Asymptotics of Nonlinear Biharmonic Eigenvalue Problems of MEMS

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Outline of the Talk

Nonlinear Biharmonic Eigenvalue Problems of MEMS

- Overview of Nonlinear Eigenvalue Problems of MEMS
- Calculation of the Pull-In Voltage Threshold. This has practical engineering applications.
- Concentration Behavior and Asymptotics of the Maximal Solution Branch. Of more mathematical interest in PDE.
**A MEMS Capacitor**

Beam or plate deflecting in the presence of an electric field.

- Top plate will contact with lower plate (i.e. touchdown) when $V > V^*$.  
- Device can act as a switch, valve or just capacitor.  
- If $V > V^*$, then no stable steady-state solutions. The threshold $V^*$ is called the pull-in voltage threshold.
The Mathematical Model of Pelesko

For small aspect ratio, the plate deflection satisfies Pelesko (2000):

\[ u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1 + u)^2} (1 + \beta |\nabla u|^2), \quad x \in \Omega \in \mathbb{R}^2, \]

\[ u = u_n = 0, \quad x \in \partial \Omega. \]

- singular nonlinearity represents a Coulomb attractive force between the deflectable surface and the fixed ground plate.
- nonlinear eigenvalue parameter \( \lambda \) is proportional to \( V^2 \).
- parameter \( \beta \) represents fringing-field effect due to the finite length of capacitor (Pelesko, Driscoll, J. Eng. Math, (2005)).
- Parameter \( \delta \) represents bending rigidity.

Main Questions:

- **Pull-in Threshold**: Of importance for applications is the saddle-node point at the end of the minimal solution branch for \( |u|_\infty \) vs. \( \lambda \).
- **Solution Multiplicity**: how does the global bifurcation diagram depend on \( \delta \) and on \( \beta \)?
The Basic Membrane Problem

Pelesko, SIAP, (2000) considered the basic membrane problem

\[ \Delta u = \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega; \quad u = 0, \quad x \in \partial \Omega. \]

For the unit disk the numerically computed bifurcation diagram is:

**Key Features:**

- In unit disk there is an infinite number of fold points with limiting behavior \( \lambda \to 4/9 \) as \( u(0) + 1 = \varepsilon \to 0^+ \).

- In contrast, for the unit slab there is either zero, one, or two steady-state solutions.
Membrane Problem in General Domains

For a general 2-D domain, the following are rigorous results:

**Theorem [Pelesko, SIAP, (2002)]:** Let $\mu_0$ be the first eigenvalue of the Laplacian, then there is no steady-state solution for $\lambda > \lambda_*$, where

$$\lambda_* \leq \bar{\lambda}_1 \equiv \frac{4\mu_0}{27}.$$ 

The lower (minimal) solution branch is linearly stable (N. Ghoussoub, Y. Guo, SIMA, (2007)).

For $\lambda \ll 1$, there is a unique solution, and there are an infinite number of fold points for $\lambda$ (Z. Guo, J. Wei, J. Lond. Math. Soc., (2008) (with no guarantee of clustering at some critical value of $\lambda$).

Extensions to $N$-dimensions with radial symmetry (Ghoussoub, Guo).

**Open Question in 2-D:** Does $\lambda \to \lambda_c$ as $|u|_\infty \to 1^-$, with an arbitrarily large number of fold points in a sufficiently small neighbourhood of some $\lambda_c$? If so, calculate $\lambda_c$, and $x_0$ for which $u(x_0) + 1 = \varepsilon \to 0^+$, and describe the asymptotics of the solution branch for $\lambda$ near $\lambda_c$.

**Remark:** This concentration question has nothing to do with critical points of regular part of a Green’s function.
Perturbation of the Membrane Problem by $\delta$

\[ u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1 + u)^2}, \quad x \in \Omega; \quad u = u_n = 0, \quad x \in \partial \Omega. \]

Numerical computations with either shooting or psuedo-arclength yield:

\[ \lambda |u(0)| \]

Left and Middle (Unit Disk): $\delta = 0.0001, 0.01, 0.05, 0.1$. Right (Unit Square): $\delta = 0.0001, 0.001, 0.01$.

- Infinite fold point structured destroyed when $\delta > 0$.
- Maximal solution branch with $\lambda \to 0$ and $|u|_\infty \to 1^-$.
- Pull-in voltage increases with $\delta$.
- Similar phenomena under effect of fringing field with $\beta > 0$. 
Bounds for the Pull-In Voltage

Ref: [LW1]: A. Lindsay, MJW, Asymptotics of Some Nonlinear Eigenvalue Problems for a MEMS Capacitor: Part I: Fold Point Asymptotics, Methods and Applications of Analysis, (2008), (28 pages)

**Theorem [LW1]:** Let $\Omega$ be the unit slab or the unit disk, and let $\mu_0 > 0$ be the first eigenvalue of

$$-\delta \Delta^2 \phi + \Delta \phi = -\mu \phi, \quad x \in \Omega; \quad \phi = \partial_n \phi = 0, \quad x \in \partial \Omega.$$  

Then, there is no steady-state solution for $\lambda > \lambda^*$, where $\lambda^* \leq \bar{\lambda} \equiv 4\mu_0/27$.

**Key:** proof requires positivity of first eigenfunction, which is guaranteed for slab and disk, but not other domains.

For the unit disk, we have

$$\phi = J_0(\xi_- r) - \frac{J_0(\xi_-)}{I_0(\xi_+)} I_0(\xi_+ r), \quad \xi_{\pm} \equiv \sqrt{\pm1 + \sqrt{1 + 4\mu \delta}}/2\delta,$$

where $\mu_0 > 0$ is the smallest root of $\xi_+ I_1(\xi_+) + \xi_- \frac{I_0(\xi_+)}{J_0(\xi_-)} J_1(\xi_-) = 0$.

Similar formula can be derived for the unit slab.
Asymptotics for the Pull-In Voltage

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<th>Slab</th>
<th>Unit Disk</th>
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Asymptotics of the Pull-In Threshold for $\delta \ll 1$: Let $\alpha = ||u||_\infty$. Assume that we know the fold point $(\lambda_0(\alpha_0), \alpha_0)$ for the unperturbed problem

$$\Delta u_0 = \frac{\lambda_0}{(1 + u_0)^2}, \quad x \in \Omega; \quad u_0 = 0 \quad x \in \partial \Omega.$$

Goal: Derive formulae for the corrections to the fold point for an arbitrary 2-D domain. How does it depend on curvature of $\partial \Omega$?

Then, calculate coefficients in this expansion for slab and disk.
Asymptotics for the Pull-In Voltage

For $\delta \ll 1$, there is a $O(\delta^{1/2})$ boundary layer near $\partial \Omega$. Away from $\partial \Omega$ we expand

$$u = u_0 + \delta^{1/2} u_1 + \delta u_2 + \cdots, \quad \lambda = \lambda_0 + \delta^{1/2} \lambda_1 + \delta \lambda_2 + \cdots,$$

to derive PDE’s for $u_1$ and $u_2$ with effective boundary conditions from matching to the boundary layer solution. For $\delta \ll 1$, the fold point location, defined by $d\lambda/d\alpha = 0$, is

$$\lambda_c = \lambda_{0c} + \delta^{1/2} \lambda_1(\alpha_0) + \delta \left[ \lambda_2(\alpha_0) - \frac{\lambda_1^2(\alpha_0)}{2\lambda_{0\alpha\alpha}(\alpha_0)} \right] + O(\delta^{3/2}).$$

At the unperturbed fold point $\alpha_0$, $\lambda_{0c} \equiv \lambda_0(\alpha_0)$, the linearized operator

$$\mathcal{L}\phi \equiv \Delta \phi + \frac{2\lambda_0}{(1 + u_0)^3} \phi = \frac{\lambda_{0\alpha}}{(1 + u_0)^2},$$

has the one-dimensional nullspace $\phi = u_{0\alpha}$. By invoking solvability conditions to evaluate the various terms:

$$\lambda_c = 1.4 + 5.6\delta^{1/2} + 25.45\delta + O(\delta^{3/2}) \quad \text{(Unit Slab)}$$

$$\lambda_c = 0.789 + 1.578\delta^{1/2} + 6.26\delta + O(\delta^{3/2}) \quad \text{(Unit Disk)}$$
Asymptotics for the Pull-In Voltage

For $\delta \rightarrow 0$ in an arbitrary 2-D domain, we obtain

**Principal Result: [LW1]:** Let $\Omega$ have a smooth boundary. Then, for $\delta \ll 1$,

$$
\lambda_c = \lambda_0 + 3\lambda_0 \delta^{1/2} \left( \frac{\int_{\partial \Omega} (\partial_n u_0) (\partial_n u_0 \alpha) \, dx}{\int_{\partial \Omega} \partial_n u_{0\alpha} \, dx} \right) + \delta \Lambda_2 + O(\delta^{3/2}),
$$

where $\Lambda_2$ involves the curvature of the boundary $\partial \Omega$.

**Asymptotics of the Pull-In Threshold for $\delta \gg 1$:** We first write

$$
-\Delta^2 u + \frac{1}{\delta} \Delta u = \frac{\tilde{\lambda}}{(1 + u)^2}, \quad x \in \Omega; \quad u = \partial_n u = 0, \quad x \in \partial \Omega,
$$

where $\tilde{\lambda} \equiv \lambda/\delta$. With $u_0$ satisfying pure Biharmonic, we expand

$$
u = u_0 + \delta^{-1} u_1 + \cdots; \quad \tilde{\lambda} = \tilde{\lambda}_0 + \delta^{-1} \tilde{\lambda}_1 + \cdots, \quad \delta \rightarrow \infty.
$$

By invoking appropriate solvability conditions, we get for $\delta \gg 1$ that

$$
\lambda_c \sim 70.1\delta + 1.7, \quad (\text{Unit Slab}); \quad \lambda_c \sim 15.4\delta + 1.0, \quad (\text{Unit Disk}).
$$
Comparison of Asymptotics and Numerics

For $\delta \ll 1$ comparison of asymptotics and numerics for $\lambda_c$

Left: Unit Slab; Right: Unit Disk. Numerics (heavy solid), two-term (dashed) and three-term (solid) asymptotics.

For $\delta \gg 1$ comparison of asymptotics and numerics for $\lambda_c$
Concentration Phenomena in Unit Disk

Construct the limiting asymptotics of the maximal solution branch to
\[ \Delta^2 u = -\lambda/(1 + u)^2, \]
with \( u = u_r = 0 \) on \( r = 1 \), for which \( \lambda \to 0 \) as \( u(0) + 1 = \varepsilon \to 0^+ \).

For \( \varepsilon \to 0^+ \) it is a singular perturbation problem since \( \lambda/(1 + u)^2 \to 0 \) except in a narrow zone near \( r = 0 \) where \( u = -1 + O(\varepsilon) \).

Leading-order term \( u_0 \) in outer region satisfies \( \Delta^2 u_0 = 0 \) in \( 0 < r < 1 \) with \( u_0 = u_0 r = 0 \) on \( r = 1 \). We must impose the point constraint \( u_0(0) = -1 \) in order to match to inner solution. Thus,

\[ u_0 = -1 + r^2 - 2r^2 \log r. \]

If we expand \( u = u_0 + \nu u_1 \) and \( \lambda = \nu \lambda_0 \), then \( \Delta^2 u_1 = -\lambda_0/(1 + u_0)^2 \), for which \( u_{1p} \sim \frac{\lambda_0}{16} \log(-\log r) \) as \( r \to 0 \). This divergence of the particular solution as \( r \to 0 \) requires the inclusion of switchback terms.

To find boundary layer width set \( \rho = r/\gamma \) to obtain
\[
\frac{u}{\gamma^2 \log \gamma} \sim \left( -\gamma^2 \log \gamma \right) (2\rho^2) + \gamma^2 (\rho^2 - 2\rho^2 \log \rho).
\]
Set \( u = -1 + \varepsilon v(\rho) \) in inner, to obtain that \( \gamma \) is given implicitly by \( \varepsilon = -\gamma^2 \log \gamma \).
Biharmonic BVP and Point Constraints

Model Problem: Consider the Biharmonic BVP in an Annulus

\[ \Delta^2 u = 0, \quad \varepsilon < r < 1, \]
\[ u = 1, \quad u_r = 0, \quad \text{on} \quad r = 1; \quad u = u_r = 0, \quad r = \varepsilon. \]

Since \( u \) is a linear combination of \( \{r^2, r^2 \log r, \log r, 1\} \), then

\[ u = A(r^2 - 1) + Br^2 \log r - (2A + B) \log r + 1, \]

for some \( A(\varepsilon) \) and \( B(\varepsilon) \). By expanding exact solution for \( \varepsilon \to 0 \) then

\[ u \sim u_0(r) + \varepsilon^2 (\log \varepsilon)^2 u_1(r) + O(\varepsilon^2 \log \varepsilon), \]
\[ u_0(r) = 1 - 16\pi G(r; 0) \equiv r^2 - 2r^2 \log r, \quad u_1 = 4(r^2 - 1) - 8r^2 \log r, \]

where \( G(r; 0) \) is the Biharmonic Green’s function; \( \Delta^2 G = \delta(x) \) and \( G = G_r = 0 \) on \( r = 1 \), given by \( G(r; 0) = (r^2 \log r - r^2/2 + 1/2) / (8\pi) \).

Remark: In fact, Biharmonic spline interpolation is based on solving linear systems of the form in \( \mathbb{R}^2 \):

\[ u_0(x_j) = \sum_{i=1}^{N} f_i G(x_j; x_i). \]
Concentration Behavior in Unit Disk

Ref: [LW2]: A. Lindsay, MJW, Asymptotics of Some Nonlinear Eigenvalue Problems for a MEMS Capacitor: Part II: Multiple Solutions and Singular Asymptotics, under review, EJAM, (2010), (34 pages)

Consider $\Delta^2 u = -\lambda/(1 + u)^2$, with $u = u_r = 0$ on $r = 1$.

Principal Result: [LW2]: For $\varepsilon \equiv u(0) + 1 \to 0^+$, the limiting asymptotics of the maximal solution branch in the outer region, away from $r = 0$, is

$$u = u_0 + \frac{\varepsilon}{\sigma} \log \sigma u_{1/2} + \frac{\varepsilon}{\sigma} u_1 + \varepsilon \log \sigma u_{3/2} + \varepsilon u_2 + O(\varepsilon \sigma \log \sigma),$$

$$\lambda = \frac{\varepsilon}{\sigma} \left[ \lambda_0 + \sigma \lambda_1 + O(\sigma^2) \right], \quad \lambda_0 = 32, \quad \lambda_1 = 16 \left( \log 2 - \frac{\pi^2}{6} \right).$$

where $\sigma = -1/\log \gamma$ and the boundary layer width $\gamma$ is determined implicitly by $-\gamma^2 \log \gamma = \varepsilon$. The point constraint $u_0(0) = -1$ holds, and

$$u_0 = -1 + r^2 - 2r^2 \log r, \quad u_{1/2} = -\frac{\lambda_0}{16} u_0, \quad u_{3/2} = -\frac{\lambda_1}{16} u_0.$$

Note: $u_{1/2}$ and $u_{3/2}$ are switchback terms.
Concentration Behavior in Unit Disk

In addition, $u_1$ and $u_2$ are the unique solutions of

$$\Delta^2 u_1 = -\frac{\lambda_0}{(1 + u_0)^2}, \quad 0 < r < 1; \quad u_1(1) = u_1 r(1) = 0,$$

$$u_1 = \frac{\lambda_0}{16} \log(-\log r) + \frac{\lambda_0}{16} + O(\log^{-1} r), \quad r \to 0,$$

$$\Delta^2 u_2 = -\frac{\lambda_1}{(1 + u_0)^2}, \quad 0 < r < 1; \quad u_2(1) = u_2 r(1) = 0,$$

$$u_2 = \frac{\lambda_1}{16} \log(-\log r) + \frac{1}{16} (\lambda_0 + \lambda_1) - \log 2 + \frac{\lambda_0}{16} \log r + O(\log^{-1} r), \quad r \to 0.$$

Comparison of Asymptotics 1-term (dotted), 2-term (dashed) and Numerics (solid)
Concentration in Unit Disk: Mixed Biharmonic

Consider $-\delta \Delta^2 u + \Delta u = \lambda/(1 + u)^2$, with $u = u_r = 0$ on $r = 1$:

Then, $u_0$ with point constraint $u_0(0) = -1$ satisfies

$$-\delta \Delta^2 u_0 + \Delta u_0 = 0, \quad 0 < r < 1; \quad u_0(1) = u_0r(1) = 0,$$

$$u_0 = -1 + \alpha r^2 \log r + \varphi r^2 + o(r^2), \quad \text{as} \quad r \to 0.$$  

We can calculate $\alpha < 0$ and $\varphi$ in terms of modified Bessel functions. Then,

$$\lambda_0 = 8\alpha^2,$$

$$\lambda_1 = -\frac{\lambda_0}{2} \left[ \frac{\pi^2}{6} - \log(-\alpha) + \left( 1 + \frac{2\varphi}{\alpha} \right) \right].$$

Comparison of Asymptotics and Numerics

Left: Small $\delta$  
Right: Larger $\delta$
Concentration in Arbitrary 2-D Domain


Consider $\Delta^2 u = -\lambda/(1 + u)^2$, with $u = \partial_n u = 0$ on $\partial \Omega$.

**Principal Result:** [KLW]: For $\varepsilon \equiv u(x_0) + 1 \to 0^+$, the limiting asymptotics of the maximal solution branch in the outer region, away from $x_0$, and $\lambda$ is

$$u = u_0 + O\left(\varepsilon \sigma^{-1} \log \sigma\right), \quad \lambda = \frac{\varepsilon}{\sigma} \lambda_0 + \varepsilon \lambda_1 + O(\varepsilon \sigma),$$

where $\sigma = -1/\log \gamma$ and the boundary layer width $\gamma$ is given implicitly by $-\gamma^2 \log \gamma = \varepsilon$. Here,

$$u_0(x; x_0) = -\frac{G(x; x_0)}{R(x_0; x_0)},$$

with point constraint $u_0(x_0) = -1$, where $G(x; \xi)$ satisfies

$$\Delta^2 G = \delta(x - \xi), \quad x \in \Omega; \quad G = \partial_n G = 0, \quad x \in \partial \Omega,$$

$$G(x, \xi) = \frac{1}{8\pi} |x - \xi|^2 \log |x - \xi| + R(x; \xi).$$
Concentration in Arbitrary 2-D Domain

To leading-order, the concentration point \( x_0 \in \Omega \) satisfies

\[
\nabla_x R(x; x_0) \big|_{x=x_0} = 0, \quad \text{provided that} \quad R(x_0; x_0) > 0.
\]

As \( x \to x_0 \), with \( r = |x - x_0| \), we identify \( \alpha \) and \( \beta \) by

\[
u_0 \sim -1 + \alpha r^2 \log r + r^2 (\beta + a_c \cos 2\theta + a_s \sin 2\theta) + \cdots,
\]

where \( \alpha < 0 \) by assumption, and \( \beta \) (sign \( \pm \)) is

\[
\alpha \equiv \frac{-1}{8\pi R(x_0; x_0)}, \quad \beta \equiv \frac{-1}{4R(x_0; x_0)} \left[ \frac{\partial^2 R}{\partial x_1^2} + \frac{\partial^2 R}{\partial x_2^2} \right]_{x=x_0}.
\]

Finally, \( \lambda_0 \) and \( \lambda_1 \) are given by

\[
\lambda_0 = 8\alpha^2, \quad \lambda_1 = -\frac{\lambda_0}{2} \left[ \frac{\pi^2}{6} - \log(-\alpha) + \left( 1 + \frac{2\beta}{\alpha} \right) \right].
\]

Asymptotics of \( \lambda \) determined in terms of properties of the regular part of the biharmonic Green’s function.

Note: \( R_{00} \equiv R(x_0; x_0) \) and the trace Trace \((\mathcal{R}_{00})\) computed by fast multipole methods for Low Reynolds number flow (Kropinski).
Numerics: Concentration in 2-D Domains

Comparison of Asymptotics and Numerics in Square Domain: For the square $[-1, 1]^2$, then $x_0 = 0$, and we compute

$$R_{00} \approx 0.0226\ldots, \quad \text{Trace } (R_{00}) \approx -0.0892\ldots.$$ 

Class of Dumbell-Shaped Domains:
Let $z \in \mathcal{B}$, where $\mathcal{B}$ is the unit disk, and define the complex mapping

$$w = f(z; b) = \frac{(1 - b^2)z}{z^2 - b^2},$$
For various values of \( b \), numerical values for \( R(x_0; x_0) \) and Trace (\( \mathcal{R}_{00} \)) at the points \( x_0 = (x_0, 0) \) where \( dR/dx_0 = 0 \).

<table>
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<tr>
<th>( b )</th>
<th>( x_0 )</th>
<th>( R(x_0, x_0) )</th>
<th>Trace (( \mathcal{R}_{00} ))</th>
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<td>( 3.11557 \times 10^{-2} )</td>
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Numerics: $R_{00}$ and the Dumbbell-Shape

(a) $b = 2.0$

(b) $b = b_c = 1.83995$

(c) $b = 1.5$

(d) $b = 1.05$
Further Directions

For $\delta \to 0$, calculate the number of solutions for the mixed Biharmonic problem, and describe the breakup of infinite fold point structure analytically.

For fringing-field problem:
1. Describe breakup of infinite fold point structure for $\beta \to 0$.
2. Calculate limiting asymptotics of maximal solution branch for $\beta = O(1)$.

For pure biharmonic, find an $\Omega$ for which $\nabla_x R(x_0, x_0) = 0$ but $R(x_0; x_0) < 0$. Related to non-positivity properties of $G$.

Time-dependent quenching behavior beyond the pull-in instability.

\begin{align*}
\text{(e) } \lambda &= 2.0, \delta = 0.003 \\
\text{(f) } \lambda &= 40.0, \delta = 0.003
\end{align*}