The Stability of Hot Spot Patterns for Reaction-Diffusion Models of Urban Crime

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Modeling Urban Crime I
Multidisciplinary efforts to model patterns of urban crime lead by UCLA group; A. Bertozzi, P. Brantingham, L. Chayes, M. Short, etc.. (since 2008); field data from LA police; What is best policing strategy?

UC MASC Project: http://paleo.sscnet.ucla.edu/ucmasc.html

Fig. 1. Dynamic changes in residential burglary hotspots for two consecutive three-month periods beginning June 2001 in Long Beach, CA. These density maps were created using ArcGIS.
Modeling Urban Crime II

Key References:


Observations: Criminal activity concentrates non-uniformly (good versus bad neighborhoods). Often “hot-spots” of crime are observed. Need to incorporate near-repeat victimization and elevated risk of re-victimization in a short time period.
Agent Based Models: City is represented by a square lattice. At each lattice site there is an “attractiveness” $A(x, t)$ and a “number” $N(x, t)$ of criminals. Criminals exhibit biased random walk and are more likely to move to a neighboring site with a higher attractiveness.

Criminal Behavior: Burglarize the house at site $x$ between times $t$ and $t + \delta t$ with probability

$$p_v(x, t) = 1 - e^{-A(x, t)\delta t}.$$  

If site $x$ is robbed, the burglar is removed from the lattice. After the attractiveness is updated, burglars are re-introduced at each lattice site at a rate $\Gamma$.

If a burglary at $x$ does not occur, the burglar moves to a neighbouring site $x'$ with probability

$$p_m(x'; t, x) = \frac{A(x', t)}{\sum_{x'' \sim x} A(x'', t)}.$$
An Agent-Based Model II

Modeling attractiveness: It has a static and dynamic component:

\[ A(x, t) = A^0 + B(x, t) . \]

Elevated risk of re-victimization in a short time-period with decay rate \( \omega \) and with a parameter \( \eta \ll 1 \) that models how attractiveness is spread to its neighbours:

\[
B(x, t + \delta t) = \left[ (1 - \eta)B(x, t) + \frac{\eta}{4} \sum_{x' \sim x} B(x', t) \right] (1 - \omega \delta t) + \theta E(x, t) .
\]

Here \( \theta > 0 \) and \( E(x, t) \) is the number of burglary events at site \( x \) in a time interval \((t, t + \delta t)\). Then, update attractiveness

\[
A(x, t + \delta t) = A^0 + B(x, t + \delta t)
\]

Remark: Expected value\((E) = N(x, t)p_v(x, t)\).

Numerics (Agent-Based): (stationary hot-spots, moving hot-spots, creation of new spots, etc..) Agent Based Simulation (M. Short et al:) (Movie)
The Basic Urban RD Crime Model

In the continuum limit, the resulting dimensionless PDE RD model with no flux b.c. is (Short et al., M3AS, (2008)):

\[
A_t = \varepsilon^2 \Delta A - A + PA + \alpha, \quad x \in \Omega,
\]

\[
\tau P_t = D \nabla \cdot \left( \nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha, \quad x \in \Omega.
\]

\( \varepsilon \ll 1 \) (results from \( \eta \ll 1 \))

\( P(x, t) \) is criminal density; \( A(x, t) \) is “attractiveness” to burglary.

The chemotactic drift term \( -2D \nabla \cdot \left( P \frac{\nabla A}{A} \right) \) represents the tendency of criminals to move towards sites with a higher attractiveness.

Here \( \alpha \) is the baseline attractiveness, while \( \gamma - \alpha > 0 \) is the constant rate of re-introduction of criminals after a burglary.

The spatially homogeneous equilibrium state is

\[
P_e = (\gamma - \alpha)/\gamma, \quad A_e = \gamma.
\]
2-D Numerics: Take $P(x, 0) = P_e$, $A(x, 0) = \gamma(1 + \text{rand} \times 0.001)$ in a square domain of width 4 with $\alpha = 1$, $\gamma = 2$, $\varepsilon = 0.08$, $\tau = 1$, and $D = 1$.

A hot-spot pattern emerges on an $O(1)$ time-scale, which then persists.
Numerical: Formation of 1-D Hot-Spots

1-D Numerics: Take $P(x, 0) = P_e, A(x, 0) = \gamma(1 - 0.01 \cos(6\pi x))$ on an interval of length 1 with $\alpha = 1, \gamma = 2, \varepsilon = 0.02, \tau = 1$. Plot $A(x, t)$ at different $t$. Left: $D = 1.0$; Right: $D = 0.5$. Lowering $D$ has increased the number of stable hot spots. (Explained in more detail later).
Outline and Perspective I

Mathematical Modeling Remarks:

- **Formulation**: formulation of a *stochastic agent-based model* based on *observational trends* in crime, behavior of criminals, etc... easy to incorporate many effects...

- **Age of Discovery**: *numerical realizations* of the agent-based model to observe qualitatively interesting phenomena (stationary hot-spots, moving hot-spots, creation of new spots, etc.)

- **Continuum Limit**: Derivation of a “simpler” PDE reaction–diffusion (RD) system. *Usual First Step*: Numerical simulations of PDE system, Turing and weakly nonlinear analysis of patterns.....

- **Rigorous PDE Analysis**: existence, regularity, and bifurcation-theoretic results for the PDE model (Rodriguez, Cantrell, Cosner and Manasevich).

- **Model Validation**: matching model predictions with field observations. Improving the model. Developing new models (some game-theoretic)... (Berestycki-Nadal, Pilcher, Short and D’Orsogna).
Outline and Perspective II

However: many seemingly “simple-looking” RD systems can exhibit extremely complex dynamics in the nonlinear regime in different parameter ranges; i.e. witness the Gray-Scott model of chemical physics (1996–date).

Our Approach: For the RD model of urban crime, use a combination of formal asymptotics, rigorous analysis, and computation, to obtain analytical results for stability thresholds, delineating in parameter space where different solution behaviors occur, etc...

Specific Goal: Analyze the existence and stability of localized patterns of criminal activity for this model in the limit $\varepsilon \to 0$. We also consider extensions of the basic model to include the effect of “police”. These patterns are “far-from-equilibrium” (Y. Nishiura..), and not amenable to standard Turing stability analysis.

Particle-like solutions to PDE’s; vortices, skyrmions, hot-spots, ...

 amor
Turing-Stability Analysis

The uniform state \( A_e, P_e \) on \( \mathbb{R}^1 \) is unstable for \( \varepsilon \to 0 \) when \( \gamma > 3\alpha/2 \). With an \( e^{imx+\lambda t} \) perturbation, the Turing instability band is

\[
D^{-1/2}\gamma(2\gamma - 3\alpha)^{-1/2} \sim m_{\text{lower}} < m < m_{\text{upper}} \sim \varepsilon^{-1}\gamma^{-1/2}(2\gamma - 3\alpha)^{1/2}.
\]

For \( \varepsilon \to 0 \), the maximum growth rate is \( \lambda_{\text{max}} \sim O(1) \), with the most unstable mode

\[
m_{\text{max}} \sim \varepsilon^{-1/2}D^{-1/4}\gamma^{-1/2}[(\gamma - \alpha)(3\gamma^2 + 2\tau(2\gamma - 3\alpha))]^{1/4}.
\]

**Remark 1:** For perturbations of the uniform state the preferred pattern has a characteristic half-length \( l_{\text{turing}} \sim \pi/m_{\text{max}} \), where

\[
l_{\text{turing}} \sim \varepsilon^{1/2}D^{1/4}\gamma^{1/2}[(\gamma - \alpha)(3\gamma^2 + 2\tau(2\gamma - 3\alpha))]^{-1/4}\pi.
\]

Notice that \( l_{\text{turing}} = O(1) \) when \( D = O(\varepsilon^{-2}) \).

**Remark 2:** Turing and weakly nonlinear analysis in 1-D and 2-D given in Short et al. (SIADS 2010), together with full numerical computations.
Global Bifurcation Diagram in 1-D

Bifurcation Diagram: $A(0)$ versus $\gamma$ on $0 < x < 1$ for $\alpha = 1$, $\varepsilon = 0.05$, $D = 2$. Notice that a localized hot-spot occurs when $A(0)$ is large.

Caption: $A_e = \gamma$ is the solid line with shallow slope; The hot-spot asymptotics (to be derived) is the dotted line $A(0) \sim 2(\gamma - \alpha)/(\varepsilon \pi)$. Subcritical Turing occurs at $\gamma = 3\alpha/2 + O(\varepsilon) \approx 1.5$, with weakly nonlinear asymptotics (dashed parabolic curve). Inserts plot $A(x)$ versus $x$. 
Basic RD Crime Model: Qualitative I

There are two key parameter regimes: \( D \gg 1 \) and \( D = O(1) \)

**Regime 1: \( D \gg 1 \)**

- Localized patterns in the form of *pulses or spikes* are readily constructed using singular perturbation techniques for the regime \( O(1) \ll D \leq O(\varepsilon^{-2}) \) in 1-D and \( O(1) \ll D \leq O(\varepsilon^{-4}) \) in 2-D.

- The stability threshold for these patterns occurs when \( D = O(\varepsilon^{-2}) \) in 1-D and \( D = O(\varepsilon^{-4}) \) in 2-D.

- The stability theory is based primarily on an *exactly solvable nonlocal eigenvalue problem*.

- A further stability threshold in \( D \) with respect to instabilities developing over a long time scale \( t = O(\varepsilon^{-2}) \) must also be calculated.

**Implication:** The *stability threshold in terms of \( D \) determines the minimum spacing between localized elevated regions of criminal activity* that allows for a stable pattern, i.e. If \( D_{\text{crit}} \) is large, stable hot-spots are closely spaced.
Localized Hot-Spot Patterns in 1-D: History

A rather extensive literature on the stability of pulses for two-component RD systems without gradient terms. Prototypical is

\[ v_t = \varepsilon^2 v_{xx} - v + v^2 / u, \quad \tau u_t = Du_{xx} - u + \varepsilon^{-1}v^2, \quad \text{(GM model)}. \]

**History:** NLEP stability theory (1999-date) (Iron, Kolokolnikov, Ward, Wei, Winter; Doelman, Gardner, Kaper, Van der Ploeg,..).

Let \( w'' - w + w^2 = 0 \) be the homoclinic. To study the stability on an \( O(1) \) time-scale need to analyze the spectrum of the NLEP

\[
\Phi_{yy} - \Phi + 2w\Phi - \chi(\tau \lambda) w^2 \frac{\int_{-\infty}^{\infty} w\Phi \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} = \lambda \Phi; \quad \Phi \to 0 \text{ as } |y| \to \infty.
\]

If \( D > D_c, K > 1, \) and \( \tau < \tau_c, \) then there is a sign-fluctuating instability of the spike amplitudes due to a positive real eigenvalue; this is a competition instability (Iron-Ward-Wei (Physica D, 2001)).

If \( D < D_c, K > 1, \) but \( \tau > \tau_c, \) then there is a synchronous oscillatory instability of the spike amplitudes due to a Hopf bifurcation. (Ward-Wei, (J. Nonl. Sci, 2003)).
Regime 1: Hot-Spot Equilibria in 1-D: I

Basic Cell Problem: Consider WLOG the interval $|x| < l$.

Since $P_x - \frac{2P}{A} A_x = (P/A^2)_x A^2$, we let $V = P/A^2$. Then, on $|x| < l$

$$A_t = \varepsilon^2 A_{xx} - A + V A^3 + \alpha,$$

$$\tau (A^2 V)_t = D (A^2 V_x)_x - V A^3 + \gamma - \alpha.$$

For $D = O(\varepsilon^{-2})$, the correct scaling is

$$V = \varepsilon^2 v, \quad D = D_0/\varepsilon^2.$$ 

Therefore, our re-scaled RD system on the basic cell $|x| < l$ is

$$A_t = \varepsilon^2 A_{xx} - A + \varepsilon^2 v A^3 + \alpha; \quad A_x(\pm l, t) = 0,$$

$$\varepsilon^2 \tau (A^2 v)_t = D_0 (A^2 v_x)_x - \varepsilon^2 v A^3 + \gamma - \alpha; \quad v_x(\pm l, t) = 0.$$

Remark: $A = O(\varepsilon^{-1})$ in the core of a hot-spot while $A = O(1)$ away from the core. We obtain that $v = O(1)$ globally.
Regime 1: Hot-Spot Equilibria in 1-D: II

From a matched asymptotic analysis:

**Principal Result:** Let \( \varepsilon \to 0 \) and \( O(1) \ll D \leq O(\varepsilon^{-2}) \) with \( D_0 \equiv \varepsilon^2 D \). Then, for a one-hot-spot solution centered at \( x = 0 \) on \( |x| \leq l \), the leading-order uniform asymptotics for \( A \) and \( P \) are

\[
A \sim \frac{1}{\sqrt{2}} \left( \frac{2l(\gamma - \alpha)}{\pi \varepsilon} - \alpha \right) w(x/\varepsilon) + \alpha, \quad P \sim [w(x/\varepsilon)]^2.
\]

Here \( w(y) = \sqrt{2} \text{sech}(y) \) is the homoclinic of \( w'' - w + w^3 = 0 \). The inner and outer approximations for \( v \) are

\[
v \sim v_0 + \varepsilon v_1, \quad |x| = O(\varepsilon); \quad v \sim \frac{\zeta}{2} ((l - |x|)^2 - l^2) + v_0, \quad O(\varepsilon) < |x| < l,
\]

Here \( \zeta \equiv (\alpha - \gamma)/(D_0 \alpha^2) < 0 \) and \( v_0 = \pi^2 \left[ 2l^2 (\gamma - \alpha)^2 \right]^{-1} \).

Remarks:

- \( A(0) \sim 2(\gamma - \alpha)/(\varepsilon \pi) \), as plotted previously on bifurcation diagram.
- For a symmetric \( K \)-hot-spot pattern with spots of equal height on a domain of length \( S \), simply set \( l = S/2K \) and use gluing.
Regime 1: Hot-Spot Equilibria in 1-D: III

Comparison with Full Numerics: with $D_0 = 1$, $\gamma = 1$, $\epsilon = 0.05$, and $\alpha = 2$:

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\epsilon$ & $A(0)$ (num) & $A(0)$ (asy1) & $v(0)$ (num) & $v(0)$ (asy1) & $v(0)$ (asy2) \\
\hline
0.1 & 6.281 & 6.366 & 3.5844 & 4.935 & 2.961 \\
0.05 & 12.805 & 12.732 & 4.1474 & 4.935 & 3.948 \\
0.025 & 25.628 & 25.465 & 4.4993 & 4.935 & 4.441 \\
0.0125 & 51.145 & 50.930 & 4.7039 & 4.935 & 4.688 \\
\hline
\end{tabular}
\end{table}

Remark: A one-term approximation for $A(0)$ is accurate even for $\epsilon = 0.1$. 
NLEP Stability Analysis: I

Let $A_e, v_e$ be one-spike solution on the basic cell $|x| < l$. We introduce

$$A = A_e + \phi e^{\lambda t}, \quad v = v_e + \varepsilon \psi e^{\lambda t}.$$  

The singularly perturbed eigenvalue problem is

$$\varepsilon^2 \phi_{xx} - \phi + 3\varepsilon^2 v_e A_e^2 \phi + \varepsilon^3 A_e^3 \psi = \lambda \phi,$$

$$D_0 (\varepsilon A_e^2 \psi_x + 2A_e v_e \phi)_x - 3\varepsilon^2 A_e^2 v_e \phi - \varepsilon^3 \psi A_e^3 = \lambda \varepsilon^2 (\varepsilon A_e^2 \psi + 2A_e v_e \phi).$$

For $z$ complex, we impose the Floquet boundary conditions

$$\phi(l) = z \phi(-l), \quad \phi'(l) = z \phi'(-l), \quad \psi(l) = z \psi(-l), \quad \psi'(l) = z \psi'(-l),$$

To obtain the spectrum of a $K$-spike pattern on a domain of length $2Kl$ subject to periodic boundary conditions we set $z^K = 1$, so that

$$z_j = e^{2\pi i j/K}, \quad j = 0, \ldots, K - 1.$$

Remark: An NLEP is derived for the periodic b.c. problem by using asymptotics to determine jump conditions for $\psi$ across $x = 0$. Then, the Neumann spectra is extracted from Periodic spectra.
NLEP Stability Analysis: II

An asymptotic analysis shows that $\phi \sim \Phi(y)$ with $y = \varepsilon^{-1}x$, where $\Phi(y)$ on $-\infty < y < \infty$ satisfies

$$L_0 \Phi \equiv \Phi'' - \Phi + 3w^2\Phi = -v_0^{-3/2} w^3 \psi(0) + \lambda \Phi; \quad \Phi \to 0 \text{ as } |y| \to \infty.$$ 

Then, for $\tau = O(1)$, $\psi(0)$ is determined from

$$\psi_{xx} = 0, \quad 0 < |x| \leq l; \quad \psi(l) = z\psi(-l), \quad \psi'(l) = z\psi'(-l),$$

subject to $\psi(0^+) = \psi(0^-) \equiv \psi(0)$ and the jump condition across $x = 0$:

$$a_0 [\psi_x]_0 + a_1 \psi(0) = a_2;$$

$$a_0 \equiv D_0 \alpha^2, \quad a_1 = -v_0^{-3/2} \int_{-\infty}^{\infty} w^3 dy, \quad a_2 = 3 \int_{-\infty}^{\infty} w^2 \Phi dy$$

Remark: $\psi(0)$ involves one nonlocal term.
NLEP Stability Analysis: III

**Principal Result:** Consider a $K > 1$ equilibrium hot-spots on an interval of length $S$ with no-flux conditions. For $\varepsilon \to 0$, $\tau = O(1)$, and with $D_0 = \varepsilon^2 D$, the stability of this solution with respect to the “large” eigenvalues $\lambda = O(1)$ of the linearization is determined by the spectrum of the NLEP

$$L_0 \Phi - \chi_j w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \lambda \Phi; \quad \Phi \to 0 \quad \text{as} \quad |y| \to \infty,$$

$$\chi_j = 3 \left[ 1 + \frac{D_0 \alpha^2 \pi^2 K^4}{4(\gamma - \alpha)^3} \left( \frac{2}{S} \right)^4 \left( 1 - \cos \left( \pi j / K \right) \right) \right]^{-1}, \quad j = 0, \ldots, K - 1.$$

For $K = 1$ we have $\chi_0 = 3$. Here $L_0$ (local operator) is

$$L_0 \Phi \equiv \Phi'' - \Phi + 3 w^2 \Phi$$

**Remark:** In contrast to the NLEP’s for the GM and GS models, the discrete spectrum for this NLEP is explicitly available.
Lemma: Let $c$ be real, and consider the NLEP on $-\infty < y < \infty$:

$$L_0 \Phi - cw^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy = \lambda \Phi; \quad \Phi \rightarrow 0 \quad \text{as} \quad |y| \rightarrow \infty,$$

corresponding to $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$. On the range $\text{Re}(\lambda) > -1$, there is a unique discrete eigenvalue given by

$$\lambda = 3 - c \int_{-\infty}^{\infty} w^5 \, dy.$$

Thus, $\lambda$ is real and $\lambda < 0$ when $c > 3 / \int w^5 \, dy$.

Idea of Proof: We use the key identity $L_0 w^2 = 3w^2$ and Green’s theorem

$$\int_{-\infty}^{\infty} \left( w^2 L_0 \Phi - \Phi L_0 w^2 \right) \, dy = 0 \quad \text{with} \quad L_0 \Phi = cw^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy + \lambda \Phi.$$

Thus,

$$\left( \lambda - 3 + c \int_{-\infty}^{\infty} w^5 \, dy \right) \int_{-\infty}^{\infty} w^2 \Phi \, dy = 0,$$

which yields the result.
**NLEP Stability Analysis: V**

By applying this Lemma to our NLEP, we get:

**Principal Result:** *On a domain of length $S$ with Neumann b.c., for $\tau = O(1)$ and $D_0 = \varepsilon^2 D$ a one-hot-spot solution is stable on an $O(1)$ time-scale $\forall D_0 > 0$. For $K > 1$ it is stable on an $O(1)$ time-scale iff $D_0 < D_{0K}^L$, where*

$$D_{0K}^L \equiv \frac{2(\gamma - \alpha)^3 (S/(2K))^{4}}{\alpha^2 \pi^2 [1 + \cos (\pi/K)]}.$$  

*In terms of the original diffusivity $D$, given by $D = \varepsilon^{-2} D_0$, the stability threshold is $D_{K}^L = \varepsilon^{-2} D_{0K}^L$ when $K > 1$.*

**Small Eigenvalues:** There are “small” $o(1)$ eigenvalues in the linearization that are difficult to asymptotically calculate directly. The threshold in $D$ for this critical spectrum is obtained indirectly by determining the value $D_{K}^S$ of $D$ for which a asymmetric $K$-hot-spot equilibrium branch bifurcates from a symmetric $K$-hot-spot branch. We readily calculate that

$$D_{K}^S = \frac{(\gamma - \alpha)^3}{\varepsilon^2 \pi^2 \alpha^2} \left( \frac{S}{2K} \right)^4.$$
Summary: The small and large (NLEP) eigenvalue stability thresholds are

\[ D^S_K \sim \left( \frac{S}{2K} \right)^4 \frac{(\gamma - \alpha)^3}{\varepsilon^2 \pi^2 \alpha^2}, \quad D^L_K = D^S_K \left( \frac{2}{1 + \cos(\pi/K)} \right) > D^S_K. \]

**Remark 1:** Thus, we have stability wrt both classes of eigenvalues when \( D < D^S_K \); a weak translational instability when \( D^S_K < D < D^L_K \); a fast \( O(1) \) time-scale when \( D > D^L_K \).

**Remark 2 (KEY):** For stability, we need the inter-hot-spot spacing \( l \) to satisfy \( l > l_c \), where

\[ l_c \sim \sqrt{\pi} D^{1/4} \varepsilon^{1/2} \alpha^{1/2} (\gamma - \alpha)^{-3/4} \]

**Remark 3 (KEY):** Since \( l_c \) and \( l_{\text{turing}} \) are both \( O(1) \) when \( D = O(\varepsilon^{-2}) \), the maximum number of stable hot-spots corresponds (roughly) to the most unstable Turing mode.
Numerical Validation I

**Full Numerics:** Let $\varepsilon = 0.07$, $\alpha = 1$, $\gamma = 2$, $S = 4$, $\tau = 1$, and $K = 2$, so that $D^S_K \approx 20.67$ and $D^K_L \approx 41.33$. The results below confirm the stability theory.

Left: $D = 15$  
Middle: $D = 30$  
Right: $D = 50$
Qualitative Implications: Stability Analysis

On an interval of length $S$, the stability properties of a $K$-hot-spot equilibrium pattern can be phrased in terms of the maximum number of hot-spots:

- Unstable wrt a competition instability developing on an $O(1)$ time scale if $K > K_{c+}$, where $K_{c+} > 0$ is the unique root of

$$K \left( 1 + \cos \left( \frac{\pi}{K} \right) \right)^{1/4} = \left( \frac{S}{2} \right) \left( \frac{2}{D} \right)^{1/4} \frac{(\gamma - \alpha)^{3/4}}{\sqrt{\pi \varepsilon \alpha}}.$$

- Stable with respect to slow translational instabilities developing on an $O(\varepsilon^{-2})$ time-scale if $K < K_{c-} < K_{c+}$, where

$$K_{c-} = \left( \frac{S'}{2} \right) D^{-1/4} \frac{(\gamma - \alpha)^{3/4}}{\sqrt{\pi \varepsilon \alpha}}.$$

- Summary: stability when $K < K_{c-}$; stability wrt $O(1)$ time-scale instabilities but unstable wrt slow translation instabilities when $K_{c-} < K < K_{c+}$; a fast $O(1)$ time-scale instability when $K > K_{c+}$.
Numerical Validation II

1-D Numerics (Revisited): Take $\alpha = 1$, $\gamma = 2$, and $\varepsilon = 0.02$.

Left ($D = 1$): $K_{c+} \approx 2.27$ and $K_{c-} \approx 1.995$. Thus, 3-spots are NLEP unstable ($O(1)$ time-scale instability), and 2-spots (unstable on very long time-interval).

Right ($D = 0.5$): $K_{c+} \approx 2.61$ and $K_{c-} \approx 2.37$. Thus, 3-spots are NLEP unstable, but 2-spots are stable wrt NLEP and translations.
Quasi-Equilibria and NLEP Stability in 2-D

In a 2-D domain $\Omega$ with area $|\Omega|$.

**Principal Result:** For $\varepsilon \to 0$ and $\tau = O(1)$, a symmetric $K$-hot-spot quasi-steady-state solution for $D = \varepsilon^{-4}D_0/\sigma$ with $\sigma = -1/\log \varepsilon$, is characterized near the $j$-th hot-spot by

$$A \sim \frac{w(\rho)}{\varepsilon^2 \sqrt{v_0}}, \quad P \sim [w(\rho)]^2,$$

with $\rho = \varepsilon^{-1}|x - x_j|$. Here $w(\rho)$ is the radially symmetric ground-state of $\Delta \rho w - w + w^3 = 0$. In addition,

$$v_0 \equiv \frac{4\pi^2 b^2 K^2}{|\Omega|^2 (\gamma - \alpha)^2} \quad b = \int_0^\infty \rho w^2 d\rho.$$

This quasi-steady-state for $K > 1$ is stable wrt $O(1)$ instabilities of the associated NLEP iff

$$D < \left(\frac{\varepsilon^{-4}}{\sigma}\right) D_{0K}^L,$$

where $D_{0K}^L \equiv \frac{|\Omega|^3 (\gamma - \alpha)^3}{4\pi \alpha^2 K^3 \left(\int_{\mathbb{R}^2} w^3 dy\right)^2}$.

For $K = 1$ a single hot-spot is stable for all $D_0$ independent of $\varepsilon$. 
Regime 2: \( D = O(1) \)

\[
A_t = \varepsilon^2 \Delta A - A + PA + \alpha, \quad x \in \Omega,
\]

\[
\tau P_t = D \nabla \cdot \left( \nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha, \quad x \in \Omega.
\]

Localized hot-spots still exist, but

- In 1-D, localized regions of criminal activity can be nucleated in the region between neighboring hot-spots when the inter hot-spot spacing exceeds some threshold.
- Leads to the “spontaneous” creation of new hot-spots, i.e. new regions of elevated criminal activity.
- This is called peak-insertion in R-D theory, and it arises from a saddle-nose bifurcation point in terms of \( D \).
- In 1-D the hot-spot dynamics is repulsive, and a reduction to finite-dimensional dynamics can be done.
**Peak-Insertion or Nucleation:** $D = O(1)$

**Numerics:** $\epsilon = 0.02$, $\alpha = 1$, $\gamma = 2$, $\tau = 1$, with $x \in (0, 2)$. when $D$ is slowly decreasing in time as $D = 1/(1 + 0.01t)$. Plot of $P(x)$. Peak insertion occurs whenever $D$ is quartered.
Peak-Insertion Behavior: Global AUTO

Global Bifurcation Diagram (AUTO): A Two-Boundary and One-Interior Spike Solution for $\gamma = 2$, $\alpha = 1$, $\varepsilon = 0.02$, on a domain of length 2. Top: $|A|_2$ versus $D$: Bottom: $P(x)$. 
Analysis of Peak-Insertion Behavior: I

**Basic Cell Problem:** Consider WLOG a one-spike pattern centered at the midpoint of the interval $|x| < l$. Since $P_x - \frac{2P}{A} A_x = (P/A^2)_x A^2$, we define

$$V = P/A^2.$$ 

Then, on $|x| < l$, the equilibrium problem is

$$\varepsilon^2 A_{xx} - A + VA^3 + \alpha = 0, \quad A_x(\pm l) = 0,$$

$$D \left( A^2 V_x \right)_x - VA^3 + \gamma - \alpha = 0, \quad V_x(\pm l) = 0.$$ 

**Goal:** Use $\varepsilon \ll 1$ asymptotics to determine $A(l)$ versus $l/\sqrt{D}$.

As $D$ decreases, peak insertion for $P(x)$ at the boundary occurs:
Analysis of Peak-Insertion Behavior: II

Inner Region: We set $y = x/\varepsilon$, and expand

$$A \sim \varepsilon^{-1} A_0 + A_1 + \cdots, \quad V \sim \varepsilon^2 \tilde{v}_0 + \cdots$$

We obtain that

$$A_0 = w/\sqrt{\tilde{v}_0}, \quad w = \sqrt{2}\text{sech}(y),$$

where $w'' - w + w^3 = 0$. Here $\tilde{v}_0$ is to be found, and $A_1 \to \alpha$ as $y \to \pm\infty$.

Outer Region: WLOG consider $0^+ < x \leq l$. Key: The leading order outer problem is nonlinear. With, $A \sim a_0$ and $V \sim v_0$, then

$$v_0 a_0^3 = a_0 - \alpha, \quad D (a_0^2 v_0 x)_x = v_0 a_0^3 - (\gamma - \alpha),$$

which leads to

$$D (f(a_0) a_0 x)_x = a_0 - \gamma, \quad 0 < x \leq l; \quad a_0(0^+) = \alpha, \quad a_0 x(l) = 0,$$

where

$$f(a_0) \equiv a_0^{-2}(3\alpha - 2a_0).$$
**Analysis of Peak-Insertion Behavior: III**

**Remark 1:** We have $f(a_0) > 0$ and $a_{0x} > 0$ when $a_0 < 3\alpha/2 < \gamma$. **Note:** for $\gamma > 3\alpha/2$ the spatially homogeneous steady-state is Turing unstable.

**Remark 2:** For the existence of a solution $a_0$ we require $a_0(l) \equiv \mu \leq 3\alpha/2$.

Upon integrating the BVP for $a_0$, we obtain an implicit relation for $\mu \equiv a_0(l)$ in terms of $l/\sqrt{D}$. Namely,

$$
\sqrt{\frac{2}{D}} l = \chi(\mu) \equiv \frac{2}{\gamma - \alpha} \sqrt{G(\alpha; \mu)} + 2 \int_{\alpha}^{\mu} \frac{1}{(\eta - \gamma)^2} \sqrt{G(\eta; \mu)} \, d\eta,
$$

where

$$
G(\eta; \mu) \equiv \int_{\eta}^{\mu} f(s)(\gamma - s) \, ds = 2(\mu - \eta) - (2\gamma + 3\alpha) \log \left( \frac{\mu}{\eta} \right) + 3\alpha \gamma \left( \frac{1}{\eta} - \frac{1}{\mu} \right).
$$

In addition, the unknown constant $\tilde{v}_0$ for the inner solution is

$$
\tilde{v}_0 = \frac{\pi^2}{4D} [G(\alpha; \mu)]^{-1}.
$$
Analysis of Peak-Insertion Behavior: IV

**Key Monotonicity Properties:** $G_{\mu}(\eta; \mu) > 0$ and $\chi'(\mu) > 0$.

**Upshot:** As $\mu = a_0(l)$ increases on $\alpha < \mu < 3\alpha/2$, then $l/\sqrt{D}$ increases.

**Main Result:** For $l$ fixed, we require $D > D_{\text{min}}$, where

$$D_{\text{min}} = \frac{2l^2}{[\chi(3\alpha/2)]^2}.$$  

As $D \to D_{\text{min}}^+$, then $a_{0x}(l) \to +\infty$, and we predict peak insertion.

**Corollary:** For $D$ fixed, we require $l < l_{\text{max}}$, where

$$l_{\text{max}} = \chi \left(\frac{3\alpha}{2}\right) \sqrt{D/2}.$$  

As $l \to l_{\text{max}}^-$, then $a_{0x}(l) \to +\infty$, and we predict peak insertion.

**Comparison:** The analytical theory gives $D_{\text{min}} \approx 0.445$ for $l = 1/2$. Full numerics with AUTO gives $D_{\text{min}} \approx 0.27$ for $\varepsilon = 0.02$ and $D_{\text{min}} \approx 0.44$ for $\varepsilon = 0.0027$.

**Remark** The error in the outer approximation is $O(-\varepsilon \log \varepsilon)$. 

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Analysis of Peak-Insertion Behavior: V

Goal: perform a local analysis near $x = l$ when $D \approx D_{\text{min}}$ in order to describe the nucleation of the new peak.

Q1: Can we analytically uncover solution multiplicity arising from a saddle-node bifurcation near $D_{\text{min}}$?

Q2: If so, is there some normal form-type equation that can be rigorously analyzed describing the local behavior of solutions near the saddle-node transition?

Local Analysis: Define the constants $A_c$, $V_c$, $\beta$, $\sigma$, and $\zeta$ by

$$A_c \equiv \frac{3\alpha}{2}, \quad V_c \equiv \mathcal{F}(A_c), \quad \beta \equiv \frac{(\gamma - 3\alpha/2)}{2D_{\text{min}}A_c^2} > 0,$$

$$\sigma = \left( \frac{-2}{A_c^2 \mathcal{F}''(A_c) \beta} \right)^{1/6}, \quad \zeta = A_c^2 \beta \left( \frac{-2}{A_c^2 \mathcal{F}''(A_c) \beta} \right)^{2/3},$$

where $\mathcal{F}(A) \equiv (A - \alpha)/A^3$ with $\mathcal{F}''(A_c) < 0$. 

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Analysis of Peak-Insertion Behavior: VI

Main Result: Then, near the endpoint \( x = l \), we obtain the local approximation

\[
A \sim A_c - \varepsilon^{2/3} \zeta U(y), \quad V \sim V_c - \varepsilon^{4/3} \beta \sigma^2 \left( A^* + y^2 \right),
\]

where \( y = (l - x)/(\varepsilon^{2/3} \sigma) \). The function \( U(y) \) on \( y \geq 0 \) satisfies the normal form non-autonomous ODE

\[
U'' = U^2 - A^* - y^2, \quad y \geq 0; \quad U'(0) = 0; \quad U' \rightarrow +1, \quad y \rightarrow +\infty.
\]

Remark: If \( U(0) < 0 \), then \( A > A_c = 3\alpha/2 \).

Main Result: For \( A^* \gg 1 \), there are two solutions \( U^\pm(y) \) with \( U' > 0 \) for \( y > 0 \) given asymptotically by

\[
U^+ \sim \sqrt{A^* + y^2}, \quad U^+(0) \sim \sqrt{A^*},
\]

\[
U^- \sim \sqrt{A^* + y^2} \left( 1 - 3 \text{ sech}^2 \left( \frac{\sqrt{A^*} y}{\sqrt{2}} \right) \right), \quad U^-(0) \sim -2\sqrt{A^*}.
\]

Rigorous: these solutions are connected via a saddle-node bifurcation.
Analysis of Peak-Insertion Behavior: VII


Plot of $A^*$ versus $s \equiv U(0)$:

$s = -5, \ A^* \sim \frac{25}{4}$

$s \approx -0.615, \ A^* = -1.466$

$s = 2.5, \ A^* \sim \frac{25}{4}$
Finite-Dimensional Dynamics: $D = O(1)$: I

On the basic cell, $|x| \leq l$, one can construct a quasi-equilibrium one-spike solution centered at $x = x_0$, where $x_0 = x_0(\varepsilon^2 t)$ moves slowly in time.

**Main Result:** Provided that no peak-insertion effects occur, then for $\varepsilon \to 0$ the dynamics on the slow time-scale $\sigma = \varepsilon^2 t$ is characterized by

$$A \sim \varepsilon^{-1} w(y) / \sqrt{\tilde{v}_0}, \quad y = \varepsilon^{-1} (x - x_0(\sigma)),$$

$$\frac{dx_0}{d\sigma} \sim \frac{3}{8\alpha} \mathcal{F}(x_0), \quad \mathcal{F}(x_0) \equiv \left[ a_{0x}(x_0^+) + a_{0x}(x_0^-) \right].$$

Here $a_0(x)$ is the solution to the multi-point BVP

$$D \left[ f(a_0) a_{0x} \right]_x = a_0 - \gamma, \quad 0 < |x| < l; \quad a_{0x}(\pm l) = 0, \quad a_0(0) = \alpha,$$

where $f(a_0) \equiv a_0^{-2} (3\alpha - 2a_0)$.

**Remark 1:** The derivation of this is delicate in that one must resolve a corner layer or knee region for $V$ that allows for matching between the inner and outer approximations for $V$. 
Finite-Dimensional Dynamics $D = O(1)$: II

From an integration of the BVP for $a_0$, one gets explicit dynamics:

$$\frac{dx_0}{d\sigma} = \frac{3}{8} \left( \frac{2}{D} \right) \left[ \sqrt{G(\alpha; \mu_r)} - \sqrt{G(\alpha; \mu_l)} \right],$$

where $\mu_r \equiv a_0(l)$ and $\mu_l \equiv a_0(-l)$ are determined implicitly by

$$\sqrt{\frac{2}{D}} (l - x_0) = \chi(\mu_r), \quad \sqrt{\frac{2}{D}} (l + x_0) = \chi(\mu_l).$$

**Qualitative I:** The dynamics is repulsive: If $x_0(0) > 0$, then $x_0 \to 0$ as $t \to \infty$. By using reflection through the Neumann B.C., two adjacent hot-spots will repel.

**Qualitative II:** Since the hot-spot dynamics is repulsive, then dynamic peak-insertion events due to large inter-hot-spot separations are unlikely.

**Remark:** Peak insertion events in the presence of mutually attracting localized pulses, leads to spatial temporal chaos in a Keller-Segel model with logistic growth in 1-D (Painter and Hillen, Physica D 2011).
The Effect of Police: 3-Component Systems

In 1-D, an RD system incorporating police is $U = U(x, t)$:

$$
A_t = \varepsilon^2 \Delta A - A + PA + \alpha, \quad x \in \Omega,
$$

$$
P_t = D \nabla \cdot \left( \nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha - f, \quad x \in \Omega,
$$

$$
\tau_u U_t = D \nabla \cdot \left( \nabla U - \frac{qU}{A} \nabla A \right), \quad x \in \Omega,
$$

with $\partial_n(A, P, U) = 0$ on $\partial \Omega$. Police are conserved; $U_0 = \int_\Omega U \, dx > 0$ for all $t$.

- Police Model I: $f = U$ (simple interaction) L. Ricketson (UCLA).
- Police Model II: $f = UP$ (standard “predator-prey” type interaction)

Remark: The police drift velocity is $\mathcal{V} = \frac{d}{dx} \ln(A^q)$.

- If $q = 2$, police drift exhibits mimicry (L. Ricketson, UCLA) Referred to as “Cops on the Dots” (Jones, Brantingham, L. Chayes, M3As, 2011). Can be derived from an agent-based model.
- If $q > 2$, police focus more on attractive sites than do criminals.
- If $0 < q < 2$, the police are less focused, and more “diffusive”.

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The Effect of Police: Qualitative

**Question I:** Optimal Police Strategy: For a given $U_0$ find the optimal $q$ (parameter in police drift velocity) that minimizes the NLEP stability threshold of $D$ with $D = D_0 / \varepsilon^2$. In this way, we maximize over $q$ the distance between stable localized hot-spots.

- Investigate this question for both Police models I and II.
- Is the optimal strategy the same for both models?

**Question II:** For $\tau_u$ sufficiently large on the regime $D = O(\varepsilon^{-2})$, corresponding to police diffusivity $D/\tau_u$, can a two-hot spot solution undergo a Hopf bifurcation leading to asynchronous temporal oscillations in the hot-spot amplitudes?

- If $\tau_u > 1$, then police ‘diffuse” more slowly than do criminals.
- Typically, only synchronous oscillatory instabilities of the spike amplitudes occur in RD systems (GM, Gray-Scott, etc..)

**Question III:** Open: For $D = O(1)$ can police prevent or limit the nucleation of new hot-spots of criminal activity arising from peak-insertion?
Police Model I: Simple-Interaction Model

**Principal Result (Equilibrium):** On the basic cell $|x| > l$, and with $q > 1$, the leading order asymptotics for $A$, $P$, and $U$, in the hot-spot region near $x = 0$ is

$$A \sim \frac{1}{\varepsilon \sqrt{v_0}} w(x/\varepsilon), \quad P \sim [w(x/\varepsilon)]^2, \quad U \sim \frac{U_0}{\varepsilon b} [w(x/\varepsilon)]^q.$$

Here $b \equiv \int w^q \, dy$, and $w = w(y) = \sqrt{2} \text{sech}(y)$ is the homoclinic of $w'' - w + w^3 = 0$. The amplitude of the hot-spot is determined by $v_0$, where

$$\frac{1}{\sqrt{v_0}} \int w^3 \, dy = 2l(\gamma - \alpha) - U_0.$$

**Remark I:** A hot-spot solution ceases to exist if the total number of police satisfies

$$U_0 \geq U_{0c} \equiv 2l(\gamma - \alpha).$$

**Remark II:** For $U_0 < U_{0c}$, for a $K$-spot equilibrium on a domain of length $S$, let $l = S/(2K)$ and replace $U_0 \rightarrow U_0/K$. Then, use a glueing technique to construct the multi-pulse pattern.
Police Model I: Stability I

For \( q > 1 \), a Floquet-based analysis leads to an NLEP with two nonlocal terms:

\[
L_0 \Phi - 3 \chi_0 j w^3 \int \frac{w^2 \Phi \, dy}{\int w^3 \, dy} - \chi_1 j w^3 \int w^{q-1} \Phi \, dy = \lambda \Phi,
\]

where for \( j = 1, \ldots, K - 1 \),

\[
\chi_{0j} \equiv \left[ 1 + v_0^{3/2} D_{j2} / \int w^3 \, dy \right]^{-1}, \quad \chi_{1j} \equiv \frac{C_q(\lambda)}{\int w^3 \, dy} \chi_{0j}.
\]

Here \( v_0, D_{jq}, \) and \( C_q(\lambda) \) are defined by

\[
\int w^3 \, dy = \frac{S}{K} (\gamma - \alpha) - \frac{U_0}{K}, \quad D_{jq} \equiv D_0 \alpha^q \left( \frac{2K}{S} \right) \left( 1 - \cos \left( \frac{\pi j}{K} \right) \right),
\]

\[
C_q(\lambda) \equiv \frac{q \kappa_p D_{jq}}{D_{jq} + \hat{\tau} \lambda}, \quad \hat{\tau} \equiv \varepsilon^{q-3} \tau \left( \frac{\int w^q \, dy}{v_0^{q/2}} \right), \quad \kappa_p \equiv \frac{U_0 \sqrt{v_0}}{K \int w^q \, dy}.
\]

**Remark 1:** For \( q = 3 \) it is an exactly solvable NLEP.

**Remark 2:** By using \( L_0(w^2) = 3w^2 \), the NLEP can be transformed into one with a single nonlocal term.
Police Model I: Stability IV

Main Result: For $q > 1$, the NLEP governing the stability of a $K$-hot-spot pattern is for $j = 1, \ldots, K - 1$:

$$L_0 \Phi - \frac{1}{C_j(\lambda)} w^3 \int w^{q-1} \Phi \, dy = \lambda \Phi,$$

$$C_j(\lambda) = \frac{1}{\chi_{1j} \int w^q \, dy} \left[ 1 - \frac{9 \chi_{0j}}{2(3 - \lambda)} \right].$$

The discrete eigenvalues are the roots of $g_j(\lambda) = 0$, where

$$g_j(\lambda) \equiv C_j(\lambda) - \mathcal{F}(\lambda), \quad \mathcal{F}(\lambda) \equiv \frac{\int w^{q-1} (L_0 - \lambda)^{-1} w^3 \, dy}{\int w^q \, dy}.$$

Rigorous: If $C_j(0) > 1/2$, then there exists an unstable real eigenvalue on $0 < \lambda < 3$. Setting $C_j(0) = 0$, then gives

$$D_{j2} = \frac{1}{2} \left( 1 + \frac{qU_0 \sqrt{v_0}}{K \int w^3 \, dy} \right) \frac{\int w^3 \, dy}{v_0^{3/2}}.$$
Police Model I: Stability V

**Main Result:** For $q > 1$, a $K$-hot-spot equilibrium on an interval of length $S$ is unstable on an $O(1)$ time-scale for any $\tau_u > 0$ when $D > D^L_K$, where

$$D^L_K \equiv \frac{S^4}{8\varepsilon^2\pi^2\alpha^2K^4(1 + \cos(\frac{\pi}{K}))}\omega^3 \left(1 + \frac{qU_0}{\omega}\right).$$

Here $\omega$ is defined by

$$\omega = S(\gamma - \alpha) - U_0.$$

By a separate analysis involving the construction of asymmetric patterns, the stability threshold with respect to the $o(1)$ “small eigenvalues” is

$$D^S_K \equiv \frac{S^4}{16\varepsilon^2\pi^2\alpha^2K^4}\omega^3 \left(1 + \frac{qU_0}{\omega}\right) < D^L_K.$$

Notice that $D^S_K$ is monotonically increasing in $q$. But, the optimal police strategy is one that minimizes the stability threshold.

**Qualitative Result:** For a fixed $U_0$, it is not optimal for the police to be overly focussed on drifting towards hot spots (observed numerically by L. Ricketson).
Police Model I: A Hopf Bifurcation I

Consider two hot-spots, and let $q = 3$ so that the NLEP is solvable. Does there exist a Hopf bifurcation when $D < D_L^L$ whereby the amplitude of the two hot-spots oscillate asynchronously on an $O(1)$ time-scale?

For $q = 3$ and two-hot-spots, the function $F(\lambda)$ in the NLEP problem

$$g_1(\lambda) \equiv C_j(\lambda) - F(\lambda), \quad F(\lambda) \equiv \frac{\int w^{q-1} (L_0 - \lambda)^{-1} w^3 \, dy}{\int w^q \, dy},$$

is $F(\lambda) = 3/ [2(3 - \lambda)]$, and so $\lambda$ is a root of

$$\lambda = a + \frac{b}{1 + \hat{\tau} \lambda}, \quad a \equiv 3 \left(1 - \frac{3\chi_01}{2}\right), \quad b \equiv -\frac{9\chi_01\kappa_p}{2}.$$

Recall:

$$\hat{\tau} \equiv \varepsilon^{q-3} \tau_u \left(\frac{\int w^q \, dy}{v_0^{q/2}}\right), \quad \kappa_p \equiv \frac{U_0 \sqrt{v_0}}{K \int w^q \, dy}.$$

(simply set $q = 3$ and $K = 2$).
Main Result: For \( q = 3 \) and \( K = 2 \), there exists a Hopf Bifurcation at \( \tau_u = \tau_{uH} \) with \( \lambda = \pm i \lambda_I \), when \( 0 < D^H_K < D < D^L_K \). We have,

\[
\tau_{uH} = \frac{v_0^{3/2} D_{13}}{a \int w^3 \, dy}, \quad \lambda_I = \sqrt{-a(a + b)}.
\]

There is an explicit formula for \( D^H_K \). No Hopf bif. if \( D < D^H_K \).

Qualitative: If \( \tau_u > \tau_{uH} \) on \( 0 < D^H_K < D < D^L_K \), we predict asynchronous oscillatory instability of the two spots. Numerically, the bifurcation looks supercritical. If \( D < D^H_K \), then we have stability for all \( \tau_u \).

Interpretation: There is an intermediate range of police diffusivity (slower than criminals) for which the two hot-spots exhibit asynchronous oscillations. Maximum criminal activity is displaced periodically in time from one hot-spot to its neighbour.

Remark: Similar behavior for is expected for \( q \neq 3 \), but is more difficult to analyze. (no explicit analytical formulas available).
Police Model II: Predator-Prey Interaction

Principal Result (Equilibrium): On the basic cell $|x| > l$, and with $q > 1$, the leading order asymptotics for $A$, $P$, and $U$, in the hot-spot region near $x = 0$ is

$$A \sim \frac{1}{\varepsilon \sqrt{v_0}} w(x/\varepsilon), \quad P \sim [w(x/\varepsilon)]^2, \quad U \sim \frac{U_0}{\varepsilon b} [w(x/\varepsilon)]^q.$$ 

Here $b \equiv \int w^q dy$, and $w = w(y) = \sqrt{2} \text{sech}(y)$ is the homoclinic of $w'' - w + w^3 = 0$. The amplitude of the hot-spot is determined by $v_0$, where

$$\frac{1}{\sqrt{v_0}} \int w^3 dy = 2l(\gamma - \alpha) - U_0 \frac{\int w^{2+q} dy}{\int w^q dy}, \quad \frac{\int w^{2+q} dy}{\int w^q dy} = \frac{2q}{q + 1}.$$ 

Remark I: A hot-spot solution ceases to exist if

$$U_0 \geq U_{0c} \equiv (q + 1)l(\gamma - \alpha)/q.$$ 

Remark II: For $U_0 < U_{0c}$, for a $K$-spot equilibrium on a domain of length $S$, let $l = S/(2K)$ and replace $U_0 \rightarrow U_0/K$. Then, use a gluing technique to construct the multi-pulse pattern.
Police Model II: Stability I

Let $q > 1$, and $U_0 < U_{0c}$ so that a $K$-hot-spot equilibrium exists. Then:

**Small Eigenvalue Threshold:** Asymmetric hot-spot equilibria bifurcate from the symmetric $K$-hot-spot branch at the threshold

$$D^S_K \equiv \frac{\omega^3 S}{16\varepsilon^2 \pi^2 \alpha^2 K^4} \left(1 + \frac{2q^2 U_0}{(q + 1)\omega}\right), \quad \omega \equiv S(\gamma - \alpha) - \frac{2U_0q}{q + 1} > 0.$$  
(Note that the amplitude of the hot-spot is proportional to $\omega$.)

**NLEP Analysis:** The NLEP now involves 3 separate nonlocal terms. When $\tau_u \ll O(\varepsilon^{q-3})$, the stability threshold for this NLEP is

$$D^L_K = D^S_K \left(1 + \frac{2}{1 + \cos(\pi/K)}\right) > D^S_K.$$  

By simple calculus we study $D^S_K$ as a function of $q$ for $U_0$ fixed with $U_0 < U_{0c}$. **Optimal strategy is to minimize the threshold.**
Police Model II: Stability II

Key Qualitative Features (Optimal Strategy):

- For $q \to \infty$, then $D^S_K \to \infty$, which gives a poor police strategy. Overly focused aggressive police can lead (paradoxically) to rather closely spaced stable hot-spots.

- $D^S_K$ has a minimum wrt $q$ at some interior point in $1 < q < \infty$. The minimum value can be rather small only if $U_0$ is relatively large. Thus, only if there is enough police should they really focus their effort towards hot-spots.

Example: $S = 1, \gamma = 2, \alpha = 1, K = 2, \varepsilon = 0.02$ Top Curve: $U_0 = 0.28$, Bottom Curve $U_0 = 0.44$: Note: $(U_{0c} = 0.5)$.
Further Directions

- For the basic urban crime model in 2-D with $D = O(1)$:
  - Investigate effect of spatial heterogeneity of $\gamma$, $\alpha$.
  - Derive the scalar nonlinear elliptic PDE, associated with a saddle-node point, governing peak insertion.

- Dynamics in 1-D and 2-D: Derive ODE’s for the locations of centers of a collection of hot-spots (repulsive interactions?).

- Are dynamic peak-insertion events for a collection of hot-spots possible? If hot-spots become too closely spaced, we anticipate annihilation. Can annihilation and creation events lead to spatial-temporal chaos?

- Analyze other models of the effect of police, incorporating dynamic deterrence, i.e. A. Pilcher, EJAM (2010).

References: