k. \tag{2.10}
\]

Here c.c denotes complex conjugate and \( \delta \ll 1 \). Alternatively, suppose that \( A_{ke} < A < A_{kl} \). Then, for any \( \tau > 0 \), the dominant initial instability was shown in [1] to have the form

\[
v = v_- + \delta e^{i \lambda_{kl} t} \phi, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi [e^{-l}(x-x_n)], \quad c_n = \cos \left(\frac{\pi (k - 1)}{k} (n - 1/2)\right), \quad n = 1, \ldots, k. \tag{2.11a}
\]

Here \( \delta \ll 1 \), and \( \lambda_{kl} > 0 \) is the unique root of \( g_k(\lambda_R) = 0 \). Since \( \sum_{n=1}^{k} c_n = 0 \), this instability locally conserves the sum of the heights of the spikes. Hence, it is referred to as a **competition instability**. The numerical experiments in §3.3 of [1] show that this instability leads to a spike competition process whereby certain spikes in a spike sequence are ultimately annihilated. Finally, in Proposition 3.3 of [1] it was shown there there is a spectral equivalence principle between (2.5) and a corresponding nonlocal eigenvalue problem derived in Proposition 2.3 of [7] for the Gierer-Meinhardt (GM) model.
3 The Pulse-Splitting Regime: $A = O(1)$

In this regime, where $A = O(1)$, it was shown in [2] that equilibrium $k$-spike solutions can be constructed in terms of the solutions $V(y) > 0$ and $U(y) > 0$ to a certain core problem defined by

$$
V'' - V + V^2 U = 0, \quad U'' = UV^2, \quad 0 < y < \infty, \quad (3.1a)
$$

$$
V'(0) = U'(0) = 0; \quad V \to 0, \quad U \sim By, \quad \text{as} \quad y \to \infty; \quad B = A \tanh(\theta_0/k). \quad (3.1b)
$$

Here $\theta_0 \equiv D^{-1/2}$. This core problem, without the term $B = A \tanh(\theta_0/k)$ associated with the finite domain, was first derived in [5] for the infinite-line problem. The equilibrium result given in Proposition 3.1 of [2] is as follows:

**Proposition 4:** Let $\varepsilon \to 0$, $A = O(1)$, $\varepsilon A/\sqrt{D} \ll 1$, and suppose that (3.1) has a solution. Then, the $v$-component for a $k$-spike equilibrium solution to (1.1) is given by

$$
v \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} \left( V \left[ \varepsilon^{-1} (x - x_j) \right] + O \left( \frac{\varepsilon A}{\sqrt{D}} \right) \right). \quad (3.2)
$$

In the $j^{th}$ inner region, where $|x - x_j| = O(\varepsilon)$, $u$ satisfies $u \sim \frac{\varepsilon}{A \sqrt{D}} U \left[ \varepsilon^{-1} (x - x_j) \right]$. The outer solution for $u$, valid for $|x - x_j| \gg O(\varepsilon)$, and $j = 1, \ldots, k$, has the form $u \sim 1 - \frac{1}{a_g} \sum_{j=1}^{k} G(x; x_j)$, where $a_g$, $x_j$, and $G(x; x_j)$ are as defined in Proposition 1.

The core problem (3.1) was studied qualitatively and numerically in [2] in terms of $B$. In [2] it was shown that $0 < \gamma < \frac{3}{2}$, where $\gamma \equiv U(0)V(0)$. Numerical solutions to (3.1) were computed for which $V$ has a single maximum at $y = 0$ as $\gamma \to 3/2$ from below, and a resulting curve $B = B(\gamma)$ was computed. This limiting solution for $V$ asymptotically matches onto the solution constructed in the low feed-rate regime in §2. As shown in [2], the curve $B = B(\gamma)$ is double-valued with $B \to 0$ as $\gamma \to 0$ and as $\gamma \to 3/2$, and it has a saddle-node bifurcation point at the maximum value $B_c$ of $B$ given by $B_c = 1.347$ (see also [5]), where $\gamma = \gamma_c = 1.92$. We refer to the range $\gamma_c < \gamma < 3/2$ and $0 < \gamma < \gamma_c$ as the primary and secondary branches of the $B = B(\gamma)$ bifurcation diagram. Since $B \equiv A \tanh(\theta_0/k)$ from (3.1b), these results of [2] show that equilibrium $k$-spike solutions exist only when $A$ is small enough. In particular, we have (cf. Proposition 3.2 of [2]):

**Proposition 5:** Let $\varepsilon \ll 1$, $A = O(1)$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, there will be no $k$-spike equilibrium solution to (1.1) that merges onto the low feed-rate regime solution when

$$
A > A_{pk} \equiv 1.347 \coth \left( \frac{1}{k \sqrt{D}} \right). \quad (3.3)
$$
Next, the following three pulse-splitting criteria of [14] were verified in [1]: 1) each $k$-spike equilibrium branch must have a saddle-node bifurcation that occurs at approximately the same bifurcation value (i.e. the lining-up property); 2) for each $k$, one branch of equilibria is unstable while the other is stable; 3) the linearization around the equilibrium solution at each saddle-node value has a dimple-shaped eigenfunction associated with a zero eigenvalue. When these criteria of [14] are satisfied, and when the bifurcation parameter is taken to be only slightly beyond the saddle-node value, pulse-splitting should occur from a single localized initial pulse due to the ghost of the dimple eigenfunction. In Fig. 1(a) we plot the norm $|v_2|$, defined by $|v_2|^2 \equiv \int_{-1}^{1} v^2 dx \sim \varepsilon^{-1} kD \int_{-\infty}^{\infty} V^2 dy$, versus $A$ when $D = 0.1$, to show that the approximate lining-up condition 1) holds. For $\tau \ll O(\varepsilon^{-1})$ the stability condition 2) of [14] is verified in §5 and §6 of [2] by showing that the primary and secondary branches are stable and unstable, respectively. Finally, the dimple eigenfunction property is verified in §6 of [2]. Therefore, we conjecture:

**Conjecture 1:** Let $\varepsilon \ll 1$, $\tau \ll O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Suppose that we have even one-spike initial data centered at the origin. Then, the final equilibrium state is stable, and it has $2^m$ spikes where, for some integer $m \geq 0$, $A$ is related to $A_{pk}$ by

$$A_{pk^{2^m-1}} < A < A_{pk^m}.$$  \hspace{1cm} (3.4)

To verify this conjecture, we computed numerical solutions to (1.1) starting from an initial pulse in $v$ centered at $x = 0$. In Fig. 1(b) we show pulse-splitting behavior for $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and
\[ \tau = 2.0. \] For these values we calculate \( A_{p1} = 2.045 < A < A_{p8} = 3.583. \) Thus, (3.4) correctly predicts the eight-spike final state in Fig. 1(b). In addition, in Experiment 5 of [2], Conjecture 1 correctly predicts the final eight-spike solution for the pulse-splitting example in Fig. 4 of [11].

References


