Delayed Reaction Kinetics and The Stability of Spikes

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Outline

For $\epsilon \to 0$ spatially localized solutions can occur for

$$v_t = \epsilon^2 v_{xx} + g(u,v); \quad \tau u_t = u_{xx} + f(u,v), \quad x \in \mathbb{R}.$$ 

Gene Expression Time Delays: In applications to morphogenesis, one might need to account for gene expression time delays whereby

$g(u,v) \to g(u_T, v_T)$ and $f(u,v) \to f(u_T, v_T)$ with $u_T = u(x, t - T)$ and $v_T = v(x, t - T)$, where $T > 0$ is fixed delay.


Our Goal: Analyze GM Model and Spike Stability with Delay

- [Iron, Khalil] (pulse stability and dynamics with delay).
1-D GM Infinite-Line Model With Delay I

Consider a 1-spike steady-state solution to the GM RD model for $x \in \mathbb{R}$ with delay:

$$
\begin{align*}
    v_t &= \epsilon^2 v_{xx} - v + v_T^p / u_T^q, \\
    \tau u_t &= u_{xx} - u + \epsilon^{-1} v_T^m / u_T^s
\end{align*}
$$

where $u_T \equiv u(x, t - T)$, $v_T \equiv v(x, t - T)$.

Remarks: Assume $0 < \epsilon^2 \ll 1$, $D = \mathcal{O}(1)$, and $\tau > 0$.

- Assume the usual conditions on $(p, q, m, s)$ that

$$
p > 1, \quad q > 0, \quad m > 1, \quad s \geq 0, \quad \xi \equiv \frac{qm}{p - 1} - s - 1 > 0.
$$

- Prototypical exponent set $(p, q, m, s) = (2, 1, 2, 0)$.

- Explicitly solvable NLEP set $p = 2m - 3$, $m > 2$, where the spectral problem is highly tractable when delay in only inhibitor kinetics.

Goal: Understand the Effect of Delay in Different Terms

- **Case I:** Delay in inhibitor only $v_T^p / u_T^q$ and $v_T^m / u_T^s$.

- **Case II:** Delay in inhibitor and activator $v_T^p / u_T^q$ and $v_T^m / u_T^s$.

- **Case III:** Delay in activator only $v_T^p / u_T^q$ and $v_T^m / u_T^s$. 
Infinite-Line Model: Equilibrium

Steady-State [IWW]: For $\epsilon \to 0$, a one-spike steady-state solution is

$$v_e(x) \sim U_0^{q/(p-1)} w(\epsilon^{-1} x) ; \quad u_e(x) \sim U_0 \frac{G_0(x)}{G_0(0)} ; \quad U_0 = \left( \frac{1}{2} \int_{-\infty}^{\infty} w^m \, dy \right)^{-1/\xi} ,$$

where $w(y)$ is the homoclinic satisfying

$$w'' - w + w^p = 0 , \quad w(0) > 0 , \quad w'(0) = 0 , \quad w \to 0 \quad \text{as} \quad |y| \to \infty .$$

Here $G_0(x) = e^{-|x|}/2$ is the Green's function satisfying $G_{0xx} - G_0 = -\delta(x)$ with $G_0 \to 0$ as $|x| \to \infty$. 

Caption: A one-pulse solution.
Infinite-Line Model: Linearization

**Linearize:** Assume delay in both activator and inhibitor:

\[ v = v_e + e^{\lambda t} \Phi(x/\epsilon), \quad u = u_e + e^{\lambda t} \eta. \]

**Case II:** We get the NLEP spectral problem for \( \Phi(y) \) for \( \Phi \in H^1(\mathbb{R}) \):

\[
L_\mu \Phi - \frac{mq\mu^2}{\sqrt{1 + \tau\lambda + s\mu}} w^p \frac{\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} = \lambda \Phi, \quad -\infty < y < \infty,
\]

where

\[
L_\mu \Phi \equiv \Phi'' - \Phi + pw^{p-1} \mu \Phi, \quad \mu \equiv e^{-\lambda T}.
\]

We refer to \( L_\mu \) as the delayed local operator and \( L_1 \) as the local operator which corresponds to setting \( T = 0 \).

The NLEP with \( L_1 \) and \( \mu = 1 \) has been extensively studied ([IWW2001], [WW2003]) and a Hopf bifurcation can occur when \( \tau \) exceeds a threshold.
Case I: Inhibitor Delay Only

Case I (Delay in Inhibitor Only): Suppose $v_T = v$ in reaction kinetics. The NLEP problem for $y \in \mathbb{R}$ with $\Phi \in H^1(\mathbb{R})$ is (with $\mu = e^{-\lambda T}$):

$$L_1 \Phi - \chi w^p \frac{\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} = \lambda \Phi, \quad \chi \equiv \frac{mq\mu}{\sqrt{1 + \tau \lambda + se^{-\lambda T}}}.$$ 

Question: Under what conditions is $\text{Re}(\lambda) > 0$ in the $\tau$ versus $T$ plane due to a Hopf Bifurcation crossing? Easiest: “Explicitly Solvable One”.

Lemma [NW], [FWW] Set $p = 2m - 3$ and $m > 2$. Then, for $\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy \neq 0$, any eigenvalues in $\text{Re}(\lambda) > 0$ must satisfy

$$\lambda = (m^2 - 2m) - \frac{m}{2} \chi.$$ 

The other eigenpairs are simply $(\Phi, \lambda) = (w', 0)$ and any negative real eigenvalues of $L_1$.

Proof: Uses the key identity $L_1 (w^{m-1}) = (m^2 - 2m)(w^{m-1})$ together with Green’s second identity.
Case I: Inhibitor Delay Only (Explicit)

When there is no delay $T = 0$ we can have a HB.

Main Result: [NW] When $T = 0$, ∃ a unique Hopf bifurcation value $\tau^0_H$, with corresponding eigenvalue $\lambda = i\lambda^0_{IH}$, given by

$$\tau^0_H = \frac{(m^2 q)^2}{2\zeta^2} \left( \beta - \frac{2s}{m^2 q} \right), \quad \lambda^0_{IH} = \sqrt{\zeta - \beta^2},$$

where $\zeta > \beta^2$ with $\beta \equiv m^2 - 2m$ is the smallest root of the quadratic

$$4(s^2 - 1)\zeta^2 - [(m^2 q)^2 + 4\beta s(m^2 q)] \zeta + 2\beta^2(m^2 q)^2 = 0.$$

This result yields the following special cases:

- For $(p, q, m, s) = (3, 2, 3, 1)$, then $\tau^0_H = 2.5$ and $\lambda^0_{IH} = 3/\sqrt{5} \approx 1.34$.
- For $(p, q, m, s) = (3, 2, 3, 0)$, then $\tau^0_H = [13 + 3\sqrt{17}]/12 \approx 2.114$ and $\lambda^0_{IH} \approx 3\sqrt{3\sqrt{17} - 11}/\sqrt{2} \approx 2.482$.

[NW], Y. Nec, MJW, An Explicitly Solvable NLEP... [2013].
Case I: Inhibitor Delay Only (Explicit)

**Question:** Is there a HB value of $T$ even when $\tau = 0$? If so, problem is more unstable with delay.

**Main Result:** [FWW] *When $\tau = 0$, then only if* $0 < s < 1$, *there exists a minimum HB threshold* $T = T^f > 0$ *and corresponding eigenvalue* $\lambda_{IH}^f$ *given by*

$$T^f \equiv \frac{1}{\lambda_{IH}^f} \sin^{-1} \left( (1 - s^2)^{1/2} \left[ (\xi + 1)^2 - 1 \right]^{1/2} \right),$$

$$\lambda_{IH}^f \equiv \frac{\beta}{\sqrt{1 - s^2}} \left[ (\xi + 1)^2 - 1 \right]^{1/2}.$$

*where*

$$\beta = m^2 - 2m, \quad \xi \equiv \frac{mq}{2m - 4} - (1 + s).$$

In particular, for $(p, q, m, s) = (3, 2, 3, 0)$, where $\beta = 3$ and $\xi = 2$, we get

$$\lambda_{IH}^f = 6\sqrt{2}, \quad T^f = \frac{1}{6\sqrt{2}} \sin^{-1} \left( \frac{2\sqrt{2}}{3} \right) \approx 0.145.$$
Case I: Inhibitor Delay Only (Explicit)

Parameterization: We parameterize the HB threshold in the $\tau$ versus $T$ plane to determine the HB boundary in parameter space. Let $\lambda = i\lambda_{IH}$, where $\lambda_{IH} \equiv \beta \omega_0$. We calculate that:

Main Result: [FWW] In terms of $\lambda_{IH} = \beta \omega_0$ with $\beta = m^2 - 2m$, we have that a HB occurs on the boundary

$$\tau = \frac{\sqrt{\alpha^2 - 1}}{\beta \omega_0}, \quad \alpha = \frac{(\xi + 1)^2 + s^2 \omega_0^2}{1 + \omega_0^2}.$$  

$$T = \frac{1}{\beta \omega_0} \sin^{-1} \left[ \frac{B_+ \omega_0 (\xi + s + 1) + B_- (s \omega_0^2 - \xi - 1)}{\alpha (1 + \omega_0^2)} \right],$$

where

$$B_\pm \equiv \sqrt{\frac{\alpha \pm 1}{2}}, \quad \alpha \equiv \sqrt{1 + \tau^2 \beta^2 \omega_0^2 \lambda_{IH}^2}.$$  

For $s > 1$, we can show that $T \to 0$ as $\tau \to 0$, with asymptotics:

$$T \sim \frac{\tau}{\sqrt{s^4 - 1}} \sin^{-1} \left( \frac{\sqrt{s^2 - 1}}{\sqrt{2s}} \right), \quad \lambda_{IH} \sim \frac{\sqrt{s^4 - 1}}{\tau}, \quad s > 1,$$
Caption: The shaded region in the $\tau$ versus $T$ plane is linearly stable. A HB occurs on the boundary. Left: $(p, q, m, s) = (3, 2, 3, 0)$. Right: $(p, q, m, s) = (3, 2, 3, 1)$. The dashed line is the limiting approximation.

Conclusion (Explicit Case): With only inhibitor delay, the spike is more unstable than with no delay.
Case I: Inhibitor Delay Only (Explicit)

Caption: \((p, q, m, s) = (3, 2, 3, 0)\). Left: HB curves (dashed curves) where additional complex conjugate pairs, indexed by \(n\), enter the region \(\text{Re}(\lambda) > 0\). Thick line: when eigenvalues first appear as a double root on the positive real axis. Right: Eigenvalue paths for \(\tau = 1\). Heavy solid curve gives the minimum threshold.

**Eigenvalues near** \(\lambda = 0\): For \((p, q, m, s) = (3, 2, 3, 0)\)

\[
\lambda \sim \frac{[\ln 3 + 2n\pi i]}{T} \left(1 + \frac{1}{T} \left[\frac{1}{3} - \frac{\tau}{2}\right] + \cdots\right).
\]
Case I: Inhibitor Delay (Validation)

Numerics: HB threshold $\tau_H$ vs $T$ on $|x| \leq L$ for $L = 1$, $L = 2$ and $L = \infty$. Exponent set $(3, 2, 3, 0)$

Caption: $v(0, t)$ vs. $t$ for $L = 2$, $T = 0.05$, $\epsilon = 0.05$, computed by MOL and $dde23$ of MATLAB: $\tau = 1.0$ (left), $\tau = 1.3$ (middle), and $\tau = 2.0$ (right). Theory yields $\tau_H \approx 1.23$. 
Case I and Case II Comparison

For \((p, q, m, s) = (2, 1, 2, 0)\) compare stability region in the \(\tau\) versus \(T\) plane when only inhibitor delay (Case I) and when both activator and inhibitor delay (Case II).

Caption: Case I (left); Case II (right)

Key Conclusion: The stability regions are qualitatively similar for the two cases. With delayed inhibitor kinetics (regardless of activator), a one-pulse solution is more unstable.
Case III: Activator Delay Only

Consider a 1-spike solution on the infinite line for

\[ v_t = \epsilon^2 v_{xx} - v + v^2_T / u, \quad \tau u_t = u_{xx} - u + \epsilon^{-1} v^2_T, \]

By linearizing, \( v = v_e + e^{\lambda t} \Phi(y) \), with \( y = x / \epsilon \), we get the NLEP

\[ L_\mu \Phi - \chi(\tau \lambda) \mu w^2 \frac{\int_{-\infty}^{\infty} w \Phi \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} = \lambda \Phi ; \quad \chi(\tau \lambda) \equiv \frac{2}{\sqrt{1 + \tau \lambda}}, \]

with \( \Phi \to 0 \) as \( |y| \to \infty \). The delayed local operator, \( L_\mu \), is

\[ L_\mu \Phi \equiv \Phi'' - \Phi + 2w\mu \Phi, \quad \mu \equiv e^{-\lambda T}, \]

The eigenvalues of the NLEP are the roots of \( g(\lambda) = 0 \), where

\[ g(\lambda) \equiv \frac{1}{\mu \chi(\tau \lambda)} - F_\mu(\lambda); \quad F_\mu(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w \left[ (L_\mu - \lambda)^{-1} w^2 \right] \, dy}{\int_{-\infty}^{\infty} w^2 \, dy}. \]

**Step 1**: Need results for delayed local eigenvalue problem (DLEP):

\[ L_\mu \Phi = \lambda \Phi, \quad \Phi \in H^1(\mathbb{R}). \]
The Delayed Local Eigenvalue Problem I

Main Result:[FWW2] Any eigenvalue $\lambda$ of the DLEP must be a root of one of the transcendental equations $K_l(\lambda) = 0$, for $l = 0, 1, 2, \ldots$, defined by

$$K_l(\lambda) \equiv 4\sqrt{1 + \lambda + 1} - \sqrt{1 + 48\mu + 2l} = 0, \quad l = 0, 1, 2, \ldots; \quad \mu \equiv e^{-\lambda T}.$$ 

The translation mode $\lambda = 0, \Phi = w'$, must be an eigenpair for all $T \geq 0$. This corresponds to $l = 1$. The continuous spectrum is $\lambda < -1$ with $\lambda$ real.

Proof: With $\mu = e^{-\lambda T}$ and $w(y) = \frac{3}{2} \text{sech}^2 (y/2)$, $\Phi(y)$ satisfies

$$\Phi'' - \Phi + 2\mu w\Phi = \lambda \Phi, \quad \Phi \in H^1(\mathbb{R}^1).$$

Put $\gamma \equiv \sqrt{1 + \lambda}$ and $\Phi(y) = w^\gamma \mathcal{Y}(z)$, with $z = z(y)$ defined $1 \rightarrow 1$ from $0 < z < 1$ to $-\infty < y < \infty$ by $2z = \left(1 - \frac{w'}{w}\right)$. This yields

$$z(1 - z)\mathcal{Y}'' + (c - (a + b + 1)z) \mathcal{Y}' - ab\mathcal{Y} = 0,$$

where the coefficients $a, b, c$ are

$$a + b + 1 = 4\gamma + 2, \quad ab = 4\gamma(\gamma - 1) + 6\gamma - 12\mu, \quad c = 1 + 2\gamma.$$

Finally, choose $a, b, c$ so that $\mathcal{Y}(z)$ is bounded as $z \rightarrow 0, 1$. 

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The Delayed Local Eigenvalue Problem II

Real Eigenvalues: First characterize any non-zero real-valued eigenvalue satisfying $\lambda > -1$ that exists for all $T \geq 0$.

- **l=0 branch:** $\mathcal{K}_0(0) < 0$, $\mathcal{K}_0(\lambda) \to +\infty$ as $\lambda \to +\infty$, and $\mathcal{K}_0'(\lambda) < 0$ on $\lambda > 0$. Thus, $\exists$ a unique $\lambda_0 = \lambda_0(T) > 0$ for any $T \geq 0$. We calculate $\lambda_0(0) = 5/4$, $\lambda_0 \sim \log(2)/T$ for $T \gg 1$, and $\lambda_0'(T) > 0$.

- **l=2 branch:** We get $\mathcal{K}_2(-1) < 0$, $\mathcal{K}_2(0) = 2 > 0$, and $\mathcal{K}_2'(\lambda) > 0$. Thus $\exists$ a unique root $\lambda_2(T)$ in $-1 < \lambda_2(T) < 0$ for any $T \geq 0$. We get $\lambda_2(0) = -3/4$, $\lambda_2 \sim \log(3/5)/T$ for $T \gg 1$.

Plot of $\lambda_0(T)$:
The Delayed Local Eigenvalue Problem III

**Question:** Can we get edge bifurcations as $T$ increases from $\lambda = -1$ of the continuous spectrum? What about complex-valued spectra?

**Edge Bifurcations:** For $l \geq 3$, a real eigenvalue bifurcates from the edge of the continuous spectrum $\lambda \leq -1$ when $T$ exceeds $T_{\text{edge}}^l \geq 0$, given by

$$T_{\text{edge}}^l \equiv \log \left( \frac{l^2 + l}{12} \right), \quad l = 3, 4, \ldots; \quad \text{with} \quad T_{\text{edge}}^{l+1} > T_{\text{edge}}^l.$$

Notice that $T_{\text{edge}}^3 = 0$.

**Proof:** set $K_l(-1) = 0$ and solve for $T$.

Since $K_l(0) > 0$, $K_l(-1) < 0$ when $T > T_{\text{edge}}^l$, $\exists$ a unique $\lambda_l(T)$ in $-1 < \lambda < 0$ when $T > T_{\text{edge}}^l$. For $T \gg 1$,

$$\lambda_l(T) \sim c_l/T, \quad c_l \equiv \log \left( \frac{12}{(l + 3)(l + 2)} \right) < 0, \quad \text{as} \quad T \to \infty.$$

**Key:** Edge bifurcations occur, but only produce stable spectra, as they remain in $\text{Re}(\lambda) < 0$ for all $T$. 
Complex Spectra: Can only occur for $l = 0$ mode. Set $\lambda = i\lambda_I$ for $l = 0$. Such a pure imaginary eigenvalue occurs when $T = T_H^n$, given by

$$T = T_H^n \equiv \frac{(\theta_0 + 2\pi n)}{\lambda_I}, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenvalue $\lambda_I$ (independent of $n$) is

$$\lambda_I = \frac{1}{8} \text{Re} \left(-1 + \sqrt{1 + 48\mu}\right) \text{Im} \left(\sqrt{1 + 48\mu}\right), \quad \mu \equiv \cos \theta_0 - i \sin \theta_0.$$

Here $\theta_0$ in $-\pi/3 \leq \theta \leq 0$ is unique root of $\mathcal{H}(\theta) = 0$:

$$\mathcal{H}(\theta) \equiv (24 \cos \theta - 7)^2 \left(12 \cos^2 \theta - 8 \cos \theta + 1\right) - 12 \sin^2 \theta.$$

Note: Uniqueness follows since $\mathcal{H}(0) > 0$, $\mathcal{H}(-\pi/3) < 0$, and $\mathcal{H}(\theta)$ monotonic on $-\pi/3 < \theta < 0$. We compute

$$\theta_0 \approx -0.99046, \quad \lambda_I \approx 2.1015,$$

and

$$T_H^1 \approx 2.5185, \quad T_H^2 \approx 5.5084, \quad T_H^3 \approx 8.4982, \quad T_H^4 \approx 11.488.$$
Caption: Complex spectra of $L_\mu$ in Re$(\lambda) \geq 0$ for $T_{H_1}^1 \leq T \leq T_{H_5}^5$. For $T \to \infty$, the paths all tend to the origin but in Re$(\lambda) > 0$. For each path, we also plot its continuation into Re$(\lambda) < 0$ for smaller delays.
Delayed Nonlocal Problem I

With $\chi(\tau \lambda) = 2/\sqrt{1 + \tau \lambda}$, the eigenvalues of the NLEP are the roots of $g(\lambda) = 0$:

$$g(\lambda) \equiv \frac{1}{\mu \chi(\tau \lambda)} - \mathcal{F}_\mu(\lambda); \quad \mathcal{F}_\mu(\lambda) \equiv \frac{\int_\infty^{-\infty} w \left[ (L_\mu - \lambda)^{-1} w^2 \right] dy}{\int_\infty^{-\infty} w^2 dy};$$

**Homotopy in $\chi(0)$:** Consider the NLEP with $\tau = 0$ for which $\chi(0) = 2$. By a homotopy in the value $\chi(0)$ from $0 < \chi(0) < 2$, we conclude that no HB is possible for the NLEP when $\tau = 0$ for any fixed $T \geq 0$.

**Caption:** Left: min value $T_H$ of $T$ vs $\chi(0)$ for a HB. Right: $\lambda_{IH}$ vs $\chi(0)$.
Delayed Nonlocal Problem II

Implication: HB occurs only on $0 \leq \chi(0) < 1$ with $\lambda_{IH} \to 0^+$ and $T_H \to +\infty$ as $\chi(0) \to 1^-$. For $\chi(0) > 1$ the NLEP does not have any HB as $T$ is increased. Since $\chi(0) = 2$ when $\tau = 0$, we conclude that no HB occurs when $\tau = 0$ as $T$ is increased.

Next, we fix $T > 0$, take $\chi = \chi(\tau \lambda)$ and let $\tau \to +\infty$. The next result shows that $\exists$ at least two positive real eigenvalues to the NLEP when $\tau \gg 1$.

Lemma: For $\lambda > 0$ real and $\lambda \neq \lambda_0(T)$, we have $F'_\mu(\lambda) > 0$. Moreover, $F_1(0) = 1$ and $F_\mu \to +\infty$ as $\lambda \to \lambda_0(T)^-$. Therefore $1/(\mu \chi) = e^{-\lambda T} \sqrt{1 + \tau \lambda/2}$ must intersect $F_\mu(\lambda)$ at least twice on the positive real axis when $\tau$ is large enough.

By continuous dependence of eigenvalue paths in $\tau$ for any fixed $T$ we obtain that:

Main Result: Let $T > 0$ be fixed. There must be a HB at some $\tau = \tau_H > 0$ depending on $T$. 
Numerics: To determine the HB threshold set $\text{Re}(g(i\lambda_{IH})) = 0$ and $\text{Im}(g(i\lambda_{IH})) = 0$ and solve $2 \times 2$ nonlinear system for the HB values $\tau_H$ and $\lambda_{IH}$ at a fixed $T \geq 0$. Path-follow in $T$.

Caption: Left: $\tau_H$ vs $T$. Middle: $\lambda_{IH}$ vs $T$. (large $T$ asymptotics (dashed) $\lambda_{IH} \sim 0.78/T$.)
Scaling Law with Activator Delay I

Question: Is there a scaling law for the HB when $T \gg 1$?

Scaling Law: For $T \gg 1$, put $\tau_H \sim \tau_0 T$ and $\lambda \sim ic_0/T$ for some $c_0 > 0$ and $\tau_0 > 0$. Then, the NLEP becomes

$$\Phi'' - \Phi + 2w [e^{-ic_0} + \cdots] \Phi - [\chi_0 e^{-ic_0} + \cdots] w^2 \int_{-\infty}^{\infty} w \Phi \, dy \int_{-\infty}^{\infty} w^2 \, dy = \left[ \frac{ic_0}{T} + \cdots \right] \Phi,$$

where $\chi_0 \equiv \chi(ic_0 \tau_0)$. Since $\Phi \sim w$ and $w'' - w + w^2 = 0$, we get

$$e^{ic_0} = 2 - \chi_0 = \frac{2 \left[ \sqrt{1 + i\tau_0 c_0} - 1 \right]}{\sqrt{1 + i\tau_0 c_0}}.$$

From the real and imaginary parts of this expression:

$$c_0 = \sin^{-1} \left( \frac{\sqrt{2}}{\alpha} \sqrt{\alpha - 1} \right) \approx 0.782106, \quad \tau_0 = \frac{\sqrt{95 + 32\sqrt{10}}}{9c_0} \approx 1.9899,$$

where $\alpha = 4(1 + \sqrt{10})/9$. Thus, $\tau_H \sim 1.99T$ and $\lambda \sim 0.782i/T$ for $T \gg 1$. 

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Caption: Right: validation of theoretical asymptotes $\lim_{T \to \infty} \frac{\tau_H}{T} \approx 1.99$ and $\lim_{T \to \infty} \lambda_{IH} T \approx 0.782$.

Key Qualitative Conclusion: A spike is stabilized by the presence of only activator delay in the reaction kinetics. The region in $\tau$ where no HB occur is larger with increasing delay.
Numerical Validation on Finite Domain

Numerics: HB threshold $\tau_H$ vs $T$ on $|x| \leq L$ for $\epsilon = 0.05$: $L = 0.2$ (dashed), $L = 1$ (dashed-dotted), $L = 2$ (solid), and $L = 10$ (heavy solid).

Caption: $v(0, t)$ vs. $t$ for $L = 2$, $T = 2$, computed by MOL and $dde23$ of MATLAB: $\tau = 5.3$ (left), $\tau = 5.6$ (middle), and $\tau = 10$ (right). Theory yields $\tau_H \approx 5.573$. 
Conclusion and Outlook

Conclusions:

- When the delay is in the activator kinetics only, a one-pulse solution has better stability properties than with no delay.
- A delay in the inhibitor kinetics is very destabilizing.

Extension to 2-D: A similar analysis can be done for 2-D multi-spot patterns to the GM model in the regime of large inhibitor diffusivity, by analyzing a 2-D NLEP problem.

Open: Oscillations with delayed kinetics appear to be subcritical. With no delay they are supercritical (Veerman, Nonlinearity, 2015). Normal form theory with delayed kinetics?

References:
