I) Optimization of the Persistence Threshold in Diffusive Logistic Model; II) Concentration Behavior in Nonlinear Biharmonic Eigenvalue Problems of MEMS

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Persistence Problem: Introduction I

Consider the diffusive logistic equation for $u(x, t)$ with $x \in \Omega \in \mathbb{R}^2$

$$u_t = D \Delta u + u [m_\varepsilon(x) - c(x)u], \quad x \in \Omega; \quad \partial_n u = 0, \quad x \in \partial\Omega.$$ 

Here $D$ is the constant diffusivity.

We linearize around the zero solution with $u = e^{\mu Dt} \phi(x)$ and set $\mu = 0$

$$\Delta \phi + \lambda m_\varepsilon(x) \phi = 0, \quad x \in \Omega; \quad \partial_n \phi = 0, \quad x \in \partial\Omega.$$ 

The bifurcation parameter $\lambda \geq 0$ is defined by

$$\lambda = 1/D.$$ 

The extinct solution $u = 0$ exists for all $\lambda \geq 0$. Depending on the form of the spatially dependent growth rate $m_\varepsilon(x)$, at some critical value of $\lambda$ there can be a transcritical bifurcation to a spatially dependent solution. This leads to the idea of a persistence threshold.

Note: Growth rate $m_\varepsilon$ changes sign $\rightarrow$ indefinite weight eig. problem (no standard oscillation theory, or standard variational characterization of eigenvalues, etc.).
**Persistence Problem: Introduction II**

**Key Previous Result I:** Assume that \( \int_{\Omega} m_\varepsilon \, dx < 0 \), but that \( m_\varepsilon > 0 \) on a set of positive measure. Then, there exists a positive principal eigenvalue \( \lambda_1 = \lambda^* \), i.e. the extinction threshold, with corresponding positive eigenfunction \( \phi \) (Brown and Lin, (1980)).

**Key Previous Result II:** Transcritical bifurcation: \( u \to u_\infty(x) \neq 0 \) as \( t \to \infty \) if \( \lambda > \lambda^* \), while \( u \to 0 \) as \( t \to \infty \) if \( 0 < \lambda < \lambda^* \). (many authors; Cantrell, Cosner, Berestycki, etc..)
Persistence Problem: Introduction III

Key Previous Result II: The optimal growth rate \( m_\varepsilon(x) \) is of bang-bang type. (Theorem 1.1 of Lou and Yanagida, JJAM, 2006, for 2-D).

Main Goal: Minimize \( \lambda_1 \) wrt \( m_\varepsilon(x) \), subject to a fixed \( \int_{\Omega} m_\varepsilon \, dx < 0 \): i.e. determine the largest \( D \) that can still allow for the persistence of the species. This is a long-standing open problem of determining the optimal shape of \( m_\varepsilon(x) \) in a 2-D domain. (Cantrell and Cosner 1990’s, Lou and Yanagida, (2006); Kao, Lou, and Yanagida, (2008); Roques and Stoica, (2007); Berestycki, Hamel, (2005,2006)).

Remark: In a 1-D domain, this problem has been solved (Lou and Yanagida, JJAM (2006)). The optimal \( m_\varepsilon(x) \) in 1-D is to concentrate favorable resources near one of the endpoints of the domain, and to have only one favorable patch.
Persistence Problem: Patch Model I

Patch Model: The eigenvalue problem for the persistence threshold is

\[ \Delta \phi + \lambda m_\varepsilon(x)\phi = 0, \quad x \in \Omega; \quad \partial_n \phi = 0, \quad x \in \partial \Omega; \quad \int_\Omega \phi^2 \, dx = 1. \]

Note that the population is confined in the domain (reflecting boundary condition). The piecewise-constant growth rate \( m_\varepsilon(x) \) is defined as

\[ m_\varepsilon(x) = \begin{cases} 
  m_j / \varepsilon^2, & x \in \Omega_{\varepsilon_j} \equiv \{x \mid |x - x_j| = \varepsilon \rho_j \cap \Omega\}, \quad j = 1, \ldots, n, \\
  -m_b, & x \in \Omega \setminus \bigcup_{j=1}^n \Omega_{\varepsilon_j}. 
\end{cases} \]

Assume that at least one \( m_j > 0 \), and \( \int_\Omega m_\varepsilon \, dx < 0 \). Then, there is a positive principal eigenvalue \( \lambda_1 > 0 \).

Biologically: On the whole the environment is hostile, but there is at least one region that can support growth.

Remarks and Terminology:

- Patches \( \Omega_{\varepsilon_j} \) of radius \( O(\varepsilon) \) are portions of small circular disks strictly inside \( \Omega \). Circular patches are locally optimal (Hamel, Roques, 2007).

- The constant \( m_j \) is the local growth rate of the \( j^{\text{th}} \) patch, with \( m_j > 0 \) for a favorable habitat and \( m_j < 0 \) for a non-favorable habitat.

- The constant \( m_b \) is the background bulk decay rate.

- The boundary \( \partial \Omega \) is piecewise smooth, with possible corner points.
Persistence Problem: Patch Model III

Define $\Omega^I \equiv \{x_1, \ldots, x_n\} \cap \Omega$ to be the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \ldots, x_n\} \cap \partial \Omega$ is the set of the centers of the boundary patches. We assume patches are well-separated, i.e. $|x_i - x_j| \gg O(\varepsilon)$ for $i \neq j$ and that $\text{dist}(x_j, \partial \Omega) \gg O(\varepsilon)$ if $x_j \in \Omega^I$.

To accommodate a boundary patch, we assign with each $x_j$ for $j = 1, \ldots, n$, an angle $\pi \alpha_j$ representing the angular fraction of a circular patch that is contained within $\Omega$. For example, $\alpha_j = 2$ when $x_j = \Omega^I$, and $\alpha_j = 1$ when $x_j \in \Omega^B$ and $x_j$ is a point where $\partial \Omega$ is smooth, and $\alpha_j = 1/2$ when $x_j \in \partial \Omega$ is at a $\pi/2$ corner of $\partial \Omega$, etc.
Persistence Problem: Patch Model IV

The condition $\int_\Omega m_\varepsilon \, dx < 0$ is asymptotically equivalent for $\varepsilon \to 0$ to

$$\int_\Omega m_\varepsilon \, dx = -m_b |\Omega| + \frac{\pi}{2} \sum_{j=1}^{n} \alpha_j m_j \rho_j^2 + O(\varepsilon^2) = C < 0.$$  

**Assumption I:** Assume that this holds and that one $m_j$ is positive. Then, by the Key Previous Result I, there exists a positive principal eigenvalue $\lambda_1$.

**Main Goal:** Calculate $\lambda_1$ as $\varepsilon \to 0$ by using an approach based on strong localized perturbation theory. Then, minimize it for a fixed $\int_\Omega m_\varepsilon \, dx < 0$. The parameter set is $\{m_1, \ldots, m_n\}$, $\{\rho_1, \ldots, \rho_n\}$, $\{x_1, \ldots, x_n\}$, and $\{\alpha_1, \ldots, \alpha_n\}$.
Persistence Problem: Qualitative Questions

After calculating $\lambda_1$ as $\varepsilon \to 0$ we then address several interesting qualitative questions:

**Q1:** What is the effect on $\lambda_1$ of resource location. Are boundary habitats preferable to interior habitats with regards to decreasing the extinction threshold?

**Q2:** What is the effect of resource fragmentation? Does fragmentation lead to larger persistence thresholds? To maintain the value of $\int_\Omega m_\varepsilon \, dx$, we need that $m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2$. 

\[ \Omega \]
Persistence Problem: Green Functions

In the analysis, the following Green functions play an important role:

**Modified G-Function:** Define the modified G-function $G_m$ by

$$G_m(x; x_j) \equiv G(x; x_j), \quad x_j \in \Omega; \quad G_m(x; x_j) \equiv G_s(x; x_j), \quad x_j \in \partial \Omega.$$ 

Here $G(x; x_j)$ is the unique Neumann Green function satisfying

$$\Delta G = \frac{1}{|\Omega|} - \delta(x - x_j), \quad x \in \Omega; \quad \partial_n G = 0, \quad x \in \partial \Omega; \quad \int_{\Omega} G \, dx = 0,$$

$$G(x; x_j) \sim -\frac{1}{2\pi} \log |x - x_j| + R(x_j; x_j), \quad \text{as} \quad x \to x_j,$$

while $G_s(x; x_j)$ is the unique surface Neumann Green function

$$\Delta G_s = \frac{1}{|\Omega|}, \quad x \in \Omega; \quad \partial_n G_s = 0, \quad x \in \partial \Omega \setminus \{x_j\}; \quad \int_{\Omega} G_s \, dx = 0,$$

$$G_s(x; x_j) \sim -\frac{1}{\alpha_j \pi} \log |x - x_j| + R_s(x_j; x_j), \quad \text{as} \quad x \to x_j \in \partial \Omega.$$
**Persistence Problem: Main Result I**

**Principal Result 4.3:** In the limit $\varepsilon \to 0$, the positive principal eigenvalue $\lambda_1$ has the following two-term asymptotic expansion

$$
\lambda_1 = \mu_0 \nu - \mu_0 \nu^2 \left( \frac{\kappa^T (\pi G_m - P) \kappa}{\kappa^T \kappa} + \frac{1}{4} \right) + \mathcal{O}(\nu^3), \quad \nu = -1 / \log \varepsilon.
$$

Here $\kappa = (\kappa_1, \ldots, \kappa_n)^T$ and $\mu_0 > 0$ is the first positive root of $B(\mu_0) = 0$

$$
B(\mu_0) \equiv -m_b |\Omega| + \pi \sum_{j=1}^{n} \sqrt{\alpha_j} \kappa_j, \quad \kappa_j \equiv \frac{\sqrt{\alpha_j} m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0}.
$$

Finally, the $n \times n$ matrix $G_m$ and diagonal matrix $P$ are defined by

$$
G_{mi} = \sqrt{\alpha_i \alpha_j} G_{mj}, \quad i \neq j; \quad G_{mjj} = \alpha_j R_{mjj}; \quad P_{jj} = \log \rho_j.
$$

**Remarks:**

- The coefficient $\mu_0$ is independent of the precise relative locations of the patches within the domain.

- The coefficient of order $\mathcal{O}(\nu^2)$ has this spatial information through the Green interaction matrix.
**Persistence Problem: Main Result II**

**Principal Result:** There exists a unique root $\mu_0$ to $B(x) = 0$ on the range $0 < x < \mu_{0u} \equiv 2/(m_J \rho_j^2)$, where $m_J \rho_j^2 = \max_{m_j>0} \{m_j \rho_j^2 \mid j = 1, \ldots, n\}$. The corresponding eigenfunction has one sign.

**Proof:** $B(0) = \int_{\Omega} m_\varepsilon(x) \, dx \sim C < 0$ by Assumption I. In addition, $B(x) \to +\infty$ as $x \to \mu_{0u}^-$, and

$$B'(x) = \sum_{j=1}^{n} \frac{\alpha_j m_j^2 \rho_j^4}{(2 - m_j \rho_j^2 x)^2} > 0, \quad 0 < x < \mu_{0u}.$$

Notice also that $\mu_{0u}$ is the smallest vertical asymptote of $B(x)$. Hence, there exists a unique root $\mu_0 > 0$. With $x^* = \mu_{0u}$ we plot:
Persistence Problem: Derivation of $\mu_0$ I

We now sketch the derivation of the leading-order term in the Principal Result.

We expand the positive principal eigenvalue $\lambda_1$ as

$$\lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \cdots, \quad \nu = -1/\log \varepsilon,$$

for some unknown $\mu_0$ and $\mu_1$ to be found. In the outer region, defined away from an $O(\varepsilon)$ neighborhoods of $x_j$, we expand the corresponding eigenfunction as

$$\phi \sim \phi_0 + \nu \phi_1 + \nu^2 \phi_2 + \cdots.$$

We obtain that $\phi_0 = |\Omega|^{-1/2}$ is a constant, and that $\phi_1$ satisfies

$$\Delta \phi_1 = \mu_0 m_b \phi_0, \quad x \in \Omega \setminus \Omega^I;$$

$$\partial_n \phi_1 = 0, \quad x \in \partial \Omega \setminus \Omega^B; \quad \int_{\Omega} \phi_1 \, dx = 0.$$

Here $\Omega^I \equiv \{x_1, \ldots, x_n\} \cap \Omega$ is the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \ldots, x_n\} \cap \partial \Omega$ is the set of the centers of the boundary patches.
Persistence Problem: Derivation of $\mu_0$ II

In the inner region, near the $j^{th}$ patch we introduce $y = \varepsilon^{-1}(x - x_j)$ and

$$\psi(y) = \phi(x_j + \varepsilon y),$$

and expand

$$\psi \sim \psi_0 + \nu \psi_1 + \nu^2 \psi_2 + \cdots,$$

where $\psi_0$ is a constant to be determined. For an interior patch with $x_j \in \Omega^I$, we obtain that $\psi_1$ satisfies

$$\Delta \psi_1 = \begin{cases} 
F_{1j}, & |y| \leq \rho_j, \\
0, & |y| \geq \rho_j,
\end{cases}$$

where $F_{1j} = -\mu_0 m_j \psi_0$. The solution for $\psi_1$, with $\rho = |y|$, is

$$\psi_1 = \begin{cases} 
A_{1j} \left( \frac{\rho^2}{2\rho_j^2} \right) + \bar{\psi}_1, & 0 \leq \rho \leq \rho_j, \\
A_{1j} \log \left( \frac{\rho}{\rho_j} \right) + \frac{A_{1j}}{2} + \bar{\psi}_1, & \rho \geq \rho_j,
\end{cases}$$

where $\bar{\psi}_1$ is an unknown constant.
Persistence Problem: Derivation of $\mu_0$ III

The divergence theorem yields $A_{1j}$ as

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j},$$

for both boundary and interior patches.

The matching condition between the outer solution as $x \to x_j$ and the inner solution as $|y| = \varepsilon^{-1}|x - x_j| \to \infty$ is

$$\phi_0 + \nu \phi_1 + \cdots \sim$$

$$\psi_{0j} + A_{1j} + \nu \left( A_{1j} \log |x - x_j| - A_{1j} \log \rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j} \right) + \cdots .$$

The leading-order matching condition (blue terms) yields

$$\phi_0 = \psi_{0j} + A_{1j}, \quad j = 1, \ldots, n.$$
Persistenve Problem: Derivation of $\mu_0$ IV

Recall that the problem for $\phi_1$ is

$$\Delta \phi_1 = \mu_0 m_b \phi_0 , \quad x \in \Omega \setminus \Omega^I ;$$

$$\partial_n \phi_1 = 0 , \quad x \in \partial \Omega \setminus \Omega^B ; \quad \int_{\Omega} \phi_1 \, dx = 0 .$$

From the $O(\nu)$ red terms in the matching condition we obtain that $\phi_1$ has the following singular behavior as $x \to x_j$

$$\phi_1 \sim A_{1j} \log |x - x_j| - A_{1j} \log \rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j} , \quad \text{as} \quad x \to x_j .$$

Next, by using the divergence theorem on the solution $\phi_1$ to we get

$$\mu_0 m_b |\Omega| \phi_0 = -\pi \sum_{j=1}^{n} \alpha_j A_{1j} .$$
Persistence Problem: Derivation of $\mu_0 V$

Then, by recalling that

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j}, \quad \phi_0 = \psi_{0j} + A_{1j},$$

we get that

$$\psi_{0j} = \frac{2\phi_0}{2 - m_j \rho_j^2 \mu_0}, \quad A_{1j} = -\frac{m_j \rho_j^2 \mu_0 \phi_0}{2 - m_j \rho_j^2 \mu_0}, \quad j = 1, \ldots, n.$$

From the equation above we obtain that the leading-order eigenvalue correction $\mu_0$ is a root of the nonlinear algebraic equation

$$\frac{m_b |\Omega|}{\pi} = \sum_{j=1}^{n} \frac{\alpha_j m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0}.$$

This yields the nonlinear algebraic equation for the leading-order term $\mu_0$ in the expansion of the eigenvalue, as given in the Principal Result.
Persistence Problem: Derivation of $\mu_0$ VI

Remarks:

- The calculation of the higher-order term of order $O(\nu^2)$, is more involved and is given [LW]. This second-order term has the spatial information on the location of the traps.

- Note that $\psi_{0j} > 0$ if $\mu_0 < \mu_{0u}$. This is the positivity property of the principal eigenfunction.

- We emphasize, that in contrast to the Laplacian eigenvalue problems for the MFPT, the equation for $\mu_0$ does contain some key qualitative information, which we now illustrate.

By optimizing the leading-order coefficient $\mu_0$ subject to $\int_{\Omega} m_{\varepsilon} dx < 0$, we can obtain key qualitative results regarding the optimal resource distribution.
Persistence Problem: Implications I

The following very simple Lemma is needed:

**Lemma**: Consider two smooth functions $C_{\text{old}}(\zeta)$ and $C_{\text{new}}(\zeta)$ defined on $0 \leq \zeta < \mu_{\text{old}}$ and $0 \leq \zeta < \mu_{\text{new}}$, respectively, with $C_{\text{old}}(0) = C_{\text{new}}(0) < 0$, and $C_{\text{old}}(\zeta) \to +\infty$ as $\zeta \to \mu_{\text{old}}$ from below, and $C_{\text{new}}(\zeta) \to +\infty$ as $\zeta \to \mu_{\text{new}}$ from below. Suppose further that there exist unique roots $\zeta = \mu_{0\text{old}}$ and $\zeta = \mu_{0\text{new}}$ to $C_{\text{old}}(\zeta) = 0$ and $C_{\text{new}}(\zeta) = 0$ on the intervals $0 < \zeta < \mu_{\text{old}}$ and $0 < \zeta < \mu_{\text{new}}$, respectively. Then,

- **Case I**: If $\mu_{\text{new}} \leq \mu_{\text{old}}$ and $C_{\text{new}}(\zeta) > C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{new}}$, then $\mu_{0\text{new}} < \mu_{0\text{old}}$.
- **Case II**: If $\mu_{\text{new}} \geq \mu_{\text{old}}$ and $C_{\text{new}}(\zeta) < C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{old}}$, then $\mu_{0\text{new}} > \mu_{0\text{old}}$. 

Persistence Problem: Implications II

**Qualitative Result I:** The movement of a single favorable habitat to the boundary of the domain is advantageous for species persistence.

**Proof:** Move the \( j \)th interior favorable patch with \( m_j > 0 \) of radius \( \varepsilon \rho_j \) and angle \( 2\pi \) (i.e. \( \alpha_j = 2 \)) to an unoccupied boundary location with patch radius \( \varepsilon \rho_k \), “mass” \( m_k > 0 \), and angle \( \pi \alpha_k \), with \( \alpha_k < 2 \). To maintain \( \int_{\Omega} m_{\xi} \, dx \), we need \( m_j \rho_j^2 = \alpha_k m_k \rho_k^2 \), which implies \( m_k \rho_k^2 > m_j \rho_j^2 \). Then,

\[
B_{\text{new}}(\zeta) - B_{\text{old}}(\zeta) = \frac{\pi \alpha_k m_k \rho_k^2}{2 - \zeta m_k \rho_k^2} - \frac{2\pi m_j \rho_j^2}{2 - \zeta m_j \rho_j^2} = \pi \left( \frac{2}{\alpha_k} \right) \frac{m_j^2 \rho_j^4 \zeta}{(2 - \zeta m_j \rho_j^2)} \frac{(2 - \zeta m_k \rho_k^2)}{(2 - \zeta m_k \rho_k^2)} (2 - \alpha_k).
\]

Recall that \( B_{\text{old}}(\zeta) = 0 \) has a root \( \zeta \) on \( 0 < \zeta < \mu_{\text{old}}^m \equiv 2/(m_j \rho_j^2) \), where \( m_j \rho_j^2 = \max_j m_j \rho_j^2 \). Since \( m_k \rho_k^2 > m_j \rho_j^2 \), the first vertical asymptote for \( B_{\text{new}}(\zeta) \) cannot be larger than that of \( B_{\text{old}}(\zeta) \). Thus, there is a unique root \( \zeta = \mu_{\text{new}}^0 \) to \( B_{\text{new}}(\zeta) = 0 \) on \( 0 < \zeta < \mu_{\text{new}}^m \equiv 2/(m_k \rho_k^2) \), where \( m_k \rho_k^2 \equiv \max\{m_j \rho_j^2, m_k \rho_k^2\} \). Since \( \mu_{\text{new}}^m \leq \mu_{\text{old}}^m \), and \( B_{\text{new}}(\zeta) > B_{\text{old}}(\zeta) \) for \( 0 < \zeta < \mu_{\text{new}}^m \), Case I of the Lemma yields \( \mu_{\text{new}}^0 < \mu_{\text{old}}^0 \). \( \blacksquare \)
Persistence Problem: Implications III

Qualitative Result II: The fragmentation of one favorable interior habitat into two separate favorable interior habitats is not advantageous for species persistence. Similarly, the fragmentation of a favorable boundary habitat into two favorable boundary habitats, with each centered at a smooth point of $\partial \Omega$, is not advantageous.

Proof: Split $k^{th}$ patch of radius $\rho_k$ into two patches of radius $\rho_A$ and $\rho_B$.

The constraint that $\int_{\Omega} m_\varepsilon(x) \, dx$ is unchanged requires:

$$m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2.$$
We prove this result for $\alpha_A = \alpha_B = \alpha_k$ as follows.

Suppose that we are fragmenting one favorable habitat into two smaller favorable habitats. Then, $m_A > 0$, $m_B > 0$, and $m_k > 0$.

For the original patch distribution, $B_{\text{old}}(\zeta) = 0$ has a positive root $\zeta = \mu_{\text{old}}$ on $0 < \zeta < \mu_{\text{old}} \equiv 2/(m_J \rho_J^2)$, where $m_J \rho_J^2 = \max_{m_j > 0} (m_j \rho_j^2)$.

Since, clearly, the first vertical asymptote for $B_{\text{new}}(\zeta)$ cannot be smaller than that of $B_{\text{old}}(\zeta)$ under this fragmentation, it follows that $B_{\text{new}}(\zeta) = 0$ has a positive root $\zeta = \mu_{\text{new}}$ on $0 < \zeta < \mu_{\text{new}}$ with $\mu_{\text{new}} \geq \mu_{\text{old}}$. 
Persistence Problem: Implications V

To maintain $\int_{\Omega} m \varepsilon \, dx$ we require that $m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_A^2$. The change in $\mathcal{B}(\zeta)$ induced by this fragmentation action is

$$
\mathcal{B}_{\text{new}}(x) - \mathcal{B}_{\text{old}}(x) = \frac{\alpha_k m_A \rho_A^2}{2 - m_A \rho_A^2 x} + \frac{\alpha_k m_B \rho_B^2}{2 - m_B \rho_B^2 x} - \frac{\alpha_k m_k \rho_k^2}{2 - m_k \rho_k^2 x}
$$

Hence, we have that $\mathcal{B}_{\text{new}}(\zeta) < \mathcal{B}_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{old}}^m \equiv 2/(m_J \rho_J^2)$. Since, in addition $\mu_{\text{new}}^m \geq \mu_{\text{old}}^m$, it follows from Case II of the Lemma that $\mu_{0}^\text{new} > \mu_{0}^\text{old}$. This proves the Qualitative Result II.

Key: Fragmentation of an interior favorable habitat into two separate favorable interior habitats is deleterious to survival of the species.
Persistance Problem: Implications VI

Q3: What about a partial fragmentation scenario, whereby an interior favorable habitat is fragmented into a boundary habitat and a smaller interior favorable habitat?

Qualitative Result III: The fragmentation of one favorable interior habitat into a new smaller interior favorable habitat together with a favorable boundary habitat, is advantageous for species persistence when the boundary habitat is sufficiently strong in the sense that

\[ m_k \rho_k^2 > \frac{4}{2 - \alpha_k} m_j \rho_j^2 > 0. \]

Such a fragmentation of a favorable interior habitat is not advantageous when the new boundary habitat is too weak in the sense that

\[ 0 < m_k \rho_k^2 < m_j \rho_j^2. \]

Finally, the clumping of a favorable boundary habitat and an unfavorable interior habitat into one single interior habitat is not advantageous for species persistence when the resulting interior habitat is still unfavorable.
Persistence Problem: Examples I

Example 1: We illustrate Qualitative Result III inside the unit disk for the case \( m_b = 2 \): Fragment a single interior patch of radius \( \varepsilon \) centered at the origin into a favorable boundary patch of radius \( \varepsilon \rho_0 \) and a smaller favorable interior patch of radius \( \varepsilon \rho_1 \). Assume that \( m_j = 1 \) for each patch. To maintain the constraint \( \int_{\Omega} m_\varepsilon \, dx = -\pi \), we require that \( \rho_0 \) and \( \rho_1 \), with \( 0 < \rho_1 < 1 \), satisfy

\[
1 = \rho_1^2 + \frac{1}{2} \rho_0^2.
\]

For the new configuration, the equation for \( \mu_0 \) is simply

\[
\mathcal{B}(\mu_0) \equiv -2\pi + \pi \left( \frac{\rho_1^2}{2 - \rho_1^2 \mu_0} - \frac{\rho_0^2/2}{2 - \rho_0^2 \mu_0} \right) = 0
\]

For this two-patch problem, \( \mu_0 \) satisfies the quadratic equation

\[
\mu_0^2 \rho_1^2 (1 - \rho_1^2) + \mu_0 \left( -2 + \frac{5}{2} \rho_1^2 - \frac{3}{2} \rho_1^4 \right) + 1 = 0.
\]

Note: \( \mu_0 = 1 \) when \( \rho_1 = 1 \) (original configuration of one interior patch);
Also \( \mu_0 = 1/2 \) when \( \rho_1 = 0 \) (only a boundary patch).
Persistence Problem: Examples II

The (sufficient condition) bounds in Qualitative Result III state that:

- fragmentation of an interior patch into a boundary patch is undesirable when $\rho_1 > \rho_0$, which yields $\rho_1 > \sqrt{2/3}$.

- such a fragmentation is advantageous when $\rho_1 < 1/\sqrt{3}$.

For this simple two-patch case, we obtain that $\mu_0 = 1$ when $\rho_1 = \sqrt{2/5}$, or equivalently $\rho_0 = \sqrt{6/5}$. Thus, fragmentation is advantageous when $\rho_1 < \sqrt{2/5}$, or equivalently $\rho_0 > \sqrt{6/5}$. 

HK – p.26
**Example 2:** Illustrate Qualitative Result III for a unit disk with $m_b = 3$ that has one pre-existing favorable interior patch of radius $\varepsilon$ and growth rate $m_+ = 1$, together with one pre-existing unfavorable interior patch of radius $\varepsilon$ and growth rate $m_- = -1$.

We introduce an additional favorable resource with $m_0 = 1$ that can occupy an area $\varepsilon^2 A_0$ if it is separated from the other two patches.

We then compare three different options for using this additional favorable resource, subject to the constraint that $\int_{\Omega} m_\varepsilon \, dx = -3\pi + A_0$ remains fixed.
Persistence Problem: Examples IV

**Case I:** If we concentrate the additional favorable resource at a smooth point on the boundary, then $\mu_0$ satisfies

$$-3 + 2 \left( \frac{1}{2 - \mu_0} - \frac{1}{2 + \mu_0} \right) + \frac{A_0/\pi}{1 - \mu_0 A_0/\pi} = 0.$$ 

**Case II:** If the additional favorable resource is used to strengthen the pre-existing favorable interior patch, then $\mu_0$ satisfies

$$-3 + \frac{2 \rho_+^2}{2 - \rho_+^2 \mu_0} - \frac{2}{2 + \mu_0} = 0, \quad \rho_+^2 = 1 + A_0/\pi.$$ 

**Case III:** Finally, if the additional favorable resource is used to diminish the strength of the unfavorable pre-existing interior patch, then $\mu_0$ satisfies

$$-3 + \frac{2}{2 - \mu_0} + \frac{m_-}{2 - m_- \mu_0} = 0, \quad m_- = -1 + A_0/\pi.$$
**Persistence Problem: Examples V**

We conclude that:

- inserting a favorable boundary patch is preferable only when it has a sufficiently large size.
- if only a limited amount of an additional favorable resource is available, it is preferable to re-enforce the pre-existing favorable habitat.
- It is never optimal for any range of $A_0/\pi$ to use the additional favorable resource to mitigate the effect of the unfavorable interior patch.

**Plot:** $\mu_0$ versus $A_0/\pi$: Heavy solid (new boundary patch); Solid curve (re-inforce favorable interior patch); dashed curve (weaken unfavorable patch). Right figure is a zoom of left.
Persistence Problem: Main Comment

**Key Remark:** These qualitative results show that, given some fixed amount of favorable resources to distribute, the optimal strategy is to clump them all together at a point on the boundary of the domain, and more specifically at the corner point of the boundary (if any are present) with the smallest angle $\leq 90^\circ$. This strategy will minimize $\mu_0$, thereby maximizing the chance for the persistence of the species.

**Remark:** Starting with the algebraic equation for $\mu_0$ as derived by formal asymptotics, these qualitative results regarding fragmentation are rigorous results based on manipulating the formula for $\mu_0$. A key issue then is to give a rigorous proof of the expression for $\mu_0$. 
The persistence threshold has a two-term asymptotic expansion

\[ \lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \cdots, \quad \nu \equiv -1/ \log \varepsilon. \]

**Remark:** The minimization of \( \lambda_1 \) is typically accomplished by optimizing \( \mu_0 \). However, in certain degenerate cases, the problem of optimizing \( \lambda_1 \) requires the examination of the \( \mu_1 \) term.

In particular, suppose that the boundary \( \partial \Omega \) is smooth, and that there is one favorable patch. Then, to optimize \( \mu_0 \), we must put this patch on the boundary. To determine which boundary point to center the patch is optimal, we must optimize \( \mu_1 \). For a boundary patch of radius \( \rho_1 \), then the Principal Result becomes

\[
\begin{align*}
\mu_1 &= -\mu_0 \left( \frac{1}{4} + \pi R_s(x_0; x_0) - \log \rho_1 \right) \\
\mu_0 &\equiv \frac{2}{m + \rho_1^2} \left[ 1 - \frac{\alpha_1 \pi m + \rho_1^2}{2|\Omega|m_b} \right] > 0.
\end{align*}
\]
**Principal Result:** For a single boundary patch centered at $x_0$ on a smooth boundary $\partial \Omega$, then $\lambda_1$ is minimized at the global maximum of the regular part $R_s(x_0; x_0)$ of the surface Neumann Green function.

Recall that $R_s(x_0; x_0)$ is defined via

\[
\triangle G_s = \frac{1}{|\Omega|}, \quad x \in \Omega; \quad \partial_n G_s = 0, \quad x \in \partial \Omega \setminus \{x_0\}; \quad \int_{\Omega} G_s \, dx = 0,
\]

\[
G_s(x; x_0) \sim -\frac{1}{\alpha_0 \pi} \log |x - x_0| + R_s(x_0; x_0), \quad \text{as} \quad x \to x_0 \in \partial \Omega.
\]

**Question:** For $\partial \Omega$ smooth, is the global maximum of $R_s(x_1; x_1)$ obtained at the global maximum of the boundary curvature $\kappa$? (No; we can find a counterexample for smooth perturbations of the unit disk, by deriving a perturbation formula for $R_s$)

**Remark:** Given a pre-existing patch distribution, finding the optimal location of a new favorable habitat may also require optimizing the $\mathcal{O}(\nu^2)$ term.
2nd-Order Optimization III

Pre-Existing Patch Distribution Formulation:

Suppose that $\partial \Omega$ is smooth. Let $x_j$ for $j = 1, \ldots, n$ be the centers of $n$ pre-existing circular patches within $\Omega$ with local growth rates $m_j$ for $j = 1, \ldots, n$.

Add a new favorable habitat, centered at $x_0$, and assume that $\mu_0$ is smallest when it is located on $\partial \Omega$ rather than inside $\Omega$.

To determine the point on $\partial \Omega$ that minimizes $\lambda_1$, we must optimize

$$
\mu_1 = \mu_0 \left( -\frac{1}{4} + \frac{\kappa^T (P - \pi G_m) \kappa + \kappa_0^2 \log \rho_0 - \pi p(x_0)}{\kappa^T \kappa + \kappa_0^2} \right).
$$

where $\kappa = (\kappa_1, \ldots, \kappa_n)^T > 0$, $P$, and $G_m$ refer to the pre-existing patches. To minimize $\mu_1$, we must maximize $p(x_0)$ defined by

$$
p(x_0) = \kappa_0^2 R_m(x_0; x_0) + 2 \sum_{j=1}^{n} \sqrt{2\kappa_0 \kappa_j G_m(x_j; x_0)}, \quad \kappa_0 \equiv \frac{m_0 \rho_0^2}{2 - \mu_0 m_0 \rho_0^2},
$$

The scalar represents the interaction of the additional favorable boundary patch with the fixed patch distribution.
2nd-Order Optimization IV

Example: Let $\Omega$ be the unit disk with $n$ pre-existing favorable resources of a common radius $\varepsilon$ and growth rate $m_j = m_c > 0$ that are equally-spaced on a concentric ring of radius $r$ with $0 < r < 1$ at $x_j = \exp(2\pi ij/n)$ for $j = 0, \ldots, n - 1$. Then, $\kappa_j = \kappa_c$ for $j = 1, \ldots, n$. We add an extra favorable resource on the boundary at angle $\theta_0$. Since the $G$-functions are known for the disk, we calculate (after some lengthy algebra) that

$$p(\theta_0) = \frac{\kappa_0^2}{8\pi} + \frac{\kappa_0}{2\pi} \left( \frac{r^2 - 1}{2} \right) (m_b - \kappa_0) - \frac{\sqrt{2}\kappa_0\kappa_c}{\pi} \chi(\theta_0),$$

where $\chi(\theta_0) = \log \left[ (r^n - \cos(n\theta_0))^2 + \sin^2(n\theta_0) \right]$.

The local minima of $\chi(\theta_0)$ (local maxima of $p(\theta_0)$) are at $\theta_0 = \frac{2\pi j}{n}$ for $j = 1, \ldots, n - 1$.

Thus, for a ring of $n$ pre-exisiting equally-spaced favorable patches, the optimal boundary locations for one additional favorable boundary patch is to put it at the shortest distance to any of the $n$ favorable habits on the ring.
**Persistence Problem: Open Problems**

**Open I:** Give a rigorous proof of the formula for $\mu_0$.

**Open II:** Consider the effect of a predator $v$, modeled in $\Omega$ by

$$
\begin{align*}
    u_t &= D \Delta u + m_\varepsilon(x)u(1 - u) - \beta uv, \\
    v_t &= \Delta v - \sigma v + \mu + \beta uv,
\end{align*}
$$

with $\partial_n u = \partial_n v = 0$ for $x \in \partial \Omega$. One might guess that a predator has an advantage when its prey is concentrated. Does the optimal strategy for the prey still remain the same as for the basic problem?

**Open III:** Consider including the weak Allee effect for which

$$
\Delta u + \lambda m_\varepsilon(x)u(1 - u)(a + u) = 0, \quad x \in \Omega; \quad \partial_n u = 0 \quad x \in \partial \Omega.
$$

The extinction threshold is now a saddle node bifurcation point.

![Diagram of bifurcation diagram](HK – p.35)
Plate will deflect in the presence of an electric field

Top plate can make contact with the lower plate (i.e. touchdown) when \( V > V^* \) at some quenching time \( t = T \).

Device can act as a switch, valve, or capacitor.

If \( V > V^* \) then no stable steady-state solutions. The threshold \( V^* \) is called the pull-in voltage threshold.
Nonlinear Biharmonic Problems of MEMS II

For small aspect ratio, the plate deflection $u$ in $\mathbb{R}^2$ satisfies

$$u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega; \quad u = u_n = 0, \quad x \in \partial \Omega.$$

- We focus on clamped boundary conditions.
- The singular nonlinearity represents a Coulomb attractive force; $\lambda$ is proportional to $V^2$; $\delta$ represents bending rigidity.
- Model originally derived by Pelesko (2000) in the narrow gap limit. It neglects self-stretching term, fringing-field effect, nonlocal term due to external circuit, etc...

Main Questions:

- **Pull-in Threshold:** Of importance for applications is the location of the saddle-node point at the end of the minimal solution branch for $|u|_\infty$ versus $\lambda$. This sets the stable operating range of the device.
- **Solution Multiplicity:** An interesting theoretical question is how does the global bifurcation diagram depend on $\delta$?
Nonlinear Biharmonic Problems of MEMS III


\[ \Delta u = \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega. \]

For the unit disk the numerically computed bifurcation diagram is:

Left: Bifurcation diagram

Right: Zoom of left figure.

Key Features:

- In the unit disk there are an infinite number of fold points with limiting behavior \( \lambda \to 4/9 \) as \( u(0) + 1 \equiv \varepsilon \to 0^+ \).

- In contrast, for the unit slab there is either zero, one, or two steady-state solutions.
Perturbation of the Membrane Problem by $\delta$

$$u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega; \quad u = u_n = 0, \quad x \in \partial \Omega.$$ 

Numerical computations with either shooting or pseudo-arclength yield:

Left and Middle (Unit Disk): $\delta = 0.0001, 0.01, 0.05, 0.1$. 
Right (Unit Square): $\delta = 0.0001, 0.001, 0.01$.

- **Practical Interest for Engineers**: Derive perturbation results for $\delta \ll 1$ and for $\delta \gg 1$ for the saddle-node point on minimal branch (Lindsay, MJW, MAA (2008)).

- **Theoretical Interest**: Infinite fold point structure destroyed when $\delta > 0$, and there is a maximal solution branch with $\lambda \to 0$ and $|u|_{\infty} \to 1^-$. 

HK – p.39
Asymptotics of Maximal Solution Branch

- **In the Unit Ball in** $\mathbb{R}^2$ **construct the limiting asymptotics of the maximal solution branch to the pure Biharmonic problem**

$$\Delta^2 u = -\lambda/(1 + u)^2, \quad 0 < r < 1; \quad u(1) = u_r(1) = 0,$$

for which $\lambda \to 0$ as $u(0) + 1 = \varepsilon \to 0^+$.

- **In the Unit Ball in** $\mathbb{R}^2$ **perform a similar calculation for the mixed Biharmonic operator**

$$\delta \Delta^2 u - \Delta u = -\lambda/(1 + u)^2, \quad 0 < r < 1; \quad u(1) = u_r(1) = 0.$$

- **In a General 2-D Domain** **construct the limiting asymptotics of the maximal solution branch to**

$$\Delta^2 u = -\lambda/(1 + u)^2, \quad x \in \Omega; \quad u = \partial_n u = 0, \quad x \in \partial\Omega.$$

Where is the concentration point $x_0$? What is the asymptotics for $\lambda$ as $u(x_0) + 1 = \varepsilon \to 0^+$?
Biharmonic Operator and Point Constraints I

Simple Model Problem: For $\varepsilon \to 0$, consider in the 2-D annulus $0 < \varepsilon < r < 1$:

$$\Delta^2 u = 0, \quad 0 < \varepsilon < r < 1,$$

$$u(1) = 1, \quad u_r(1) = 0; \quad u(\varepsilon) = u_r(\varepsilon) = 0.$$

We first find the exact solution and then expand it as $\varepsilon \to 0$.

Since the radial solutions are linear combinations of \{r^2, r^2 \log r, \log r, 1\}, the solution with $u(1) = 1$ and $u_r(1) = 0$ has the form

$$u = A(r^2 - 1) + B r^2 \log r - (2A + B) \log r + 1.$$

Upon satisfying $u(\varepsilon) = u_r(\varepsilon) = 0$, we get two equations for $A$ and $B$.

By solving these equations in the limit $\varepsilon \to 0$, we get

$$B \sim -2 - 8\varepsilon^2 (\log \varepsilon)^2, \quad A \sim 1 + 4\varepsilon^2 (\log \varepsilon)^2.$$
This gives the two-term outer approximation in $r \gg O(\varepsilon)$:

$$u \sim u_0(r) + \varepsilon^2 (\log \varepsilon)^2 u_1(r) + \cdots ,$$

where

$$u_0 = r^2 - 2r^2 \log r, \quad u_1 = 4(r^2 - 1) - 8r^2 \log r .$$

**Key:** In the limit $\varepsilon \to 0$, the outer solution does not tend to the simple solution $u = 1$ that occurs when there is no hole.

**Note:** $u_0$ is not $C^2$. It does satisfy point constraint $u_0(0) = 0$. In fact $u_0 = 1 + cG(r : 0)$, where $\Delta^2 G = \delta(x)$ and $c = -1/G(0; 0)$.

**Remarks:** Point constraints are allowed since $G_f = \frac{1}{8\pi} r^2 \log r$ in $\mathbb{R}^2$.

Satisfying point constraints by using the Biharmonic Green’s function is the basis of **biharmonic spline interpolation**, i.e. writing the interpolant as $f = \sum_{i=1}^{N} \alpha_i G(x; x_i)$ where we have determined the $\alpha_i$ by fitting the data $f = f_j$ at $x = x_j$, for $j = 1, \ldots, N$. 
Consider $\Delta^2 u = -\lambda/(1 + u)^2$, with $u = u_r = 0$ on $r = 1$.

**Principal Result:** *In the limit* $\varepsilon \equiv u(0) + 1 \to 0^+$, *the limiting asymptotic behaviour of the maximal solution branch in the outer region, away from $r = 0$, is*

$$u = u_0 + \frac{\varepsilon}{\sigma} \log \sigma u_{1/2} + \frac{\varepsilon}{\sigma} u_1 + \varepsilon \log \sigma u_{3/2} + \varepsilon u_2 + O(\varepsilon \sigma \log \sigma),$$

$$\lambda = \frac{\varepsilon}{\sigma} \left[ \lambda_0 + \sigma \lambda_1 + O(\sigma^2) \right] ; \quad \lambda_0 = 32, \quad \lambda_1 = 16 \left( \log 2 - \frac{\pi^2}{6} \right),$$

*where* $\sigma = -1/\log \gamma$ *and the boundary layer width* $\gamma$ *are determined in terms of* $\varepsilon$ *by* $-\gamma^2 \log \gamma = \varepsilon$. *The point constraint* $u_0(0) = -1$ *holds, and*

$$u_0 = -1 + r^2 - 2r^2 \log r, \quad u_{1/2} = -\frac{\lambda_0}{16} u_0, \quad u_{3/2} = -\frac{\lambda_1}{16} u_0.$$

*Here* $u_{1/2}, u_{3/2}$ *are switchback terms proportional to* $u_0$. 
In addition, \( u_1 \) and \( u_2 \) are the unique solutions of

\[
\Delta^2 u_1 = -\frac{\lambda_0}{(1 + u_0)^2}, \quad 0 < r < 1; \quad u_1(1) = u_{1r}(1) = 0,
\]

\[
u_1 = \frac{\lambda_0}{16} \log(-\log r) + \frac{\lambda_0}{16} + \mathcal{O}(\log^{-1} r), \quad r \to 0,
\]

\[
\Delta^2 u_2 = -\frac{\lambda_1}{(1 + u_0)^2}, \quad 0 < r < 1; \quad u_2(1) = u_{2r}(1) = 0,
\]

\[
u_2 = \frac{\lambda_1}{16} \log(-\log r) + \frac{1}{16} (\lambda_0 + \lambda_1) - \log 2 + \frac{\lambda_0}{16} \log r + \mathcal{O}(\log^{-1} r), \quad r \to 0.
\]

In the inner region with \( \rho = r/\gamma \), we have

\[
u = -1 + \varepsilon v(\rho), \quad v(\rho) = v_0 + \sigma v_1 + \sigma^2 v_2 + \cdots; \quad \sigma = \left(\frac{-1}{\log \gamma}\right),
\]

where \( v_0 = 2\rho^2 + 1 \) is the unique solution of

\[
\Delta^2 v_0 = 0, \quad 0 < \rho < \infty; \quad v_0(0) = 1, \quad v'_0(0) = v''_0(0) = 0;
\]

\[
v_0 \sim 2\rho^2, \quad \text{as} \quad \rho \to \infty.
\]
The higher order (inner) terms $v_1$ and $v_2$ are the unique solutions of

$$\Delta^2_\rho v_1 = -\frac{\lambda_0}{v_0^2}, \quad 0 < \rho < \infty; \quad v_1(0) = v'_1(0) = v'''_1(0) = 0;$$

$$v_1 \sim -2\rho^2 \log \rho + \rho^2 + \cdots, \quad \text{as} \quad \rho \to \infty.$$

$$\Delta^2_\rho v_2 = -\frac{\lambda_1}{v_0^2} + \frac{2\lambda_0}{v_0^3} v_1, \quad 0 < \rho < \infty; \quad v_2(0) = v'_2(0) = v'''_2(0) = 0,$$

$$v_2 = O[\log^3(\rho)], \quad \text{as} \quad \rho \to \infty.$$

To calculate $\lambda_0$, we use the $v_1$ equation to obtain

$$\lim_{R \to \infty} \left[ \int_0^R \left( \Delta^2_\rho v_1 + \frac{\lambda_0}{(1 + 2\rho^2)^2} \right) \rho d\rho \right] = \lim_{R \to \infty} \left[ \rho \frac{d}{d\rho} \left( \Delta_\rho v_1 \right) \bigg|_{\rho=R} + \frac{\lambda_0}{4} \right] = 0,$$

which yields $\lambda_0 = 32$. Similarly, from the $v_2$ equation we get

$$\lambda_1 = 8\lambda_0 \int_0^\infty \frac{v_1}{v_0^3} \rho d\rho = 16 \left( \log 2 - \frac{\pi^2}{6} \right),$$

as obtained by calculating $v_1$ explicitly, and performing the integration.
Key Points in the Construction: IV

- For $\varepsilon \to 0^+$ and $\lambda \to 0$, then $\lambda/(1 + u)^2 \to 0$ except in a narrow zone near $r = 0$, where $u = -1 + O(\varepsilon)$.

- Leading order term $u_0$ in the outer region satisfies $\Delta^2 u_0 = 0$ in $0 < r < 1$, with $u_0 = u_0r = 0$ on $r = 1$. We must impose the point constraint $u_0(0) = -1$ in order to match to the inner solution. Thus,

  $$u_0 = -1 + r^2 - 2r^2 \log r.$$ 

- If we expand $u = u_0 + \nu u_1$ and $\lambda = \nu \lambda_0$, then $\Delta^2 u_1 = -\lambda_0/(1 + u_0)^2$, for which $u_1 \rho \sim \frac{\lambda_0}{16} \log (-\log r)$ as $r \to 0$. This divergence of the particular solution as $r \to 0$ requires the inclusion of switchback terms.

- To find the boundary layer width $\gamma$, set $\rho = r/\gamma$, with $\gamma \ll 1$, to obtain $u_0 \sim -1 + (-\gamma^2 \log \gamma)(2\rho^2) + \gamma^2 (\rho^2 - 2\rho^2 \log \rho)$.

- Thus, in the inner region, we set $u = -1 + \varepsilon (v_0(\rho) + \sigma v_1 + \cdots)$ with $\sigma = -1/\log \gamma$. Thus, $\gamma$ is given implicitly by $\varepsilon = -\gamma^2 \log \gamma$.

- The leading order inner problem is $\Delta^2 v_0 = 0$ with $v_0 = 2\rho^2 + 1$. The constant term 1 is then matched by adding an appropriate term in outer expansion.

- Next, we get $\Delta^2 v_1 = -\lambda_0/v_0^2$ with $v_1 \sim -2\rho^2 \log \rho + \rho^2$ as $\rho \to \infty$. 

Mixed Biharmonic in the Unit Ball: I

Let $\delta > 0$ be fixed. In the unit ball we construct a solution with $\lambda \to 0$ as $u(0) + 1 = \varepsilon \to 0^+$ for

$$\delta \Delta^2 u - \Delta u = -\lambda/(1 + u)^2, \quad 0 < r < 1; \quad u(1) = u_r(1) = 0.$$

The leading-order outer solution $u_0$ is the solution to:

$$\delta \Delta^2 u_0 - \Delta u_0 = 0, \quad 0 < r < 1; \quad u_0(1) = u_0r(1) = 0, \quad u_0(0) = -1,$$

which is given explicitly by

$$u_0 = A + B \log r + C K_0(\eta r) + D I_0(\eta r), \quad \eta \equiv 1/\sqrt{\delta}.$$ 

Here the constants are given by

$$A = [I_0(\eta) (1 + \eta K'_0(\eta)) - \eta I'_0(\eta) K_0(\eta)] \mathcal{G}(\eta),$$

$$B = C = \eta I'_0(\eta) \mathcal{G}(\eta), \quad D = -[1 + \eta K'_0(\eta)] \mathcal{G}(\eta),$$

where $\mathcal{G}(\eta)$ is defined by

$$\mathcal{G}(\eta) \equiv [\eta I'_0(\eta) (K_0(\eta) + \log (\eta/2) + \gamma_e) + (1 + \eta K'_0(\eta)) (1 - I_0(\eta))]^{-1},$$

and $\gamma_e \sim 0.5772$ is Euler’s constant.
The asymptotics as $r \to 0$ are

$$u_0 = -1 + \alpha r^2 \log r + \varphi r^2 + o(r^2), \quad \text{as} \quad r \to 0,$$

$$\alpha = -\left(\frac{\eta^3}{4}\right) I'_0(\eta) G(\eta),$$

$$\varphi = -\frac{\eta^2}{4} \left[\eta I'_0(\eta) \left(\log (\eta/2) + \gamma_e - 1\right) + (1 + \eta K'_0(\eta))\right] G(\eta).$$

Plot of $\alpha(\delta)$ and $\varphi(\delta)$ for $0 < \delta < 2.5$ showing $\alpha < 0$ are:

![Graph showing $\alpha(\delta)$ and $\varphi(\delta)$](image-url)
**Mixed Biharmonic in the Unit Ball: III**

**Principal Result:** *In the limit* $\varepsilon \equiv u(0) + 1 \to 0^+$, *the limiting asymptotic behaviour of the maximal radially symmetric solution branch is*

$$\lambda = \frac{\delta \varepsilon}{\sigma} \left[ \lambda_0 + \sigma \lambda_1 + \mathcal{O}(\sigma^2) \right],$$

*where* $\sigma = -1/\log \gamma$ *and the boundary layer width* $\gamma$ *is determined in terms of* $\varepsilon$ *by* $-\gamma^2 \log \gamma = \varepsilon$. *Here,*

$$\lambda_0 = 8\alpha^2, \quad \lambda_1 = -\frac{\lambda_0}{2} \left[ \frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\varphi}{\alpha}\right) \right],$$

*where* $\alpha(\delta)$ *and* $\varphi(\delta)$ *are determined by the local behavior of* $u_0$ *as* $r \to 0$. 
Comparison of Asymptotics and Full Numerics

Pure Biharmonic: Comparison with Full Numerics
Comparison of Asymptotics
1-term (dotted), 2-term (dashed), and Numerics (solid).

Mixed Biharmonic: Comparison of 2-term Asymptotics with Full Numerics

(a) $|u(0)|$ vs. $\lambda$ (smaller $\delta$)  
(b) $|u(0)|$ vs. $\lambda$ (larger $\delta$)
Consider $\Delta^2 u = -\lambda/(1 + u)^2$ in $\Omega$, with $u = \partial_n u = 0$ on $\partial \Omega$.

**Principal Result:** In the limit $\varepsilon \equiv u(x_0) + 1 \to 0^+$, the limiting asymptotics of the maximal solution branch in the outer region, away from $x_0$, and $\lambda$ is

$$u = u_0 + O(\varepsilon \sigma^{-1} \log \sigma), \quad \lambda = \frac{\varepsilon}{\sigma} \lambda_0 + \varepsilon \lambda_1 + O(\varepsilon),$$

where $\sigma = -1/\log \gamma$ and the boundary layer width $\gamma$ is given implicitly in terms of $\varepsilon$ by $-\gamma^2 \log \gamma = \varepsilon$. Here

$$u_0 = -\frac{G(x; x_0)}{R(x_0; x_0)},$$

with point constraint $u_0(x_0) = -1$, where $G(x; x_0)$ satisfies

$$\Delta^2 G = \delta(x - x_0), \quad x \in \Omega; \quad G = \partial_n G = 0, \quad x \in \partial \Omega,$$

$$G(x; x_0) = \frac{1}{8\pi} |x - x_0|^2 \log |x - x_0| + R(x; x_0).$$
Concentration in Arbitrary 2-D Domain II

To leading order, the concentration point $x_0 \in \Omega$ satisfies

$$\nabla_x R(x; x_0)|_{x=x_0} = 0,$$

provided that $R(x_0; x_0) > 0$.

As $x \to x_0$, with $r = |x - x_0|$, we identify $\alpha$ and $\beta$ by

$$u_0 \sim -1 + \alpha r^2 \log r + r^2 (\beta + a_c \cos 2\theta + a_s \sin 2\theta) + \cdots,$$

where $\alpha < 0$ by assumption, and $\beta$ (sign $\pm$) are

$$\alpha \equiv -\frac{1}{8\pi R(x_0; x_0)}, \quad \beta \equiv -\frac{1}{4R(x_0; x_0)} \left[ \frac{\partial^2 R}{\partial x_1^2} + \frac{\partial^2 R}{\partial x_2^2} \right]_{x=x_0}.$$

Finally, $\lambda_0$ and $\lambda_1$ are given by

$$\lambda_0 = 8\alpha^2, \quad \lambda_1 = -\frac{\lambda_0}{2} \left[ \frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\beta}{\alpha}\right) \right].$$

Thus, 2-term asymptotics of $\lambda$ are determined by properties of the regular part of the Biharmonic Green function.

Note: $R_{00} = R(x_0; x_0)$ and $\text{Trace}(R_{00})$ can be computed by fast multipole methods for Low Reynolds number flow (Kropinski).
Comparison of Asymptotics and Numerics in Square Domain: For the square $[-1, 1]^2$, then $x_0 = 0$, and to evaluate asymptotic result we computed

$$R_{00} \approx 0.0226 \ldots, \quad \text{Trace}(R_{00}) \approx -0.0892 \ldots.$$ 

Numerics (solid); 1-term asymptotics (dots); 2-term asymptotics (dashed)
Concentration in Arbitrary 2-D Domain IV

Class of Dumbbell-Shaped Domains

Determine points \( x_0 \) for which \( \nabla R|_{x_0} = 0 \) for a one-parameter family of mappings of the unit disk \( B \):

\[
w = f(z; b) = \frac{(1 - b^2)z}{z^2 - b^2}, \quad z \in B.
\]

For various values of \( b \), numerical values for \( R(x_0; x_0) \) and Trace \( (\mathcal{R}_{00}) \) at the points \( x_0 = (x_0, 0) \) where \( dR/dx_0 = 0 \).

<table>
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<th>( b )</th>
<th>( x_0 )</th>
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<td>( 9.59768 \times 10^{-5} )</td>
<td>( 0.379489 \times 10^{-2} )</td>
</tr>
<tr>
<td></td>
<td>0.494500</td>
<td>( 4.94718 \times 10^{-3} )</td>
<td>( 3.11557 \times 10^{-2} )</td>
</tr>
</tbody>
</table>
Concentration in Arbitrary 2-D Domain V
Plot of the domain and $R_{00}$ along the horizontal axis $(x_0, 0)$

(c) $b = 2.0$

(d) $b = b_c = 1.83995$

(e) $b = 1.5$

(f) $b = 1.05$
Open Questions

Questions:

- Give a rigorous proof of the limiting concentration behavior in a disk and an arbitrary domain.

- Can one find an example with $\nabla R|_{x_0} = 0$, but $R(x_0, x_0) < 0$. Then, concentration at $x_0$ cannot occur. This might be theoretically possible since the Green function $G$ is not guaranteed to be of one sign.

- Construct solutions in 2-D domains with multiple concentrations.

- Describe in detail the breakup of infinite fold point structure associated with the membrane MEMS problem when $\delta > 0$ but small. In particular, how many fold points are there for $\delta$ small but fixed?

- Can we construct limiting asymptotics of some related Biharmmonic nonlinear eigenvalue problems with other nonlinearities and different dimension $N$?
References

The following papers are available on my UBC website:

