On the Asymptotic and Numerical Analysis of Exponentially Ill-Conditioned
Singularly Perturbed Boundary Value Problems

June-Yub Lee †
Dept. of Mathematics
Courant Institute, New York University
New York, New York 10012

Michael J. Ward *
Dept. of Mathematics
University of British Columbia
Vancouver, Canada V6R 2N4

Abstract

Asymptotic and numerical methods are used to study several classes of singularly perturbed
boundary value problems for which the underlying homogeneous operators have exponentially
small eigenvalues. Examples considered include the familiar boundary layer resonance problems
and some extensions, and certain linearized equations associated with metastable internal layer
motion. For the boundary layer resonance problems, a systematic projection method, motivated
by the work of De Groen [SIAM J. Math. Anal. 11, (1980), pp. 1-22], is used to analytically
calculate high order asymptotic solutions. This method justifies and extends some previous results
obtained from the variational method of Grasman and Matkowsky [SIAM J. Appl. Math. 32,
(1977), pp. 588-597]. A numerical approach, based on an integral equation formulation, is used to
accurately compute boundary layer resonance solutions and their associated exponentially small
eigenvalues. For various examples, the numerical results are shown to compare very favorably with
two term asymptotic results. Finally, some Sturm-Liouville operators with exponentially small
spectral gap widths are studied. One such problem is applied to analyzing metastable internal
layer motion for a certain forced Burgers equation.

† The work of this author was supported by a Packard Foundation fellowship awarded to Leslie
Greengard
* Supported by NSERC under grant 5-81541
1. Introduction

For \( \epsilon \to 0^+ \), we study certain classes of ill-conditioned boundary value problems of the form

\[
L_\epsilon u(x) = \epsilon u''(x) + p(x, \epsilon)u'(x) + q(x, \epsilon)u(x) = f(x, \epsilon), \quad u(a) = \alpha, \quad u(b) = \beta. \tag{1.1}
\]

The problems we consider are those for which the eigenvalue problem \( L_\epsilon \phi = \lambda \phi \) with \( \phi(a) = \phi(b) = 0 \) has an exponentially small eigenvalue \( \lambda_0 \) with \( \lambda_0 = O(\epsilon^{2}\omega e^{-\epsilon^{-1}}) \) as \( \epsilon \to 0 \) for some \( \omega > 0 \).

Examples of operators \( L_\epsilon \) that have such a spectral property include the exit operator (cf. [22], [23], [6]) and the Hermite operator (cf. [6]). Both of these operators are associated with the well-known boundary layer resonance phenomena that was first discovered in [1] and later studied analytically, from various viewpoints, in [24], [9], [25], [15], [6] and [17] (see also the references therein). More recently, it has been shown that an exponentially small eigenvalue occurs for a certain linearized operator that arises from a nonlinear viscous shock problem exhibiting metastable internal layer motion (cf. [18], [27], [28], [19] and [20]).

For the boundary layer resonance problems, a conventional matched asymptotic expansion approach fails to determine the solutions uniquely. To overcome this deficiency, a variational principle was postulated and used in [9] to calculate leading order asymptotic solutions for these problems. This method was subsequently extended in [34] and [29] to obtain a uniformly valid leading order characterization of the extreme sensitivity of the boundary layer resonance solutions to small changes in the data for (1.1). More recently, a modified variational principle was proposed in [30] to calculate higher order corrections for the solutions.

The relationship between boundary layer resonance phenomena and exponentially small eigenvalues was uncovered and emphasized in [6] (see also [17]). Motivated by this work, our first goal is to formulate and use a systematic projection method to calculate high order asymptotic expansions for the boundary layer resonance solutions and for their associated exponentially small eigenvalues. This method, which explicitly exploits the spectral properties of the homogeneous operator, gives a justification for the postulated variational principle of [9] and its extensions. Moreover, the extreme sensitivity of the boundary layer resonance solutions to small changes in data, and the difficulty in numerically computing such solutions using standard methods, has a very natural interpretation in terms of the exponential ill-conditioning of the underlying homogeneous operator. This spectral approach also provides a unifying link to a related projection approach used recently in [27] and [28] to analyze metastable viscous shock layer motion.

Our second goal is to use the recent numerical method of [21], which is based on an integral equation formulation, to accurately compute boundary layer resonance solutions in response to exponentially small changes in the coefficients of the differential operator. The associated exponentially small eigenvalues are also computed accurately. For a series of examples, numerical results for the boundary layer resonance solutions and the associated small eigenvalues are compared with corresponding asymptotic results. In many cases, these comparisons show that two term asymptotic results are required to obtain a close quantitative agreement with the numerical values. These comparisons then justify the need for the higher order asymptotic theory. To the best of our knowledge, these examples provide the first comprehensive and accurate comparisons between asymptotic and numerical results, at small values of \( \epsilon \), for the boundary layer resonance phenomena.
The previous studies of boundary layer resonance have focussed on the homogeneous case $f = 0$ in (1.1). Our third goal is to use the projection method and the numerical method of [21] to examine some inhomogeneous problems. The analysis shows that when $f$ is not orthogonal to the eigenfunction of $L_\epsilon$ that corresponds to the exponentially small eigenvalue, the solution of (1.1) is very sensitive to an exponentially small forcing function. These results are then applied to a linearized sensitivity analysis of a model equation of [4] that arises from a diffusive regularization of the shape from shading problem.

As shown in [18], [27], [28], [19] and [20], exponentially small eigenvalues are also associated with metastable viscous shock layer motion. For a particular choice of boundary condition, an asymptotic estimate for the exponentially small eigenvalue was derived in [27]. For Burgers equation, another goal is to derive further eigenvalue estimates relevant to different boundary conditions and to verify these estimates numerically using the method of [21].

We also consider some singularly perturbed Sturm-Liouville operators with exponentially small spectral gap widths. For two exactly solvable such problems, the numerical method of [21] is shown to provide highly accurate results for the small gap widths. Our last focus is to show that a modified version of the symmetric double well potential problem of [12] is directly relevant to analyzing the slow internal layer motion for a class of forced Burgers equation studied previously using finite difference methods in [18]. For this modified double-well problem, an asymptotic formula for the exponentially small spectral gap width is obtained and is verified numerically. The reciprocal of this gap width then determines the metastable time scale for the slow internal layer motion.

The outline of the paper is as follows. In §2 we discuss the favorable properties of the numerical method of [21] for computing both the solutions to exponentially ill-conditioned boundary value problems and their corresponding exponentially small eigenvalues. In §3 and §4 we examine, using analytical and numerical methods, the boundary layer resonance phenomena associated with the exit operator and the Hermite operator, respectively. Inhomogeneous exponentially ill-conditioned problems and an application are considered in §5. In §6 eigenvalue estimates for metastable internal layer motion are derived and verified numerically. Finally, in §7, we consider some nearly degenerate Sturm-Liouville operators, and the results are applied to a a forced Burgers equation.

2. The Numerical Method

There are two inherent obstacles in solving exponentially ill-conditioned problems of the form (1.1) numerically. The first difficulty is that $u(x)$ can be exponentially sensitive to all the data in (1.1); for example, to $f(x)$. More specifically, since the norm of the mapping from $f$ to $u$ is of the order $O(\lambda^{-1}_0)$, a perturbation $\Delta f$ to $f$ may cause rather large changes $\Delta u$ in $u$, of the order $O(\lambda^{-1}_0 \Delta f)$. Thus one of the necessary conditions for a numerical method to treat these problems is that the numerical error be less than the order of the smallest eigenvalue of the homogeneous operator. However, any numerical method will certainly generate discretization errors by restricting the functional spaces containing $p, q, f$ and $u$ to finite dimensional spaces. Therefore, high order or spectral-type numerical methods, which lead to small residual errors with a moderate number of meshpoints, are preferred for these problems.

The second difficulty is universally encountered when a differential operator $L_\epsilon$ is replaced by a discrete operator $L_\epsilon^h$ with a parameter $h$, usually representing mesh size, and evaluated numerically using finite precision arithmetic. Although $||L_\epsilon^h||$ grows without bound when $L_\epsilon^h$ approaches $L_\epsilon$
as \( h \to 0 \), it is still possible to bound the \( l_2 \) norm of the numerical operator \( L_h \) by choosing a
fixed size of \( h \). Thus, in spite of the unboundedness of \( L_c \), we can get some digits of accuracy for
well-conditioned problems since for some fixed \( h \), the condition number \( \| (L_h)^{-1} \| \) might be
smaller than the reciprocal of the machine precision. However, for ill-conditioned problems where
\( \| (L_h)^{-1} \| \approx 10^{14} \), it is difficult to achieve any accuracy in double precision arithmetic unless
\( \| L_h \| \) is of the order of unity.

The requirement of having to simultaneously obtain a small value of \( \| L_h \| \) and a small residual
error provides a significant limitation on the use of standard discretizations, such as those based
on finite difference or finite element methods, to treat exponentially ill-conditioned problems of the
form (1.1). To circumvent this difficulty, we use a high order integral equation scheme for (1.1)
whose condition number does not depend on mesh size. Thus, the performance of this numerical
method is limited only by the ill-conditioning of the underlying continuous problem.

2.1 Numerical Solutions of Stiff Boundary Value Problems

The solution to (1.1) is decomposed into the sum of two terms; a linear function \( u_i \) satisfying
the boundary conditions of (1.1) and the solution \( u_h \) of (1.1) satisfying homogeneous boundary
conditions. In principle, \( u_h \) can then be written in terms of the Green’s function of \( L_c \). However,
since, for arbitrary variable coefficients in \( L_c \), it is not analytically feasible to calculate this Green’s
function, we instead derive an integral equation for \( u_h \) in terms of the Green’s function \( G_0(x,t) \)
of the simpler problem \( \phi''(x) - q_0 \phi(x) = 0 \) with \( \phi(a) = \phi(b) = 0 \), where \( q_0 \) is an arbitrary non-negative
constant (see [7] for a discussion of Green’s functions).

In terms of \( G_0(x,t) \) we define the operator \( P : L^2[a,b] \to L^2[a,b] \) by

\[
P \eta(x) = c \eta(x) + p(x) \int_a^b \frac{d}{dx} G_0(x,t) \eta(t) dt + \left( q(x) - q_0 \right) \int_a^b G_0(x,t) \eta(t) dt.
\]

(2.1)

Then, the solution of (1.1) can be written in the form

\[
u(x) = u_i(x) + u_h(x) = u_i(x) + \int_a^b G_0(x,t) \sigma(t) dt ,
\]

(2.2)

where \( \sigma \) satisfies the following Fredholm integral equation of the second kind:

\[
P \sigma = f(x) - \left( c u_i'(x) + p(x) u_i'(x) + q(x) u_i(x) \right).
\]

(2.3)

From [26], this integral equation is well conditioned in the sense that both \( P \) and its inverse
\( P^{-1} \) are bounded regardless of the mesh size and the order of discretization. In spite of this
favorable property, numerical approaches based on integral equation formulations have not been
widely used due to the fact that traditional integral equation solvers are computationally more
expensive than those for finite difference or finite element methods. Recently, however, some fast
methods for integral equations have been developed (cf. [10], [11], and [31]). In particular, a fast
adaptive numerical method for stiff two point boundary value problems, which was based on (2.3),
was studied and fully described in a paper by one of the present authors (cf. [21]). This method,
which is used for all the numerical experiments in this paper, solves (1.1) with an arbitrary order
of accuracy at a computational cost proportional to the number of mesh points. For the examples
in sections 3 through 6 below, the order of accuracy for the method of [21] was set to 24 and
the number of subintervals was set to about 64. These choices were sufficient to resolve the
exponentially small changes in the coefficient functions and to obtain full numerical accuracy to
within double precision.

2.2 The Numerical Evaluation of Eigenvalues and Eigenfunctions

In sections 3-7, exponentially small eigenvalues for the Sturm-Liouville problem

\[ L\phi + \lambda w\phi \equiv (p\phi')' + q\phi + \lambda w\phi = 0, \]  

(2.4)
satisfying certain homogeneous boundary conditions, are computed.

One classical way to compute eigenvalues for (2.4) is to use a shooting method whereby a
sequence of initial value problems with unknown parameter \( \lambda \) is solved until \( \lambda \) is adjusted so that
both boundary conditions are satisfied. Well-developed software packages that use a shooting
approach, such as SLEIGN or the NAG library code D02KDF, are available (see [2], [8], [13] for
a further discussion). The difficulty with this method is that it typically requires the numerical
determination of a root of a nonlinear shooting function, which may be difficult for ill-conditioned
problems.

Another approach for (2.4) is to iteratively approximate eigenvalues and eigenfunctions by
solving a sequence of two point boundary value problems. A discussion of subspace methods and
their generalizations can be found in the review article [33], while linear algebraic techniques using
finite difference and collocation methods are studied in [5], [16].

The approach we use to compute eigenvalues of (2.4) is based on combining an inverse
orthogonal iteration method with the integral equation solver of §2.1. Although, the algorithm is
fully described in [21], we would like to summarize some important characteristics of the method.
Firstly, the convergence of subspace iteration is, in general, linear and the convergence rate depends
on the distribution of the eigenvalues. To overcome the slow convergence for the eigenvalue com-
putations, a shifting procedure is performed every iteration to ensure quadratic convergence. 
This shifting procedure is crucial for the success of the computations in §7 involving nearly degenerate
Sturm-Liouville spectra. Roughly 15 iterations are sufficient to obtain double precision accuracy
for the examples in this paper. Secondly, the accuracy of the eigenvalues and the eigenfunctions
depends on the order of the integral equation method. Therefore, the limit of accuracy is bounded
by the conditioning of the underlying continuous problem. The relationship between the smallest
eigenvalue and the conditioning is illustrated in Fig. 1 for example 3.1 of §3.

3. The Exit Operator

For \( \epsilon \to 0 \), we obtain high order asymptotic solutions for

\[ L_{\epsilon}u = \epsilon u'' + x^{2m+1} p(x) u' = -g(x) e^{\epsilon x} e^{-\epsilon x} u, \quad -a \leq x \leq b, \quad u(-a) = \alpha, \quad u(b) = \beta. \]  

(3.1)

Here \( a > 0, b > 0, d > 0, m \) is a non-negative integer and \( p(x) > 0 \) and \( g(x) \) are smooth functions.
The asymptotic results are verified numerically.

3.1 Spectral Properties
We first study the spectral properties for the eigenvalue problem
\[ L_\epsilon \phi = \lambda \phi, \quad \phi(-a) = \phi(b) = 0. \] (3.2)
For (3.2), the eigenvalues \( \lambda_j \) for \( j \geq 0 \) are real with \( \lambda_j < 0 \) for \( j \geq 0 \) and \( \lambda_j \to -\infty \) as \( j \to \infty \). The normalized eigenfunctions \( \phi_j \) satisfy \( (\phi_j, \phi_k)_w = \delta_{jk} \), where the inner product is defined by
\[ (u, v)_w \equiv \int_a^b u v w \, dx. \]
The weight function \( w = w(x, \epsilon) \) is defined by
\[ w \equiv \exp \left( -\epsilon^{-1} \int_0^x t^{2m+1} p(t) \, dt \right). \] (3.3)
The Rayleigh quotient can be used to show the crude estimate that \( \lambda_0 = O \left( \epsilon^q e^{-\epsilon^{-1} \omega} \right) \) for some \( q \) and \( \omega > 0 \). We now calculate \( \lambda_0 \) more precisely.

Let \( v \) satisfy \( L_\epsilon v = 0 \). Then, upon integrating by parts, we derive
\[ \lambda_0 (v, \phi_0)_w = \epsilon w(b) v(b) \phi_0'(b) - \epsilon w(-a) v(-a) \phi_0'(-a). \] (3.4)
Choose \( v = 1 \) and assume that \( \lambda_0 = o(\epsilon^k) \) for any \( k > 0 \). Then, since \( L_\epsilon v = 0 \), we have that \( \phi_0 \sim M_0 \), where \( M_0 \) is a normalization constant. However, since this approximate form for \( \phi_0 \) does not satisfy the boundary conditions in (3.2), we must construct boundary layer profiles for \( \phi_0 \) near each endpoint before calculating \( \phi_0'(-a) \) and \( \phi_0'(b) \).

In the region near \( x = -a \) we define \( y = \epsilon^{-1}(a + x) \) and \( \Phi_L(y) = \phi_0(-a + \epsilon y) \). Then, by substituting \( \Phi_L(y) = \Phi_L^0(y) + \epsilon \Phi_L^1(y) + \cdots \) into (3.2) and by using the matching condition \( \Phi_L(y) \to M_0 \) as \( y \to \infty \), we obtain the following boundary layer equations
\[ \Phi_L'' + \xi_a \Phi_L' = 0, \quad \Phi_L^0(0) = 0, \quad \Phi_L^0(y) \to M_0 \text{ as } y \to \infty, \]
\[ \Phi_L''' + \xi_a \Phi_L'' = \xi_a y \left( \frac{(2m+1)}{a} - \frac{\rho'(-a)}{\rho(-a)} \right) \Phi_L', \quad \Phi_L^1(0) = 0, \quad \Phi_L^1(y) \to 0 \text{ as } y \to \infty. \] (3.5)
Here \( \xi_a \equiv a^{2m+1} p(-a) \). The solutions to (3.5) are
\[ \Phi_L^0(y) = M_0 \left( 1 - e^{-\xi_a y} \right), \]
\[ \Phi_L^1(y) = M_0 \left( \frac{\rho'(-a)}{\rho(-a)} - \frac{(2m+1)}{a} \right) \left( \frac{\xi_a y^2}{2} + y \right) e^{-\xi_a y}. \] (3.6)
Thus, upon using \( \phi_0'(-a) = \epsilon^{-1} \Phi_L^0(0) \), we obtain the two term expansion
\[ \phi_0'(-a) = M_0 \xi_a \epsilon^{-1} \left[ 1 + \frac{\epsilon}{\xi_a} \left( \frac{\rho'(-a)}{\rho(-a)} - \frac{(2m+1)}{a} \right) + \cdots \right], \quad \xi_a \equiv a^{2m+1} p(-a). \] (3.7a)
A similar calculation, which we omit, provides the following two term expansion for \( \phi_0'(b) \):
\[ \phi_0'(b) = -M_0 \xi_b \epsilon^{-1} \left[ 1 - \frac{\epsilon}{\xi_b} \left( \frac{\rho'(b)}{\rho(b)} + \frac{(2m+1)}{b} \right) + \cdots \right], \quad \xi_b \equiv b^{2m+1} p(b). \] (3.7b)
By extending the boundary layer analyses to higher order, we can in principle calculate further terms in the expansions of \( \phi_0'(-a) \) and \( \phi_0'(b) \). For \( \epsilon \to 0 \), these expansions have the form

\[
\phi_0'(-a) = M_0 \xi_a \epsilon^{-1} \gamma_a(\epsilon), \quad \gamma_a(\epsilon) \sim 1 + \sum_{j=1}^{\infty} \epsilon^j \phi_{Lj},
\]

\[
\phi_0'(b) = -M_0 \xi_b \epsilon^{-1} \gamma_b(\epsilon), \quad \gamma_b(\epsilon) \sim 1 + \sum_{j=1}^{\infty} \epsilon^j \phi_{Rj},
\]

for some \( \phi_{Lj} \) and \( \phi_{Rj} \) independent of \( \epsilon \). The terms \( \phi_{L1} \) and \( \phi_{R1} \) in (3.8) can be read from (3.7).

Next we calculate \( (v, \phi_0) \). Since \( w \) is concentrated near \( x = 0 \), this inner product can be calculated using Laplace’s method with \( v = 1 \) and \( \phi_0 \sim M_0 \) to obtain

\[
(v, \phi_0) \sim M_0 \epsilon^{1/(2m+2)} \theta_\epsilon, \quad \theta_\epsilon \sim \sum_{j=0}^{\infty} \epsilon^{j/(m+1)} \theta_j,
\]

for some coefficients \( \theta_j \) independent of \( \epsilon \). The first two coefficients are

\[
\theta_0 = \frac{\epsilon^{1/(2m+2)}}{(m+1) \Gamma \left( \frac{1}{2m+2} \right)},
\]

\[
\theta_1 = \frac{3 \epsilon^{3/(2m+2)}}{(2m+2)^2} \left( \frac{[p'(0)]^2 (2m+5)}{p(0) (2m+3)^2} - \frac{p''(0)}{(2m+4)} \right) \Gamma \left( \frac{3}{2m+2} \right),
\]

where \( r \equiv 2(m+1)/p(0) \) and \( \Gamma(z) \) is the Gamma function. Substituting (3.3), (3.8) and (3.9a) in (3.4), we obtain the following estimate for the exponentially small eigenvalue

\[
\lambda_0 \sim -\epsilon^{-1/(2m+2)} \theta_\epsilon^{-1} \left[ k^{2m+1} p(b) \gamma_b(\epsilon) e^{-\epsilon^{-1} \omega_b} + a^{2m+1} p(-a) \gamma_a(\epsilon) e^{-\epsilon^{-1} \omega_a} \right].
\]

Here \( \gamma_a(\epsilon) \) and \( \gamma_b(\epsilon) \) are defined by (3.8) while \( \omega_a \) and \( \omega_b \) are defined by

\[
\omega_a = \int_0^a t^{2m+1} p(t) \, dt, \quad \omega_b = \int_0^b t^{2m+1} p(t) \, dt.
\]

For the case \( m = 0 \) and upon replacing \( \theta_\epsilon \), \( \gamma_a \) and \( \gamma_b \) in (3.10a) by their leading behaviors \( \theta_\epsilon \sim \theta_0 \), \( \gamma_a \sim 1 \) and \( \gamma_b \sim 1 \), the resulting leading order formula for \( \lambda_0 \) was proven in [6]. Similar leading order eigenvalue estimates for some related equations were derived in [22] and [23].

We now consider two special cases of (3.10a). Assume that \( a = b \), \( m = 0 \) and that \( p(x) \) is an even function. Then, using (3.7), (3.9b) and (3.10a), we obtain the two term expansion

\[
\lambda_0 \sim - \left( \frac{2p(0)}{\pi \epsilon} \right)^{1/2} b p(b) \left[ 1 + \epsilon \left( \frac{3p''(0)}{8[p(0)]^2} - \frac{1}{b p(b)} \left( \frac{1}{b} + \frac{p'(b)}{p(b)} \right) \right) \right] \exp \left( -\epsilon^{-1} \int_0^b t p(t) \, dt \right).
\]

When \( p(x) \) is even, \( a = b \) and \( m = 1 \), we have the two term expansion

\[
\lambda_0 \sim - \left( \frac{p(0)}{4 \epsilon} \right)^{1/4} \frac{4b^3 p(b)}{\Gamma(1/4)} \left[ 1 + \epsilon^{1/2} \left( \frac{16}{p(0)} \right)^{3/4} \Gamma(3/4) \right] \exp \left( -\epsilon^{-1} \int_0^b t^3 p(t) \, dt \right).
\]
We now give illustrate (3.11) for some specific forms of \( p(x) \). The motivation for considering these forms of \( p(x) \) is given in §5.2. Let \( p(x) = 2\pi x^{-1} \sin(\pi x/2), \) \( b = 1 \) and \( m = 0 \). Then (3.11a) becomes

\[
\lambda_0 \sim - (2\pi)^{3/2} \epsilon^{-1/2} \left( 1 - \frac{\epsilon}{32} \right) e^{-\epsilon^{-1}}. \tag{3.12a}
\]

Now let \( p(x) = 2\pi (1 + \mu x^2)^{-1}, \) \( \mu \geq 0, \) \( b = 1 \) and \( m = 1 \). Then (3.11b) becomes

\[
\lambda_0 \sim - \left( \frac{\pi}{2\epsilon} \right)^{1/4} \frac{8\pi}{(1 + \mu) \Gamma(1/4)} \left( 1 - \frac{\epsilon^{1/2} \mu}{2} \pi \frac{1}{\Gamma(3/4) \Gamma(1/4)} \right) e^{-\epsilon^{-1} \omega_1}, \tag{3.12b}
\]

where \( \omega_1 = \pi \mu^{-1} [1 - \mu^{-1} \log(1 + \mu)] \). In the special case when \( \mu = 0 \) we can calculate an additional term in (3.12b). When \( \mu = 0 \) we have \( \theta_k = \theta_0 + o(\epsilon^k) \) for any \( k > 0 \). Thus, in this case, (3.8) and (3.10) yield

\[
\lambda_0 \sim - \left( \frac{\pi}{2\epsilon} \right)^{1/4} \frac{8\pi}{(1 + \mu) \Gamma(1/4)} \left( 1 - \frac{3\epsilon}{2\pi} \right) e^{-\epsilon^{-1} \pi/2}. \tag{3.12c}
\]

To compare with (3.12), we use the eigenvalue solver of [21], summarized in §2, to compute \( \lambda_0 \) directly from (3.2). In Tables 1a, 1b and 1c we display the results for \( \lambda_0 \) corresponding to (3.12a), (3.12b) with \( \mu = 1 \), and (3.12c), respectively. In each of these tables, the second column gives the numerical result for \( \lambda_0 \) while the third column gives the two term expansion for \( \lambda_0 \). In the fourth column we also display the results for \( \lambda_0 \) that are obtained by truncating the pre-exponential factors in (3.12) to one term. Several trends can be observed from these comparisons. We first note that a two term expansion for the pre-exponential factor of \( \lambda_0 \) is needed to obtain close quantitative agreement with the numerical results. In certain cases, the agreement between the two term result and the numerical result is close even at rather large values of \( \epsilon \) (see the first few rows of Table 1a and Table 1c). From Table 1b we note that when the pre-exponential factor for \( \lambda_0 \) has an expansion in powers of \( \epsilon^{1/2}, \) the two term asymptotic result gives only a moderately good determination of \( \lambda_0 \). Finally, we note that the numerical method for calculating \( \lambda_0 \) becomes inaccurate when \( \lambda_0 \approx 10^{-15} \) (see the last row in each of these tables).

### 3.2 The Projection Method

We now solve (3.1) asymptotically for \( \epsilon \to 0 \). Since \( d > 0 \), the composite matched asymptotic expansion for the solution to (3.1) is given in terms of an undetermined constant \( C_0 \) by

\[
\tilde{u}^\epsilon (x) = C_0 + (\alpha - C_0) e^{-\xi_\epsilon \epsilon^{-1}(\alpha + x)} + (\beta - C_0) e^{-\xi_\epsilon \epsilon^{-1}(b - x)}. \tag{3.13}
\]

Here \( \xi_\epsilon \equiv a^{2m+1} p(-a) \) and \( \xi_\beta = b^{2m+1} p(b) \). We now describe the projection method, motivated by the work of [6], to determine \( C_0 \). Defining \( v = u - \tilde{u}^\epsilon \), and assuming that \( v \ll \tilde{u}^\epsilon \), we find from (3.1) that \( v \) satisfies the approximate equation

\[
L_\epsilon v = -L_\epsilon \tilde{u}^\epsilon - g(x) e^{-\epsilon^{-1}d} \tilde{u}^\epsilon, \tag{3.14a}
\]

\[
v(-a) = \alpha - \tilde{u}^\epsilon (-a), \quad v(b) = \beta - \tilde{u}^\epsilon (b). \tag{3.14b}
\]

We then expand \( v \) in terms of the eigenfunctions \( \phi_j \) of (3.2) to obtain

\[
v = \sum_{j=0}^\infty A_j \phi_j, \quad A_j = \epsilon \omega v \phi_j a^b \bigg| \phi_j, L_\epsilon \phi_j \bigg| a^b - (\phi_j, L_\epsilon \tilde{u}^\epsilon) a^b - \epsilon e^{-\epsilon^{-1}d} (\phi_j, g \tilde{u}^\epsilon) a^b. \tag{3.15}
\]
Since $\lambda_0 \to 0$ as $\epsilon \to 0$, a necessary condition for (3.14) to have a solution in this limit is that $A_0 \to 0$ as $\epsilon \to 0$. The condition $A_0 = 0$, which eliminates the largest component in $v$, provides the following equation for $C_0$:

$$-\epsilon^d \epsilon^{-1/d} \left( \phi_0, g \tilde{u}^\epsilon \right)_w = \left( \phi_0, L_\epsilon \tilde{u}^\epsilon \right)_w - \epsilon w \phi_0^\epsilon \left|_{-\alpha} \right. .$$

(3.16)

For $\epsilon \to 0$, we now evaluate the terms in (3.16). Using Laplace’s method we can calculate the asymptotic expansion of $(\phi_0, g \tilde{u}^\epsilon)_w$ in terms of the behavior of $w$, $\phi_0$ and $\tilde{u}^\epsilon$ near $x = 0$. This method yields

$$\left( \phi_0, g \tilde{u}^\epsilon \right)_w \sim C_0 \left( \phi_0, g \right)_w \sim M_0 C_0 \epsilon^{1/(2m+2)} g_\epsilon , \quad g_\epsilon \sim \sum_{j=0}^{\infty} \epsilon^{j/(m+1)} B_j ,$$

(3.17a)

for some coefficients $B_j$ independent of $\epsilon$. The first two coefficients are

$$B_0 = \frac{r^{1/(2m+2)}}{(m+1) g(0) \Gamma \left( \frac{1}{2m+2} \right)},$$

$$B_1 = \frac{3 r^{3/(2m+2)}}{(2m+2)^3} \left[ \frac{(2m+2) g''(0)}{3} - r \left( \frac{2 p'(0) g'(0)}{2m+3} + \frac{p''(0) g(0)}{2m+4} \right) \right] \Gamma \left( \frac{3}{2m+2} \right) \frac{(2m+5)}{(2m+3)^3} \left( \frac{2}{2m+2} \right),$$

(3.17b)

where $r \equiv (2m+1)/p(0)$. Next, to evaluate $(\phi_0, L_\epsilon \tilde{u}^\epsilon)_w$ we first integrate by parts to derive the identity

$$\left( \phi_0, L_\epsilon \tilde{u}^\epsilon \right)_w = -\epsilon \left( \tilde{u}_L^\epsilon + \tilde{u}_R^\epsilon \right) \phi_0^\epsilon \left|_{-\alpha} \right. + \lambda_0 \left( \phi_0, \tilde{u}_L^\epsilon \right)_w + \lambda_0 \left( \phi_0, \tilde{u}_R^\epsilon \right)_w ,$$

(3.18a)

where $\tilde{u}_L^\epsilon = (\alpha - C_0) e^{-\epsilon_d \epsilon^{-1}(a-x)}$ and $\tilde{u}_R^\epsilon = (\beta - C_0) e^{-\epsilon_d \epsilon^{-1}(b-x)}$. Since $\lambda_0$ is exponentially small, (3.18a) reduces to

$$\left( \phi_0, L_\epsilon \tilde{u}^\epsilon \right)_w \sim \epsilon w(-a) [\alpha - C_0] \phi_0^\epsilon (-a) - \epsilon w(b) [\beta - C_0] \phi_0^\epsilon (b) .$$

(3.18b)

From (3.14b) and (3.13) we note that $v(-a)$ and $v(b)$ are both exponentially small. Thus, the second term on the right side of (3.16) can be neglected in comparison with $(\phi_0, L_\epsilon \tilde{u}^\epsilon)_w$. Next, we substitute (3.17a) and (3.18b) in (3.16) to obtain the following equation for $C_0$

$$-\epsilon^{d+1/(2m+2)} C_0 g_\epsilon \epsilon^{-1/d} \sim b^{2m+1} p(b) (\beta - C_0) \gamma_\beta e^{-\epsilon_d \epsilon^{-1} \omega_b} + a^{2m+1} p(-a) (\alpha - C_0) \gamma_\alpha e^{-\epsilon_d \epsilon^{-1} \omega_\alpha} .$$

(3.19)

Finally, $\tilde{u}^\epsilon$ is given in terms of $C_0$ by (3.13). We now consider some special cases of (3.19).

**Case 1:** Assume that $d > \max(\omega_\alpha, \omega_b)$. Then, from (3.19), $C_0$ satisfies

$$C_0 \sim \left( \frac{a^{2m+1} p(-a) \gamma_\alpha e^{-\epsilon_d \epsilon^{-1} (\omega_\alpha - \omega_\alpha)} + \beta}{b^{2m+1} p(b) \gamma_\beta} \right) \left( \frac{a^{2m+1} p(-a) \gamma_\alpha e^{-\epsilon_d \epsilon^{-1} (\omega_\alpha - \omega_\alpha)} + 1}{b^{2m+1} p(b) \gamma_\beta} \right)^{-1} .$$

(3.20)

When $\omega_b > \omega_\alpha$, then $C_0 \sim \alpha$ and there is no boundary layer at $x = -a$. When $\omega_\alpha < \omega_b$, then $C_0 \sim \beta$ and there is no boundary layer at $x = b$. For $|\omega_b - \omega_\alpha| = O(\epsilon)$, (3.20) provides a uniform
transition between these two limiting behaviors for \( C_0 \). For \( m = 0 \), a leading order calculation for this transition behavior, in which \( \gamma_a \sim 1 \) and \( \gamma_b \sim 1 \), was derived in [29] using the variational method of [9].

**Case 2:** Assume that \(|d - \omega_b| = O(\epsilon)\), \( \omega_{\omega_a} > \omega_b \) and \( \nu = -1/(2m + 2) \). In this case boundary layers exist at both endpoints and

\[
C_0 \sim \beta \left( 1 - \frac{g_e b^{-2m-1}}{p(b) \gamma_b} e^{\epsilon^{-1}(\omega_b-d)} \right)^{-1}. \tag{3.21}
\]

Setting \( m = 0 \) and replacing \( g_e \) and \( \gamma_b \) by their leading terms \( g_e \sim \left[ 2\pi / p(0) \right]^{1/2} g(0) \) and \( \gamma_b \sim 1 \) as \( \epsilon \to 0 \), we find that (3.21) agrees with the result in [34].

**Case 3:** Assume that \(|d - \omega_b| = O(\epsilon)\), \( \omega_{\omega_a} = \omega_b \) and \( \nu = -1/(2m + 2) \). Then, boundary layers exist at both endpoints and

\[
C_0 \sim (\beta b^{2m+1} p(b) \gamma_b + a \omega_{\omega_a} + p(-a) \gamma_a) \left( b^{2m+1} p(b) \gamma_b + a \omega_{\omega_a} + p(-a) \gamma_a - g_e e^{\epsilon^{-1}(\omega_b-d)} \right)^{-1}. \tag{3.22}
\]

If, in addition, \( p(x) \) is an even function, then \( a = b, \gamma_a = \gamma_b \) and so

\[
C_0 \sim (\beta + a) \left( 2 - \frac{g_e b^{-2m-1}}{p(b) \gamma_b} e^{\epsilon^{-1}(\omega_b-d)} \right)^{-1}. \tag{3.23}
\]

### 3.3 Examples and Computations

We now give some examples of the theory. The asymptotic results are confirmed by direct numerical computations on (3.1).

**Example 3.1:** For \( b > 0 \) and \( \mu \geq 0 \), consider

\[
\epsilon u'' - \frac{2\pi x^3}{(1 + \mu x^2)} u' = -s e^{-1/4} (1 + x^2) e^{-\epsilon^{-1}d} \quad -1 \leq x \leq b,
\]

\[
u(-1) = 1, \quad u(b) = 2. \tag{3.24}
\]

Thus \( p(x) = 2\pi(1 + \mu x^2)^{-1} \), \( a = 1 \), \( m = 1 \) and \( \nu = -1/4 \). We first let \( s = 0 \) in (3.24). Then, from (3.8), (3.10b) and (3.20) we obtain that

\[
C_0 \approx \frac{\rho e^{\epsilon^{-1}(\omega_b - \omega_a)} + 2}{\rho e^{\epsilon^{-1}(\omega_b - \omega_a)} + 1}, \quad \rho \approx \frac{(1 + \mu b^2)}{2\pi(1 + \mu)} \left[ 1 - \frac{\epsilon}{2\pi} \left( \mu + 3 - \frac{(3 + \mu b^2)}{b^4} \right) \right], \tag{3.25}
\]

where \( \omega_b = \pi \mu^{-1} \log(1 + \mu b^2) \). To compare with (3.25), solutions to (3.24) were computed using the method of [21]. For the case \( b = 1 \) and \( \mu = 0 \), in Table 2a we compare the numerical value of \( C_0 \), defined by \( C_0 = u(0) \), with the exact value \( C_0 = 1.5 \). Therefore, the third column in this table shows the loss of precision in the computations as \( \epsilon \) is decreased. From (3.12c), we note that \( \lambda_0 \approx -3.4 \times 10^{-15} \) when \( \epsilon = 1/23 \) and \( \lambda_0 \approx -1.5 \times 10^{-16} \) when \( \epsilon = 1/25 \). Even in the presence of such an extreme ill-conditioning, the numerical method is able to accurately capture the first few significant digits of \( C_0 \). For \( \mu = 0 \) and for \( \mu = 1 \), in Fig. 1 we show the loss of accuracy of the numerically computed value of \( C_0 \) as \( \epsilon \to 0 \). The eigenvalue \( \lambda_0 \) is also plotted in this figure. For
the case \( \mu = 1 \) with \( \epsilon = 1/36 \), in Table 2b we show the very close agreement between the numerical value of \( C_0 \) and the asymptotic value (3.25) as \( b \) is varied in an \( O(\epsilon) \) neighborhood of \( b = 1 \).

Now let \( s = 1 \) in (3.24). Then, from (3.10b), (3.17), (3.21) and (3.23) we have that

\[
C_0 \sim 2 \left[ 1 - \frac{(1 + \mu b^2)}{2\pi b^3 \gamma_b} \left( B_0 + \epsilon^{1/2} B_1 \right) e^{\epsilon^{-1} (\omega_b - d)} \right]^{-1}, \quad b < 1, \quad |d - \omega_b| = O(\epsilon), \quad (3.26a)
\]

\[
C_0 \sim 3 \left[ 2 - \frac{(1 + \mu)}{2\pi \gamma_1} \left( B_0 + \epsilon^{1/2} B_1 \right) e^{\epsilon^{-1} (\omega_1 - d)} \right]^{-1}, \quad b = 1, \quad |d - \omega_1| = O(\epsilon). \quad (3.26b)
\]

Here \( B_0, B_1 \) and \( \gamma_b \) are given by

\[
B_0 = \frac{1}{2} \left( \frac{2}{\pi} \right)^{1/4} \Gamma(1/4), \quad B_1 = \frac{1}{4} \left( \frac{2}{\pi} \right)^{3/4} (2 + \mu) \Gamma(3/4),
\]

\[
\gamma_b \sim 1 - \frac{\epsilon (1 + \mu b^2)}{2\pi b^4} \left( \frac{3 + \mu b^2}{1 + \mu b^2} \right). \quad (3.27)
\]

For the case \( s = 1, b = 1, d = \pi/2 \) and \( \mu = 0 \), in Table 2c we compare the numerical value of \( C_0 \) at various \( \epsilon \) with the asymptotic result (3.26b). Then, setting \( \epsilon = 1/20 \), in Table 2d we compare \( C_0 \) values as \( d \) is varied in a narrow range near \( d = \omega_1 = \pi/2 \). In Fig. 2 we plot \( C_0 \) versus \( z = \epsilon^{-1} (d - \pi/2) \) and we also show the numerical solutions to (3.24) for some selected values of \( z \). Now setting \( b = 3/4, \mu = 0 \) and \( \epsilon = 1/36 \), in Table 2e we compare the numerical \( C_0 \) with the asymptotic result (3.26a) as \( d \) is varied near \( d = \omega_b \). From Tables 2b-e we observe that (3.26a,b) provides a very close determination of the corresponding numerical result. We remark, however, that the agreement between the asymptotic and numerical values of \( C_0 \) is not nearly as close if only the leading term \( \gamma_b \sim 1 \) is used in (3.26). As seen from Table 2d, we emphasize that the numerical method of [21] gives an accurate result for \( C_0 \), even when the numerical value of the perturbing term in (3.24) is roughly \( e^{-\epsilon^{-1} d} \approx e^{-\epsilon^{-1} \pi/2} = 2.3 \times 10^{-14} \).

We now interpret the apparent singularity in the plot of \( C_0 = C_0(d) \) shown in Fig. 2. This singular-type behavior is rather analogous to a similar behavior that was found in [35] to occur for the related problem \( L_\epsilon u = \eta \lambda_0 u \) on \(-1 < x < 1\). Here \( \eta \) is a constant, \( L_\epsilon \) is given in (3.1), \( p(x) \) is even, \( \lambda_0 \) is the exponentially small eigenvalue of (3.2) and \( u(-1) = \alpha, \ u(1) = \beta \). Clearly this related problem has no solution if \( \eta = 1 \). The outer solution for this problem is easily shown to be \( u \sim C_0 = (1 - \eta)^{-1} (\beta + \alpha)/2 \). Thus as \( \eta \to 1 \), \( C_0 \) becomes unbounded and the plot of \( C_0 \) versus \( \eta \) is qualitatively rather similar to that shown in Fig. 2. This analogy is not entirely satisfactory in that we believe the true curve \( C_0 = C_0(d) \) for example 3.1 is in fact bounded on closed intervals containing \( d = \pi/2 \) for each fixed \( \epsilon \). However, as \( \epsilon \to 0 \), we expect that \( C_0 = C_0(d) \) can become very large at a certain value of \( d \).

**Example 3.2:** For \( b > 0 \), consider

\[
\epsilon u'' - x (x^2 - \frac{7}{4} x + 1) u' = -\epsilon^{-1/2} (1 + x^2) e^{-\epsilon^{-1} d} u, \quad -1 \leq x \leq b,
\]

\[
u(-1) = 1, \quad u(b) = 2. \quad (3.28)
\]

Thus \( m = 0, \nu = -1/2 \) and \( p(x) \) is not even. From (3.10b) we find that \( \omega_1 > \omega_b \) when \( b < 2 \) and that \( \omega_1 = \omega_2 = 4/3 \). First suppose that \( b = 2 \) and \( |d - 4/3| = O(\epsilon) \). Then, from (3.8), (3.17) and
we calculate that
\[
C_0 \sim (39 - 24\epsilon) \left[ 27 - 16\epsilon - 4(2\pi)^{1/2} \left( 1 + \frac{269\epsilon}{96} \right) e^{-\epsilon^{-1}(d-4/3)} \right]^{-1}, \quad b = 2, \quad |d - 4/3| = O(\epsilon). \tag{3.29}
\]
Thus, when \( d > 4/3, \ C_0 \sim 13/9 - 8\epsilon/243. \) Fixing \( d = 4/3, \) in Table 3a we compare, at decreasing values of \( \epsilon, \) the asymptotic result (3.29) with the numerical result for \( C_0 \) computed from (3.28). The numerical value of \( C_0 \) is again defined by \( C_0 = u(0). \)

Next, when \( b < 2 \) and \( |d - \omega_b| = O(\epsilon), \tag{3.8}, (3.17) \) and (3.21) provide
\[
C_0 \sim 2\rho_\epsilon \left[ \rho_\epsilon - (2\pi)^{1/2} \left( 1 + \frac{269\epsilon}{96} \right) e^{-\epsilon^{-1}(d-\omega_b)} \right]^{-1}, \quad b < 2, \quad |d - \omega_b| = O(\epsilon). \tag{3.30}
\]
Here \( \rho_\epsilon \) and \( \omega_b \) are given by
\[
\rho_\epsilon \sim b \left( b^2 - \frac{7}{4} b + 1 \right) - \epsilon \left( b^2 - \frac{7}{4} b + 1 \right)^{-1} \left( 3b^2 - \frac{7}{2} b + 1 \right), \quad \omega_b = \frac{b^2}{4} \left( b^2 - \frac{7}{3} b + 2 \right). \tag{3.31}
\]
Fixing \( b = 1 \) and \( \epsilon = 1/60, \) in Table 3b we compare the asymptotic and numerical values of \( C_0 \) as \( d \) is varied in a narrow range near \( d = \omega_b. \) From Tables 3a, 3b we note that the agreement between the asymptotic and numerical results for \( C_0 \) is moderately good but is not as close as for the previous example.

**Example 3.3:** Bohé [3] showed that the viscous shock problem
\[
ex'' = -x(x')^2, \quad \alpha \leq u \leq \beta, \quad x(\alpha) = -1, \quad x(\beta) = b > 0, \tag{3.32}
\]
for \( x = x(u) \) can be mapped, using the inverse mapping \( u = u(x), \) to the following example of (3.1):
\[
\epsilon u'' - xu' = 0, \quad -1 \leq x \leq b, \quad u(-1) = \alpha, \quad u(b) = \beta. \tag{3.33}
\]
The qualitative behavior of the shock layer location for (3.32) in response to small changes in \( b \) for \( b \) near 1 was discussed in [3]. To quantify this behavior, we note that the outer solution \( u \sim C_0 \) for (3.33), where
\[
C_0 \sim \frac{\alpha \rho_\epsilon \epsilon^{-1} (b^2 - 1)^2 + \beta}{\rho_\epsilon \epsilon^{-1} (b^2 - 1)^2 + 1}, \quad \rho_\epsilon \sim b^{-1} \left[ 1 - \epsilon (1 - b^{-2}) \right]^2, \tag{3.34}
\]
determines the shock layer location for (3.32). Thus, as \( b \) increases (decreases) past one, the shock layer tends to the left (right) endpoint according to (3.34). For several values of \( z = \epsilon^{-1} (b - 1) \) and for \( \alpha = -1, \beta = 1 \) and \( \epsilon = 1/60, \) in Fig. 3 we plot the shock layer solution to (3.32), which is computed numerically using (3.33).

4. **The Hermite Operator**

We now consider the boundary value problem
\[
L_\epsilon u \equiv \epsilon u'' - xu' + Nu = -g(x) \epsilon^\epsilon e^{-\epsilon^{-1}d} u, \quad -a \leq x \leq b, \quad u(-a) = \alpha, \quad u(b) = \beta. \tag{4.1}
\]
Here \( a > 0, b > 0, d > 0, N \) is a nonnegative integer, \( g(x) \) is a smooth function and \( \epsilon \ll 1 \).

### 4.1 Spectral Properties

As in §3.1, we must first analyze the spectral properties associated with the eigenvalue problem (3.2) where \( L_\epsilon \) is now defined in (4.1). For this problem, the eigenvalues \( \lambda_j \) for \( j \geq 0 \) are real with \( \lambda_j \to N - j \) as \( \epsilon \to 0 \). The corresponding normalized eigenfunctions \( \phi_j \) satisfy \( \langle \phi_j, \phi_k \rangle_w = \delta_{jk} \), where the weight function is given by \( w = w(x, \epsilon) = \epsilon^{-\epsilon^{-1}x^2/2} \). The eigenvalue \( \lambda_N \) is exponentially small as \( \epsilon \to 0 \) and we now estimate it precisely.

Let \( v \) satisfy \( L_\epsilon v = 0 \). Then, upon integrating by parts, we derive

\[
\lambda_N(v, \phi_N)_w = \epsilon w(b)v(b)\phi_N'(b) - \epsilon w(-a)v(-a)\phi_N'(-a) .
\]

Choose \( v = He_N(\epsilon^{-1/2}x) \), where \( He_N(z) \) is the Hermite polynomial of degree \( N \), and assume that \( \lambda_N = o(\epsilon^k) \) for any \( k > 0 \). Then, since \( L_\epsilon v = 0 \), we have that \( \phi_N \sim M_N He_N(\epsilon^{-1/2}x) \) away from \( O(\epsilon) \) regions near the endpoints at \( x = -a \) and \( x = b \). Here \( M_N \) is a normalization constant. However, since this approximate form for \( \phi_N \) does not satisfy the homogeneous boundary conditions in (3.2), we cannot use it to calculate \( \phi_N'(-a) \) and \( \phi_N'(b) \) in (4.2). Instead, these quantities are evaluated after constructing boundary layer profiles for \( \phi_N \) near each endpoint.

In the region near \( x = -a \) define \( y = \epsilon^{-1}(a + x) \) and \( \Phi_L(y) = \phi_N[-a + \epsilon y] \). Then, by expanding

\[
\Phi_L(y) = \epsilon^{-N/2}\Phi_L^0(y) + \epsilon^{-N/2}\Phi_L^1(y) + \cdots ,
\]

we obtain, after equating powers of \( \epsilon \) in \( L_\epsilon \phi_N = 0 \), the following boundary layer equations:

\[
\begin{align*}
\Phi_L'' + a\Phi_L' &= 0 , & \Phi_L^0(0) &= 0 , & y > 0 , \\
\Phi_L'' + a\Phi_L' &= y\Phi_L' - N\Phi_L , & \Phi_L^1(0) &= 0 , & y > 0 .
\end{align*}
\]

The solutions to (4.4) will match as \( y \to \infty \) to the outer solution \( \phi_N = M_N He_N(\epsilon^{-1/2}x) \) provided that

\[
\Phi_L^0(y) \sim M_N(-a)^N , \quad \text{as} \quad y \to \infty ; \quad \Phi_L^1(y) \sim M_N(-a)^{N-2}[h_N -aNy] , \quad \text{as} \quad y \to \infty .
\]

In (4.5), \( h_N \) is defined by

\[
h_N = \lim_{z \to \infty} z^{2-N} \left[ He_N(z) - z^N \right] , \quad \text{so that} \quad h_N = -N(N-1)/2 .
\]

From (4.4) and (4.5) we calculate that

\[
\begin{align*}
\Phi_L^0(y) &= M_N(-a)^N \left( 1 - e^{-ay} \right) , \\
\Phi_L^1(y) &= M_N(-a)^{N-2} \left[ h_N - aNy - \left( \frac{1}{2}a^2y^2 + a(1 + N)y + h_N \right)e^{-ay} \right] .
\end{align*}
\]

Finally, using (4.3), (4.7) and \( \phi_N'(-a) = \epsilon^{-1}\Phi_L^1(0) \) we obtain the two term expansion

\[
\phi_N'(-a) = -M_N(-a)^{N+1} \epsilon^{-1-N/2} \left[ 1 - \frac{1}{2N + 1 - h_N} \frac{\xi}{a^2} + \cdots \right] .
\]
A similar calculation provides the following two term expansion for \( \phi_N'(b) \):

\[
\phi_N'(b) = - M_N b^{N+1} \epsilon^{-1-N/2} \left[ 1 - (2N + 1 - h_N) \frac{\epsilon}{b^2} + \cdots \right].
\]  \hspace{1cm} (4.8b)

Further terms in the expansions of \( \phi_N'(-a) \) and \( \phi_N'(b) \) can, in principle, be obtained by calculating higher order boundary layer corrections for \( \phi_N \). These expansions have the form

\[
\phi_N'(-a) \sim - M_N(-a)^{N+1} \epsilon^{-1-N/2} \left( 1 + \sum_{j=1}^{\infty} \epsilon^j \phi_{Lj} \right), \quad \phi_N'(b) \sim - M_N b^{N+1} \epsilon^{-1-N/2} \left( 1 + \sum_{j=1}^{\infty} \epsilon^j \phi_{Rj} \right),
\]  \hspace{1cm} (4.9)

for some coefficients \( \phi_{Lj} \) and \( \phi_{Rj} \) independent of \( \epsilon \). From (4.8) we have \( \phi_{L1} = - a^{-2}(2N + 1 - h_N) \) and \( \phi_{R1} = - b^{-2}(2N + 1 - h_N) \).

Next, for \( \epsilon \to 0 \) we estimate the left side of (4.2) to obtain

\[
(v, \phi_N)_\omega \sim M_N \epsilon^{1/2} \int_{-\infty}^{\infty} e^{-z^2/2} [\text{He}_N(z)]^2 \, dz = M_N \epsilon^{1/2} (2\pi)^{1/2} N!.
\]  \hspace{1cm} (4.10)

Finally, substituting (4.9), (4.10), \( v = \text{He}_N(\epsilon^{-1/2} x) \) and \( w = e^{-\epsilon^{-1} x^2/2} \) in (4.2), we obtain the following estimate for the exponentially small eigenvalue \( \lambda_N \):

\[
\lambda_N \sim - \frac{\epsilon^{-1-N/2}}{(2\pi)^{1/2} N!} \left[ b^{2N+1} \gamma_b(\epsilon) e^{-\epsilon^{-1} b^2/2} + a^{2N+1} \gamma_a(\epsilon) e^{-\epsilon^{-1} a^2/2} \right].
\]  \hspace{1cm} (4.11)

In (4.11), \( \gamma_a(\epsilon) \) and \( \gamma_b(\epsilon) \) are defined by

\[
gamma_a(\epsilon) = \frac{\text{He}_N(-\epsilon^{-1/2} a)}{\epsilon^{-N/2}(-a)^N} \left( 1 + \sum_{j=1}^{\infty} \phi_{Lj} \epsilon^j \right), \quad \gamma_b(\epsilon) = \frac{\text{He}_N(\epsilon^{-1/2} b)}{\epsilon^{-N/2} b^N} \left( 1 + \sum_{j=1}^{\infty} \phi_{Rj} \epsilon^j \right).
\]  \hspace{1cm} (4.12a)

Clearly \( \gamma_a \) and \( \gamma_b \) can be written as series in powers of \( \epsilon \). In particular, from (4.6), (4.8) and (4.9) we obtain the explicit two term expansions

\[
\gamma_a(\epsilon) \sim 1 - (N^2 + N + 1) \frac{\epsilon}{a^2}, \quad \gamma_b(\epsilon) \sim 1 - (N^2 + N + 1) \frac{\epsilon}{b^2}.
\]  \hspace{1cm} (4.12b)

Thus, in the special case when \( a = b \), a two term expansion for \( \lambda_N \), obtained from (4.11), is

\[
\lambda_N \sim - \frac{2 \epsilon^{-N-1/2}}{(2\pi)^{1/2} N!} b^{2N+1} \left[ 1 - (N^2 + N + 1) \frac{\epsilon}{b^2} \right] e^{-\epsilon^{-1} b^2/2}.
\]  \hspace{1cm} (4.13)

The leading order result for \( \lambda_N \), obtained by replacing the pre-exponential factor in (4.13) by its leading order behavior, was proven in [6].

The eigenvalue solver of [21] is then used to compute \( \lambda_N \) numerically when \( a = b = 1 \). In Tables 4a and 4b we compare the numerical results for \( \lambda_N \) against the corresponding asymptotic result (4.13) (with \( b = 1 \)) for the case \( N = 1 \) and \( N = 2 \), respectively. In these tables we also give the asymptotic result for \( \lambda_N \) that occurs from truncating the pre-exponential factor in (4.13) to one term. From these tables we note that the two term result provides a significantly better
determination of $\lambda_N$ than does the one term result. As seen from the last row of Table 4a, we note that the numerical result for $\lambda_N$ becomes inaccurate only when $\lambda_0 \approx 10^{-14}$.

4.2 The Projection Method
We now solve (4.1) in the limit $\epsilon \to 0$. The composite matched asymptotic expansion for the solution to (4.1) is given in terms of an undetermined constant $C_0$ by

$$
\tilde{u}(x) = \left[ \alpha - C_0(-a)^N \right] e^{-\alpha \epsilon^{-1}(a+x)} + \left[ \beta - C_0b^N \right] e^{-\beta \epsilon^{-1}(b-x)} + C_0 \epsilon^{N/2} \text{He}_N \left( \epsilon^{-1/2} x \right). \quad (4.14)
$$

Defining $v$ by $v = u - \tilde{u}(x)$, and assuming that $v \ll \tilde{u}(x)$, we obtain from (4.1) that $v$ satisfies the approximate equation

$$
L_\epsilon v = -L_\epsilon \tilde{u}(x) - g(x) \epsilon^r \epsilon^{-1/2} \tilde{u}(x), \quad v(-a) \sim C_0 \left[ (-a)^N - \epsilon^{N/2} \text{He}_N(-\epsilon^{-1/2} a) \right], \quad v(b) \sim C_0 \left[ b^N - \epsilon^{N/2} \text{He}_N(\epsilon^{-1/2} b) \right]. \quad (4.15a)
$$

Next, we expand $v$ in terms of the eigenfunctions of $L_\epsilon \phi = \lambda \phi$ to obtain the form (3.15). Since $\lambda_N \to 0$ as $\epsilon \to 0$, a necessary condition for the solvability of (4.15) is that $A_N \to 0$ as $\epsilon \to 0$, where $A_N$ is given in (3.15). Setting $A_N = 0$ we find that $C_0$ satisfies

$$
-\epsilon^r \epsilon^{-1/2} \left( \phi_N, g \tilde{u}(x) \right)_w = \left( \phi_N, L_\epsilon \tilde{u}(x) \right)_w - \epsilon w \phi'_N \bigg|_{-a}. \quad (4.16)
$$

For $\epsilon \to 0$, we now evaluate the terms in (4.16). With $\phi_N \sim N \text{He}_N(\epsilon^{-1/2} x)$, the inner product $(\phi_N, g \tilde{u}(x))_w$ is evaluated by Laplace’s method to obtain

$$
(\phi_N, g \tilde{u}(x))_w \sim N C_0 \epsilon^{(N+1)/2} (2\pi)^{1/2} g \epsilon N!, \quad \text{where} \quad g \epsilon \sim g(0) + \sum_{j=1}^{\infty} \epsilon^j \theta_j g^{(2j)}(0). \quad (4.17a)
$$

Here $\theta_j$ is defined by

$$
\theta_j = \frac{(2\pi)^{-1/2}}{N!(2j)!} \int_{-\infty}^{\infty} z^{2j} \text{He}_N(z)^2 e^{-z^2/2} dz, \quad j = 1, 2, \ldots \quad (4.17b)
$$

In particular, $\theta_1 = N + 1/2$. To evaluate $(\phi_N, L_\epsilon \tilde{u}(x))_w$, we proceed in a similar way as in (3.18a, b) to obtain

$$
(\phi_N, L_\epsilon \tilde{u}(x))_w \sim \epsilon w(-a) \left[ \alpha - C_0(-a)^N \right] \phi'_N(-a) - \epsilon w(b) \left[ \beta - C_0b^N \right] \phi'_N(b). \quad (4.18)
$$

Then, substituting (4.9), (4.15b), (4.17a), and (4.18) in (4.16), we obtain the following explicit equation for $C_0$:

$$
-\epsilon^{r+N+1/2} \epsilon^{-1/2} C_0 \frac{1}{2} \left[ \frac{\alpha \epsilon^{-N/2}}{\text{He}_N(-\epsilon^{-1/2} a)} - C_0 \right] a^{2N+1} g \epsilon N! e^{-\epsilon^{-1} a^2/2} + \left[ \frac{\beta \epsilon^{-N/2}}{\text{He}_N(\epsilon^{-1/2} b)} - C_0 \right] b^{2N+1} g \epsilon N! e^{-\epsilon^{-1} b^2/2}. \quad (4.19)
$$
Here $\gamma_a(\epsilon)$ and $\gamma_b(\epsilon)$ are defined in (4.12a) and $g_b$ is defined in (4.17a). Finally, $\overline{u}^\epsilon$ is given in terms of $C_0$ by (4.14).

We note that in deriving (4.19) it was sufficient to retain only the leading order boundary layer correction terms in $\overline{u}^\epsilon$ as given in (4.14). However, at this stage we could improve our approximation to $u$ by adding further boundary layer correction terms to the right side of (4.14). These additional terms would be determined in by $C_0$. We now consider some special cases of (4.19).

**Case 1:** Assume that $2d > \max(a^2,b^2)$. Then, from (4.19), $C_0$ satisfies

$$C_0 \sim \epsilon^{-N/2} \left( \frac{\beta \gamma_1 + 1}{\text{He}_N(\epsilon^{-1/2}b)} e^{-1)(a^2-b^2)/2} + \frac{\alpha \gamma_1 + 1}{\text{He}_N(-\epsilon^{-1/2}a)} \left( b^2 + 1 \gamma_b e^{-1(a^2-b^2)/2} + a^2 \gamma_a \right) \right)^{-1}. \quad (4.20)$$

Thus, when $b > a$ ($b < a$) we have $C_0 \sim \alpha(-a)^{-N} (C_0 \sim \beta b^{-N})$ and so $u$ has no boundary layer at $x = -a$ ($x = b$). For $|b - a| = O(\epsilon)$, (4.20) provides a uniform transition between these two limiting behaviors for $C_0$. Using $\text{He}_N(z) = (-1)^N \text{He}_N(-z)$, we have that $\gamma_a = \gamma_b$ when $a = b$. Thus, when $a = b$, (4.20) reduces to

$$C_0 \sim \frac{\epsilon^{-N/2}}{2\text{He}_N(\epsilon^{-1/2}b)} \left[ \beta + (1-N) \alpha \right] \sim \frac{b^{-N}}{2} \left[ \beta + (1-N) \alpha \right] \left[ 1 + \frac{\epsilon}{2b^2}N(N-1) \right]. \quad (4.21)$$

**Case 2:** Assume that $|d - b^2/2| = O(\epsilon)$, $a > b$ and $\nu = -N - 1/2$. Then, $u$ has boundary layers near each endpoint and $C_0$ satisfies

$$C_0 \sim \frac{\beta \epsilon^{-N/2}}{\text{He}_N(\epsilon^{-1/2}b)} \left[ 1 - \frac{(2\pi)^{1/2}g_0 \gamma_1}{b^2 + 1 \gamma_b} e^{-1(d-b^2/2)} \right]^{-1}. \quad (4.22)$$

Using the leading behaviors $g_b \sim g(0)$, $\text{He}_N(\epsilon^{-1/2}b) \sim \epsilon^{-N/2}b^N$ and $\gamma_b \sim 1$, we find that (4.22) reduces to the leading order result of [34]. A result similar to (4.22) holds for the case $d = a^2/2$, $b > a$ and $\nu = -N - 1/2$.

**Case 3:** Assume that $|d - b^2/2| = O(\epsilon)$, $b = a$ and $\nu = -N - 1/2$. Then, $u$ has boundary layers at both endpoints, and (4.19) becomes

$$C_0 \sim \frac{\epsilon^{-N/2}}{\text{He}_N(\epsilon^{-1/2}b)} \left[ \beta + (1-N) \alpha \right] \left[ 2 - \frac{(2\pi)^{1/2}g_0 \gamma_1}{b^2 + 1 \gamma_b} e^{-1(d-b^2/2)} \right]^{-1}. \quad (4.23)$$

**4.3 Examples and Computations**

We now consider some specific examples and we verify the asymptotic results with corresponding numerical results.

**Example 4.1:** For $b > 0$, consider

$$\epsilon u'' - xu' + u = s(2\pi)^{-1/2}\epsilon^{-3/2}e^{-\epsilon^{-1}d}u, \quad -1 \leq x \leq b; \quad u(-1) = 1, \quad u(b) = 2. \quad (4.24)$$

This problem was studied in [14] when $b = 1$ and $s = 0$ and it was considered in [29] and [30] for arbitrary $b$ and $s = 1$. From (4.14), the composite expansion for the solution to (4.24) is

$$\overline{u}^\epsilon(x) = (1 + C_0) e^{-\epsilon^{-1}(1+x)} + (2 - C_0b) e^{-b \epsilon^{-1}(b-x)} + C_0 x, \quad (4.25)$$
where \( C_0 \) satisfies (4.19). We first consider the case \( s = 0 \). Then from (4.12b) and (4.20), we find that \( C_0 \) satisfies
\[
C_0 \sim \frac{2b^2 \rho_0(b) e^{\epsilon^{-1}(1-b^2)/2} - 1}{b^3 \rho_0(b) e^{\epsilon^{-1}(1-b^2)/2} + 1}, \quad \rho_0(b) \sim \frac{1 - 3eb^{-2}}{1 - 3\epsilon}.
\] (4.26)

This result for \( C_0 \) is easily verified by expanding, for \( \epsilon \rightarrow 0 \), the exact solution to (4.24) (with \( s = 0 \)) given by
\[
u(x) = a_1 x + a_2 \left( e^{\epsilon^{-1}x^2/2} - e^{-1}x J_\epsilon(x) \right), \quad J_\epsilon(x) \equiv \int_{-1}^{x} e^{\epsilon^{-1}s^2/2} ds,
\] (4.27)
\[
a_2 = (2 + b) \left[ b e^{\epsilon^{-1}/2} + e^{\epsilon^{-1}b^2/2} - e^{-1}b J_\epsilon(b) \right]^{-1}, \quad a_1 = -1 + e^{\epsilon^{-1}/2} a_2.
\]

For \( \epsilon = 1/60 \) and \( s = 0 \), in Table 5a we compare the numerical value of \( C_0 \) with the asymptotic result (4.26) as \( b \) is varied in a narrow range near \( b = 1 \). The numerical value of \( C_0 \) is defined by \( C_0 = u'(0) \), where \( u \) is computed from (4.24) using the method of [21]. From (4.13), we calculate that \( \lambda_0 \approx -3.3 \times 10^{-11} \), when \( b = 1 \) and \( \epsilon = 1/60 \). Despite this ill-conditioning, the asymptotic and numerical results in Table 5a agree to several decimal places of accuracy. For three values of \( \epsilon \), in Fig. 4 we plot \( C_0 = C_0(b) \) obtained from the asymptotic result (4.26). In this figure we also show the numerical results for \( C_0 \) for the case \( \epsilon = 1/60 \).

Now let \( s = 1 \) and \( |d - b^2/2| = O(\epsilon) \) with \( b < 1 \). Then, since \( g_\epsilon = -(2 \pi)^{-1/2} \), (4.12b) and (4.22) provide
\[
C_0 \sim 2b^2 \left[ b^3 + \left( 1 - \frac{3\epsilon}{b^2} \right)^{-1} e^{-\epsilon^{-1}d} \right]^{-1}.
\] (4.28)

Thus, when \( d = b^2/2 \), we have \( C_0 \sim 2(b^3 + 1)^{-1} \left[ b^2 - 3\epsilon(b^3 + 1)^{-1} \right]^{-1} \), which agrees with the two term result in [30] that was obtained using an extension of the variational method. Finally, when \( b = 1 \) and \( d - 1/2 = O(\epsilon) \), we obtain from (4.23) that
\[
C_0 \sim 2 \left[ 1 - 3\epsilon \right]^{-1} e^{-\epsilon^{-1}(d - 1/2)}
\] (4.29)

Setting \( d = b^2/2 \) and \( s = 1 \), in Tables 5b and 5c we show the very close agreement between the asymptotic and numerical values for \( C_0 \) as a function of \( \epsilon \) for the cases \( b = 1 \) and \( b = 3/4 \), respectively. Setting \( b = 1 \) and \( \epsilon = 1/60 \), in Table 5d we compare \( C_0 \) values as \( d \) is varied in a narrow region near \( d = 1/2 \). A plot of \( C_0 = C_0(d) \) for \( \epsilon = 1/60 \) is shown in Fig. 5. In Fig. 5 we also plot the numerical solution to (4.24) at a few values of \( d \). In Table 5e we compare \( C_0 \) for various \( d \) for the case \( b = 3/4 \) and \( \epsilon = 1/100 \).

**Example 4.2:** Our next example is
\[
u'' - xu' + Nu = -s \frac{(2 \pi)^{-1/2} e^{-N-1/2}}{[(x+1)^2 + 1]} e^{-\epsilon^{-1}d} u, \quad -1 \leq x \leq b; \quad u(-1) = 1, \quad u(b) = 2,
\] (4.30)

with \( b > 0 \). The composite expansion is given in (4.14) with \( \alpha = 1, \beta = 2 \) and \( a = 1 \). For several parameter ranges we now give results for \( C_0 \) obtained from (4.19) – (4.23). When \( s = 0 \), then
\[
C_0 \sim \frac{2b^{N+1} \rho_0(b) e^{\epsilon^{-1}(1-b^2)/2} - (-1)^N \rho_0(1)}{b^{2N+1} \rho_0(b) e^{\epsilon^{-1}(1-b^2)/2} + \eta_0(1)}
\] (4.31)
where \( \rho_\epsilon(b) \sim 1 - e\epsilon^{-2}(N + 1)(N + 2)/2 \) and \( \eta_\epsilon(b) \sim 1 - e\epsilon^{-2}(N^2 + N + 1) \) as \( \epsilon \to 0 \). Setting \( s = 0 \) and \( \epsilon = 1/60 \), in Table 6a we show the close agreement between the asymptotic and numerical values of \( C_0 \) as \( b \) is varied near \( b = 1 \). Note that the numerical value of \( C_0 \) is now defined by \( C_0 = -\epsilon^{-1}u(0) \), where \( u(0) \) is obtained from the numerical solution to (4.30).

Now when \( s = 1 \), \(|d - b^2/2| = O(\epsilon) \) and \( b < 1 \),

\[
C_0 \sim 4b^{-N} \left( 1 - \frac{\epsilon(N - 1)}{2b^2} \right)^{-1} \left[ 2 - b^{-2N-1}\zeta_\epsilon(b)N! e^{-\epsilon^{-1}(d-b^2/2)} \right]^{-1}. \tag{4.32}
\]

Finally, when \( s = 1 \), \(|d - 1/2| = O(\epsilon) \) and \( b = 1 \),

\[
C_0 \sim 2 \left( 1 - \frac{\epsilon(N - 1)}{2} \right)^{-1} \left( 2 + (-1)^N \right) \left[ 4 - \zeta_\epsilon(1)N! e^{-\epsilon^{-1}(d-1/2)} \right]^{-1}. \tag{4.33}
\]

In (4.32) and (4.33), \( \zeta_\epsilon(b) \) is given by

\[
\zeta_\epsilon(b) \sim \frac{1 + \epsilon(N + 1/2)}{1 - \epsilon(N^2 + N + 1)b^{-2}}. \tag{4.34}
\]

Setting \( d = 1/2 \), \( b = 1 \) and \( s = 1 \), in Table 6b we show the favorable agreement between the asymptotic and numerical values for \( C_0 \) as \( \epsilon \) is decreased. Setting \( b = 1 \) and \( \epsilon = 1/60 \), in Table 6c we compare \( C_0 \) values as \( d \) is varied near \( d = 1/2 \). For the case \( b = 3/4 \) and \( \epsilon = 1/100 \), in Fig. 6 we compare the asymptotic and numerical results for \( C_0 = C_0(d) \) when \( d \) is near 9/32. Numerical values for \( C_0(d) \) are given in Table 6d. In Fig. 6 we also plot the numerical solution to (4.30) at a few values of \( d \). From this figure, we note that as \( d \) increases past 9/32, the boundary layer near the right endpoint disappears.

There are two main conclusions to be drawn from the examples considered in \( \S \)3 and \( \S \)4. Firstly, the asymptotic results given for \( C_0 \) typically provide a very close determination of the numerical value, even when considering the delicate transition behavior in \( C_0 \) as certain parameters, such as \( d \), cross through critical values. However, we emphasize that to obtain a close quantitative agreement between the asymptotic and numerical values for \( C_0 \) at a small but finite \( \epsilon \), it is crucial to obtain at least a two term expansion for the pre-exponential factors in the asymptotic formulas for \( C_0 \). These expansions, which were derived in \( \S \)3 and \( \S \)4, extend some previous leading order analytical results for \( C_0 \) that were obtained using the variational principle of [9]. Our second conclusion is that, provided that \( \lambda_0 \) is not exceedingly small, the numerical method of [21] provides accurate numerical solutions to these highly ill-conditioned problems. Moreover, the numerical method is able to determine the change in \( C_0 \) that occurs as a result of exponentially small changes in the coefficient of the differential operator. Thus, for certain ranges of \( \epsilon \), the numerical results can be reliably used to verify the corresponding asymptotic results.

5. Extensions of the Theory

In \( \S \)5.1 we examine some inhomogeneous problems and in \( \S \)5.2 we apply the results to a model equation arising from the shape from shading problem.

5.1 The Inhomogeneous Problem
Hermite Operator: We now consider a forced version of (4.1) given by

\[ L_\epsilon u = \epsilon u'' - xu' + Nu = g(x)e^{\nu e^{-\epsilon^{-1}d}}, \quad -a \leq x \leq b, \]
\[ u(-a) = \alpha, \quad u(b) = \beta. \tag{5.1} \]

Here \( d > 0 \), \( N \) is a non-negative integer and \( g(x) \) is a smooth function. Since \( d > 0 \), the composite expansion \( \tilde{u}^\epsilon \) for the solution to (5.1) is given by (4.14) where \( C_0 \) is to be found. To determine \( C_0 \) we use the projection method. Substituting \( v = u - \tilde{u}^\epsilon \) in (5.1) and expanding \( v \) in terms of the eigenfunctions \( \phi_j \) of \( L_\epsilon \phi = \lambda \phi \), we obtain

\[ v = \sum_{j=0}^{\infty} A_j \phi_j, \quad A_j = \epsilon w v \phi_j^b \left|_{-a}^b \right. - (\phi_j, L_\epsilon \tilde{u}^\epsilon)_w + \epsilon \nu e^{-\epsilon^{-1}d} (\phi_j, g)_w, \tag{5.2} \]

where \( w = e^{-\epsilon^{-1}d/2} \). The limiting solvability condition \( A_N = 0 \) then gives the following equation for \( C_0 \):

\[ \epsilon \nu e^{-\epsilon^{-1}d} (\phi_N, g)_w = (\phi_N, L_\epsilon \tilde{u}^\epsilon)_w - \epsilon w v \phi_N^b \left|_{-a}^b \right. . \tag{5.3} \]

Since the right side of (5.3) was evaluated in §4.2, we need only calculate the left side of (5.3). Using \( \phi_N \sim M_N \text{He}_N(\epsilon^{-1/2}x) \) and the orthogonality properties of the Hermite polynomials, we readily calculate that

\[ (\phi_N, g)_w \sim M_N \epsilon^{(N+1)/2} (2\pi)^{1/2} g_\epsilon, \quad g_\epsilon \sim \sum_{j=0}^{\infty} \epsilon^j \theta_j g^{(N+2j)}(0). \tag{5.4a} \]

Here \( \theta_j \) is defined by

\[ \theta_j = \frac{(2\pi)^{-1/2}}{(N + 2j)!} \int_{-\infty}^{\infty} z^{2j+N} \text{He}_N(z) e^{-z^2/2} dz. \tag{5.4b} \]

In particular, \( \theta_0 = 1 \) and \( \theta_1 = 1/2 \). Substituting (4.9), (4.15b), (4.18) and (5.4) in (5.3), we obtain an equation for \( C_0 \) that can be solved to yield

\[ C_0 \sim C_{0H} - \frac{(2\pi)^{1/2} g_\epsilon \epsilon^{N+1/2} \nu e^{-\epsilon^{-1}(d-\sigma^2/2)}}{a^{2N+1} \gamma_a(\epsilon) + b^{2N+1} \gamma_b(\epsilon) e^{\epsilon^{-1}(a^2-\sigma^2)/2}}. \tag{5.5} \]

Here \( \gamma_a(\epsilon) \) and \( \gamma_b(\epsilon) \) are defined in (4.12a), and \( C_{0H} \) is defined to be the right side of (4.20). The asymptotic solution to (5.1) is then given in terms of \( C_0 \) by (4.14). Notice that \( C_0 \sim C_{0H} \) when \( g(x) = 0 \).

We now make some remarks concerning (5.1). Suppose that \( b = a, \nu = -N - 1/2 \) and \( g^{(N)}(0) \neq 0 \). Then from (5.5) and (5.1) we conclude that a forcing of order \( O(\epsilon^{-N-1/2} e^{-\epsilon^{-1}d}) \) produces an exponentially larger response in \( u \) of order \( O(e^{-\epsilon^{-1}(d-b^2/2)}) \). In particular, if \( d = b^2/2 \) the forcing induces an \( O(1) \) response. This exponential sensitivity is simply another manifestation of the exponential ill-conditioning of the underlying problem. Notice, however, that if \( g(x) \) is a polynomial of degree \( N - 1 \) then \( (\phi_N, g)_w \) is exponentially small. For this special form of \( g(x) \), it can be shown that the solution to (5.1) is not exponentially sensitive to an exponentially small forcing. We now consider a specific example.
Example 5.1: Consider (5.1) with $\beta = 2$, $\alpha = 1$, $a = 1$, $N = 2$, $\nu = -5/2$ and $g(x) = (2\pi)^{-1/2}(1 + x^2)^{-1}$. Then, using (4.12b), (4.20) and (5.4), (5.5) becomes

$$C_0 \sim \frac{2b^3(1 - 6\epsilon b^2) \epsilon^{-1}e^{-(1 - \beta^2)/2} + (1 - 6\epsilon) + (2 - 12\epsilon) \epsilon^{-1}e^{-(d - 1)/2}}{b^2(1 - 7\epsilon b^2) \epsilon^{-1}e^{-(1 - \beta^2)/2} + (1 - 7\epsilon)} .$$  

(5.6)

Therefore, when $b = 1$ and $d = 1/2$ then $C_0 \sim 5(1 + \epsilon)/2$. Setting $b = 1$ and $d = 1/2$, in Table 7a we compare the asymptotic and numerical values for $C_0$ as $\epsilon$ is decreased. The numerical value of $C_0$ is defined by $C_0 = -\epsilon^{-1}u(0)$, where $u(0)$ is computed from (5.1). In Table 7b we fix $\epsilon = 1/60$ and $b = 1$ and we compare values for $C_0 = C_0(d)$ as $d$ is varied in a neighborhood of $d = 1/2$. In Fig. 7 we plot $C_0 = C_0(d)$ and we show the numerical solution to (5.1) at a few values of $d$.

Exit Operator: We now consider a forced version of (3.1) given by

$$L \epsilon u \equiv u'' - x^{2m+1}p(x)u' = g(x)e^{\nu - \epsilon d}, \quad -a \leq x \leq b ,$$

$$u(-a) = \alpha , \quad u(b) = \beta .$$

(5.7)

Here $d > 0$, $g(x)$ and $p(x) > 0$ are smooth functions and $m$ is a non-negative integer. Since $d > 0$, the composite expansion for the solution to (5.7) is given by (3.13) where $C_0$ is to be found. Using the projection method, we readily find that $C_0$ satisfies

$$C_0 \sim C_0_H - \frac{\epsilon^{\nu + 1/(2m + 2)} g_x e^{-\epsilon^{-1}(d - \omega_a)}}{a^{2m+1}p(-a)\gamma_a(\epsilon) + b^{2m+1}p(b)\gamma_b(\epsilon) e^{-\epsilon^{-1}(\omega_b - \omega_a)}}. $$

(5.8)

Here $g_x$ is defined in (3.17a), $\gamma_a(\epsilon)$ and $\gamma_b(\epsilon)$ are defined in (3.8) while $\omega_a$ and $\omega_b$ are defined in (3.10b). In (5.8), $C_0 H$ is the value of $C_0$ for the homogeneous problem and is defined as the right side of (3.20). Thus, in analogy with the forced Hermite problem (5.1), the solution to (5.7) can be exponentially sensitive to an exponentially small forcing function. However, we note that in the special case when $g(x)$ is odd, the coefficient of order $O(\epsilon^{j/(m+1)})$ in the expansion of $g_x$ vanishes for any $j \geq 0$. It then follows that $(\phi_0, g)_\omega$ is exponentially small and, consequently, the exponential sensitivity of the solution to (5.7) is suppressed.

Example 5.2: In (5.7) let $\beta = 2$, $\alpha = 1$, $b = a = 1$, $m = 0$, $\nu = -1/2$, $p(x) = 2\pi x^{-1} \sin(\pi x/2)$ and $g(x) = 1 + x^2$. Then, using (3.8), (3.10b) and (3.17a), (5.8) becomes

$$C_0 \sim \frac{3}{2} - \left( \frac{2}{\pi} \right)^{1/2} \frac{\epsilon^{\nu - 1/2} e^{-(d - \omega_a)}}{4\pi} \left[ 1 + \epsilon \left( \frac{1}{\pi^2} + \frac{1}{32} \right) \right]. $$

(5.9)

5.2 An Exponentially Ill-Conditioned Diffusive Regularization

Following [4], we now consider the forced problem

$$N_\epsilon y \equiv - y'' + 2y = f(x, \epsilon), \quad -1 \leq x \leq 1 , \quad y(-1) = \pi , \quad y(1) = -\pi .$$

(5.10)

This BVP, which is a model problem illustrating the effect of a diffusive regularization on the more general shape from shading problem, was studied numerically in [4]. A discussion of the shape from shading problem and the specific motivation for considering (5.10) are described in [4].
Let \( y_0(x) \) be an odd, monotone non-increasing, function that is independent of \( \epsilon \) and that satisfies the boundary conditions in (5.10). In [4], the forcing \( f \) in (5.10) was chosen as \( f = f_0 = 2y_0y_0' \) where \( y_0 \) can assume either of the two forms

\[
y_0(x) = \pi \cos \left[ \frac{\pi}{2}(x + 1) \right], \quad \text{(type A)}; \quad y_0(x) = -\pi x^3, \quad \text{(type B)}.
\]  

(5.11)

For each of these two forms of \( y_0(x) \), the term \( \epsilon y'' \) is uniformly small throughout the domain as \( \epsilon \to 0 \). Therefore, it seems reasonable to expect that the solution to (5.10) with \( f \) replaced by \( f_0 \) will satisfy \( |y - y_0(x)| = O(\epsilon) \) uniformly on \([0, 1]\). In the sense it was hoped in [4] that the \( \epsilon y'' \) term in (5.10) is a diffusive regularization of the first order equation \( 2yy' = f \). We note that, since \( y \) is assumed to have a zero-crossing, this reduced problem requires some type of regularization. Rather interestingly, the computations in [4] using (5.10) and (5.11) showed that the solution to the regularized problem differs significantly from \( y_0 \) even when \( \epsilon \) is rather small. This difference is most pronounced near the turning point region at \( x = 0 \). Moreover, a rather large number of Newton iterations were required for numerical convergence of the discrete scheme used for (5.10), and in several instances convergence was not obtained.

We suggest that the qualitative effect of this diffusive regularization can largely be understood by studying the operator associated with linearizing (5.10) about \( y_0(x) \). In what follows, we make \( y_0 \) be an exact solution to (5.10) by choosing \( f = f_0 \equiv N_\epsilon y_0 \). For \( \epsilon \to 0 \), we then show that if \( f \) is perturbed from \( f_0 \) by an exponentially small amount, the solution to (5.10) can deviate from \( y_0 \) by an algebraically small amount. This exponential sensitivity in response to very small changes in the forcing function occurs as a result of the exponential ill-conditioning of the linearized operator.

We now perform such a sensitivity analysis for

\[
N_\epsilon y \equiv -\epsilon y'' + 2yy' = f_0(x, \epsilon) + \epsilon^\nu e^{-\epsilon^{-1}d} g(x), \quad -1 \leq x \leq 1, \quad y(\pm 1) = \mp \pi,
\]  

(5.12)

where \( f_0 = N_\epsilon y_0 \) and \( y_0 \) is given in (5.11). In (5.12), \( \nu > 0 \), \( g(x) \) is smooth, and \( d \) is a positive constant to be specified below. Setting \( y = y_0 + v \) in (5.12), we obtain the following linearized problem for \( v \):

\[
\epsilon v'' - 2(y_0 v)' = -\epsilon^\nu e^{-\epsilon^{-1}d} g(x), \quad -1 \leq x \leq 1, \quad v(\pm 1) = 0.
\]  

(5.13)

Now introduce \( u \) by \( v = wu \), where \( w = w(x, \epsilon) \equiv \exp \left[ 2\epsilon^{-1} \int_0^x y_0(\eta) \, d\eta \right] \). Then, from (5.13), we find that \( u \) satisfies

\[
L_\epsilon u \equiv \epsilon u'' + 2y_0u' = -\epsilon^\nu e^{-\epsilon^{-1}d} w^{-1} g(x), \quad -1 \leq x \leq 1, \quad u(\pm 1) = 0.
\]  

(5.14)

Clearly, (5.14) with (5.11) is an example of (5.7). Let \( \lambda_0 \) be the exponentially small eigenvalue of \( L_\epsilon \phi = \lambda \phi \) with \( \phi(\pm 1) = 0 \). Then, for \( y_0 \) of type A, \( \lambda_0 \) was given in (3.12a), whereas for \( y_0 \) of type B, \( \lambda_0 \) was given in (3.12c). Using the projection method, we can solve (5.14) for \( u \) and then recover \( y \) from \( y = y_0 + uw \). This calculation leads to

\[
y \sim y_0(x) + C_0 \left( 1 - e^{-2\pi \epsilon^{-1}(1+x)} - e^{-2\pi \epsilon^{-1}(1-x)} \right) \exp \left( 2\epsilon^{-1} \int_0^x y_0(\eta) \, d\eta \right),
\]  

(5.15a)
where $C_0$ satisfies
\[ C_0 \sim \frac{\epsilon^\nu}{4\pi} \exp \left[ -\epsilon^{-1}(d - \gamma) \right] \int_{-1}^{1} g(\eta) \, d\eta, \quad \text{with} \quad \gamma = -2 \int_{0}^{1} y_0(\eta) \, d\eta. \tag{5.15b} \]

For $y_0$ of type A, $\gamma = 4$ and for $y_0$ of type B, $\gamma = \pi/2$. It is clear from (5.15) that, in general, our linearization assumption is only valid when $d \geq \gamma$. Note that if $d = \gamma$, this assumption requires that $\nu > 0$.

Comparing (5.15) with (5.12), and noting that $w = \exp \left[ 2\epsilon^{-1} \int_{0}^{\pi} y_0(\eta) \, d\eta \right]$ is strongly concentrated near $x = 0$, we conclude that a perturbation in the forcing of order $O \left( \epsilon^\nu e^{-\epsilon^{-1} \gamma} \right)$ can perturb the solution near the turning point region by an amount of order $O(\epsilon^\nu)$ unless $\int_{-1}^{1} g(\eta) \, d\eta = 0$. Since $w$ is exponentially small away from $x = 0$, this perturbation in the forcing does not have nearly such a dramatic effect on the solution away from the turning point region.

We remark that if $0 < d < \gamma$, our linearization assumption is not valid. However, in this case, (5.15) certainly leads to the conjecture that $y$ can differ substantially from $y_0$ near the turning point region. Although our sensitivity analysis does not precisely explain the numerical findings of [4] obtained from the discrete form of (5.10), this analysis does show, at least at the continuous level, the severe ill-conditioning that can be associated with (5.10).

As an example, consider (5.13) - (5.15) for $g(x) = 1$, and for $y_0(x)$ of type A. Without loss of generality we can take $\nu = 0$. For $\epsilon = 1/8$, in Table 7c and Fig. 8 we compare the asymptotic and numerical values for $C_0 = C_0(d)$ for $d$ near $\gamma = 4$. In Fig. 8 we also plot, at several values of $d$, the numerical solutions $u(x)$ and $v(x)$ to (5.14) and (5.13), respectively. Although the computed solutions for $d < 4$ are not strictly relevant to the sensitivity analysis above, it is clear from Fig. 8 that the spike type behavior in $v = y - y_0$ near the turning point region becomes more pronounced as $d$ is decreased.

6. Nonlinear Problems with Exponentially Ill-Conditioned Linearizations

We now consider some nonlinear problems that are associated with small eigenvalues.

6.1 Eigenvalues for Linearized Viscous Shock Problem

Consider the viscous shock problem
\[ u_t + [f(u)]_x = \epsilon u_{xx}, \quad 0 \leq x \leq 1, \tag{6.1a} \]
\[ \epsilon u_x(0, t) - \kappa_l [u(0, t) - \alpha] = 0, \quad \epsilon u_x(1, t) + \kappa_r [u(1, t) + \alpha] = 0, \tag{6.1b} \]
where $\kappa_l > 0$, $\kappa_r > 0$ and $\alpha > 0$. We assume that $(\kappa_r - \alpha)(\kappa_l - \alpha) > 0$ and that $f(u)$ is a convex function satisfying $f(0) = f'(0) = 0$ and $f(\alpha) = f(-\alpha)$. This problem, and various related problems with special forms of $f(u)$, have been analyzed for $\epsilon \to 0^+$ in [18], [27], [28], [19] and [20]. These analyses have uncovered three main features for the solution to (6.1). Firstly, (6.1) has an equilibrium shock layer solution $u^\epsilon = u^\epsilon(x, \epsilon)$ where the shock layer location $x_0^\epsilon$, defined by $u^\epsilon(x_0^\epsilon, \epsilon) = 0$ and satisfying $0 < x_0^\epsilon < 1$, can only be determined by resolving exponentially small effects. Secondly, there is an exponentially slow shock layer motion for the evolution problem. Thirdly, the linearization of (6.1) around $u^\epsilon(x, \epsilon)$ is exponentially ill-conditioned. For a certain parameter range, an asymptotic estimate for the exponentially small eigenvalue was given in [27]. We now give further eigenvalue estimates for other parameter ranges and we relate (6.1) to the
exit problem (3.1). More importantly, the eigenvalue estimates are verified numerically for Burgers equation.

The linearization of (6.1a) around $u^\epsilon$ leads to the eigenvalue problem

$$
\epsilon v_{xx} + [q(x, \epsilon)v]_x = \lambda v, \quad q(x, \epsilon) \equiv - f' [u^\epsilon(x, \epsilon)].
$$

Thus, defining $v = \exp \left[ - \epsilon^{-1} Q(x, \epsilon) \right] \phi$ where $Q(x, \epsilon) \equiv \int_{x_0}^x q(\eta, \epsilon) \, d\eta$, we obtain

$$
L_\epsilon \phi \equiv \epsilon \phi_{xx} - q(x, \epsilon) \phi_x = \lambda \phi, \quad \epsilon \phi_x(0) - [\kappa_l + q(0, \epsilon)] \phi(0) = 0, \quad \epsilon \phi_x(1) + [\kappa_r - q(1, \epsilon)] \phi(1) = 0.
$$

Since $q(x_0^\epsilon, \epsilon) = 0$, $q_x(x_0^\epsilon, \epsilon) > 0$ and $(x - x_0^\epsilon)q(x, \epsilon) > 0$ for $0 < x < 1$, (6.3) has the form of the exit operator (3.1) with $m = 0$ and with the turning point at $x_0^\epsilon$ rather than at $x = 0$. Thus, (6.3) has an eigenpair $\phi_0$, $\lambda_0$ where $\lambda_0$ is exponentially small and where $\phi_0 \sim 1$ away from boundary layers near $x = 0$ and $x = 1$.

To calculate $\lambda_0$ we use the following identity:

$$
\lambda_0(\phi_0, 1)_w = [q(1, \epsilon) - \kappa_r ] \phi_0(1) w(1) - [q(0, \epsilon) + \kappa_l ] \phi_0(0) w(0).
$$

Here $(u, v)_w \equiv \int_0^1 u v \, dx$ and $w = \exp \left[ - \epsilon^{-1} Q(x, \epsilon) \right]$. Then, from a boundary layer analysis, we derive that $\phi_0(1) \sim q(1, \epsilon)/\kappa_r$ and $\phi_0(0) \sim -q(0, \epsilon)/\kappa_l$. Therefore, (6.4) becomes

$$
\lambda_0 \int_0^1 \exp \left[ - \epsilon^{-1} Q \right] dx \sim \left[ \frac{q(1, \epsilon)}{\kappa_r} - 1 \right] q(1, \epsilon) e^{-\epsilon^{-1} Q(1, \epsilon)} + \left[ \frac{q(0, \epsilon)}{\kappa_l} + 1 \right] q(0, \epsilon) e^{-\epsilon^{-1} Q(0, \epsilon)}.
$$

Here $Q(x, \epsilon)$ was defined following (6.2). We note that $q(x, \epsilon) = -f'(x) + O(\epsilon^{-C_+})$ as $\epsilon^{-1}(x - x_0^\epsilon) \to \pm \infty$ for some $C_+ > 0$. Thus, in contrast to the situation in §3.1, for (6.3) there are no higher order boundary layer correction terms for $\phi_0$ near $x = 0$ and $x = 1$ that are algebraic in $\epsilon$.

Consider Burgers equation where $f(u) = u^2/2$. In this case,

$$
u^\epsilon(x) = -\beta \tanh \left[ \beta \epsilon^{1/2}(x - x_0^\epsilon)/2 \right] = -q(x, \epsilon), \quad Q(x, \epsilon) = 2\epsilon \log \left( \cosh \left[ \beta \epsilon^{1/2}(x - x_0^\epsilon)/2 \right] \right),
$$

where $\beta$ and $x_0^\epsilon$ satisfy the transcendental equations

$$
\beta^2 \text{sech}^2 \left( \beta \epsilon^{1/2} x_0^\epsilon/2 \right) + 2k_l \left[ \beta \tanh \left( \beta \epsilon^{1/2} x_0^\epsilon/2 \right) - \alpha \right] = 0,
$$

$$
\beta^2 \text{sech}^2 \left[ \beta \epsilon^{-1}(1 - x_0^\epsilon)/2 \right] + 2k_r \left[ \beta \tanh \left( \beta \epsilon^{-1}(1 - x_0^\epsilon)/2 \right) - \alpha \right] = 0.
$$

Thus, the integral on the left side of (6.5) satisfies $\int_0^1 \exp \left[ - \epsilon^{-1} Q \right] \, dx \sim 4\epsilon \beta^{-1} + \text{T.S.T.}$ as $\epsilon \to 0$.

We now give asymptotic estimates for $\lambda_0$, calculated from (6.5)–(6.7), for four different ranges of $k_l$ and $k_r$. First, let $k_l = k_r \equiv k \neq \alpha$. Then, $x_0^\epsilon = 1/2$, $\beta \sim \alpha + 2\alpha(1 - \alpha/k)\epsilon^{-1/2}$ and

$$
\lambda_0 \sim 2\epsilon^{-1} \alpha^2 \left( \frac{\alpha}{k} - 1 \right) e^{-\epsilon^{-1} \alpha^2/2}, \quad (k \neq \alpha).
$$

Now, let $k_l = k_r \equiv k = \alpha$. Then, $x_0^\epsilon = 1/2$, $\beta \sim \alpha + 2\alpha \epsilon^{-1/2}$ and

$$
\lambda_0 \sim -4\epsilon^{-1} \alpha^2 \epsilon^{-\epsilon^{-1} \alpha}, \quad (k = \alpha).
$$
Next, let $\kappa_l = \infty$ and $\kappa_r > \alpha$. Then, $x_0^\circ \sim 1/2 - \epsilon(2\alpha)^{-1} \log(1 - \alpha/\kappa_r)$, $\beta \sim \alpha + 2\alpha e^{-\epsilon^{-1} \alpha x_0^\circ}$ and

$$
\lambda_0 \sim -2\epsilon^{-1} \alpha^2 (1 - \alpha/\kappa_r)^{1/2} e^{-\epsilon^{-1} \alpha^2/2}, \quad (\kappa_l = \infty, \ \kappa_r > \alpha). \tag{6.10}
$$

Finally, let $\kappa_l = \infty$ and $\kappa_r = \alpha$. Then $x_0^\circ \sim 2/3 + \text{T.S.T.}$, $\beta \sim \alpha + 2\alpha e^{-2\epsilon^{-1} \alpha^3/3}$ and

$$
\lambda_0 \sim -3\epsilon^{-1} \alpha^2 e^{-2\epsilon^{-1} \alpha^3/3}, \quad (\kappa_l = \infty, \ \kappa_r = \alpha). \tag{6.11}
$$

The result (6.8) was given in [27]. Comparing (6.8) with (6.9) and (6.10) with (6.11) we note that the estimate for $\lambda_0$ changes significantly as $\kappa_r$ (or $\kappa_l$) crosses through $\alpha$. It is significantly more difficult to provide estimates for $\lambda_0$ that uniformly incorporate these transitions.

Before comparing (6.8)–(6.11) with corresponding numerical results for $\lambda_0$, we formulate two eigenvalue problems that, for Burgers equation, are equivalent to (6.3). These equivalent problems are introduced primarily to illustrate the flexibility of the eigenvalue solver of [21] for accurately computing the spectrum associated with different types of singularly perturbed Sturm Liouville operators. The first equivalent problem is obtained by substituting $\psi = \exp[-Q/2\epsilon] \phi$ in (6.3), where $Q$ is defined in (6.6). We then obtain

$$
e^2 \psi_{xx} + \frac{\beta^2}{2} \text{sech}^2 \left[ \frac{\beta \epsilon^{-1}}{2}(x - x_0^\circ) \right] \psi = \left( \epsilon \lambda + \frac{\beta^2}{4} \right) \psi, \quad 0 < x < 1 \tag{6.12}
$$

$$
\epsilon \psi_x(1) + [\kappa_r - \frac{1}{2} q(1, \epsilon)] \psi(1) = 0, \quad \epsilon \psi_x(0) - \frac{1}{2} q(0, \epsilon)] \psi(0) = 0.
$$

The problem (6.12) can also be recast using the transformation $\xi = \tanh \left[ \beta \epsilon^{-1}(x - x_0^\circ)/2 \right]$ and $\varphi(\xi) = \psi[x(\xi)]$ to the following Legendre’s equation with eigenvalue parameter $\mu^2 = 1 + 4\epsilon \lambda \beta^{-1}$:

$$
[(1 - \xi^2) \varphi_\xi]_\xi + 2\varphi = \mu^2 (1 - \xi^2)^{-1} \varphi, \quad -\xi_l < \xi < \xi_r, \tag{6.13a}
$$

$$
\varphi_\xi(\xi_r) + F_\epsilon(1 - x_0^\circ, \kappa_r) \varphi(\xi_r) = 0, \quad \varphi_\xi(-\xi_l) - F_\epsilon(x_0^\circ, \kappa_l) \varphi(-\xi_l) = 0. \tag{6.13b}
$$

Here $\xi_l$, $\xi_r$ and $F_\epsilon$ are defined by $\xi_l = -\tanh (\beta \epsilon^{-1} x_0^\circ/2)$, $\xi_r = \tanh [\beta \epsilon^{-1}(1 - x_0^\circ)/2]$ and

$$
F_\epsilon(a_1, a_2) = \frac{2}{\beta} \cosh^2 \left( \frac{\beta \epsilon^{-1} a_1}{2} \right) \left[ a_2 - \frac{\beta}{2} \tanh \left( \frac{\beta \epsilon^{-1} a_1}{2} \right) \right]. \tag{6.13c}
$$

Since $\lambda_0$ is exponentially small, then (6.13) has an eigenvalue with $\mu$ exponentially close to one.

The eigenvalue solver of [21] allows for accurate numerical computations of $\lambda_0$ using any one of the three equivalent formulations (6.3), (6.12) or (6.13). In particular, the method can compute the spectrum associated with a strongly localized potential such as in (6.12) and the method can also treat nearly singular Sturm Liouville operators such as in (6.13). Notice, in (6.13), that since the endpoints $\xi_l$ and $\xi_r$ are exponentially close to the regular singular points $\xi = \pm 1$, it would certainly be difficult, using standard numerical methods, to accurately compute the eigenvalue $\mu$ with $\mu$ exponentially close to one.

To compute $\lambda_0$ numerically in various cases, we first accurately solve (6.7) for $x_0^\circ$ and $\beta$ using Newton’s method. The eigenvalue solver of [21] is then used to compute $\lambda_0$ from either (6.3), (6.12) or (6.13). In the computations below we set $\alpha = 1$. In Tables 8a, 8b, 8c and 8d we compare the
asymptotic and numerical values for \( \lambda_0 \) for different parameter values corresponding to (6.8), (6.9), (6.10) and (6.11), respectively. From these tables we observe that the asymptotic result closely determines \( \lambda_0 \) even at moderately small values of \( \epsilon \). Note that in each table there is a range of \( \epsilon \) where the numerically computed value of \( \lambda_0 \) differs from the asymptotic value of \( \lambda_0 \) by roughly \( 10^{-13} \). This suggests not only that the asymptotic results for \( \lambda_0 \) are very precise but also that the numerical method of [21] gives accurate results for \( \lambda_0 \) to within roughly 13 decimal places.

As a remark, for the case \( \kappa_r = \kappa_l = \infty, \alpha = 1 \) and \( \epsilon = 1/50 \), the result \( \lambda_0 = -1.24 \times 10^{-9} \) was computed in [18] using a finite difference method. This is to be contrasted with the result \( \lambda_0 = -1.38879 \times 10^{-9} \) that was computed using the method of [21]. From (6.8) the asymptotic result is \( \lambda_0 = -1.38876 \times 10^{-9} \).

6.2 The Linearized Allen-Cahn Equation

Consider the boundary value problem \( \epsilon^2 u_{xx} + 2(u - u^3) = 0 \) on \( 0 \leq x \leq 1 \) with \( u_x(0) = u_x(1) = 0 \). A monotonically increasing solution to this problem with exactly one internal layer is given, to within exponentially small terms, by \( u(x, \epsilon) = \tanh[\epsilon^{-1}(x - 1/2)] \). Linearizing around this solution we obtain the following eigenvalue problem, which is closely related to (6.12):

\[
\epsilon^2 v_{xx} + 6 \text{sech}^2 \left[ \epsilon^{-1}(x - \frac{1}{2}) \right] v = (\lambda + 4)v, \quad v_x(0) = v_x(1) = 0. \tag{6.14}
\]

For this problem, the first eigenvalue \( \lambda_0 \) satisfies \( \lambda_0 \sim 96 \epsilon^{-2}\epsilon^{-1} \) as \( \epsilon \to 0 \) (see [32]). In Table 8 we show the close agreement between this asymptotic result for \( \lambda_0 \) and the corresponding numerical result for \( \lambda_0 \) computed from (6.14).

6.3 An Ill-Conditioned System

Exponentially ill-conditioned linearizations can also be associated with systems of boundary value problems. To illustrate this consider the following simple system on \( -1 \leq x \leq 1 \):

\[
u'' = -v, \quad u(-1) = u(1) = 0, \tag{6.15a}\]
\[\epsilon v'' + f(u')v' = 0, \quad v(-1) = \alpha, \quad v(1) = \beta, \quad 0 < \alpha < \beta. \tag{6.15b}\]

We assume that \( f(0) = 0, f(z) > 0 \) for \( z > 0 \), and that \( f(z) < 0 \) for \( z < 0 \). When \( f(z) = z \), this problem qualitatively resembles a subset of the drift-diffusion equations of semiconductor physics. The composite expansion for the solution to (6.15) is given in terms of an undetermined constant \( c_0 > 0 \) by

\[
\tilde{u}(x) = -\frac{c_0}{2}(x^2 - 1), \quad \tilde{v}(x) = c_0 + (\alpha - c_0) e^{-f(c_0)\epsilon^{-1}(1+x)} + (\beta - c_0) e^{-f(-c_0)\epsilon^{-1}(1-x)}. \tag{6.16}
\]

To determine \( c_0 \) we replace \( f(u') \) in (6.15b) with \( f(-c_0x) \). Since the resulting equation is of the form given in (3.1), we can determine \( c_0 \) using the projection method. Using this method, we obtain to leading order that

\[
\frac{f(-c_0)(\beta - c_0)}{f(c_0)(\alpha - c_0)} \sim \exp \left[ (\epsilon c_0)^{-1}(F(-c_0) - F(c_0)) \right], \tag{6.17}
\]

where \( F(z) \equiv \int_0^z f(\eta) \, d\eta \). Thus if \( F(z) > F(-z) \) (\( F(z) < F(-z) \)) for all \( z > 0 \), then \( c_0 \sim \beta (c_0 \sim \alpha) \) and there is no boundary layer near \( x = 1 \) (\( x = -1 \)). When \( f(z) \) is odd, then \( c_0 = (\alpha + \beta)/2 \).
We remark that if \( v \) is replaced by \( -v \) in (6.15a), the modified system is not exponentially ill-conditioned.

7 Exponential Small Spectral Gap Widths

We now consider three problems with exponentially small spectral gap widths.

7.1 Exponential Gap Widths and the Viscous Shock Problem

Consider (6.1a) for Burgers equation \((f(u) = u^2/2)\) with boundary conditions

\[
u(0, t) = \alpha - A_t e^{-\epsilon^{-1} c_t}, \quad u(1, t) = -\alpha + A_r e^{-\epsilon^{-1} c_r}.
\] (7.1)

Here \( c_t > 0, c_r > 0 \) and \( \alpha > 0 \). The slow internal layer motion for this problem was studied in [28] and [19]. Applying the Cole-Hopf transformation \( u = -2\epsilon v_x/v \) to (6.1a), (7.1), and separating variables by writing \( v = e^{\lambda t} \phi(y) \), we obtain the eigenvalue problem

\[
\epsilon \phi'' - \lambda \phi = 0, \quad 0 \leq y \leq 1,
\]

\[
\phi'(0) = \frac{1}{2\epsilon} \left( -\alpha + A_t e^{-\epsilon^{-1} c_t} \right) \phi(0), \quad \phi'(1) = \frac{1}{2\epsilon} \left( \alpha - A_r e^{-\epsilon^{-1} c_r} \right) \phi(1).
\] (7.2)

The first two eigenfunctions are \( \phi_0 = \cosh[\mu_0(x - x_0)] \) and \( \phi_1 = \sinh[\mu_1(x - x_1)] \), where \( \mu_0 = (\lambda_0/\epsilon)^{1/2}, \mu_1 = (\lambda_1/\epsilon)^{1/2}, x_0 \) and \( x_1 \) satisfy some explicit transcendental equations. By analyzing these equations, we obtain the following estimate as \( \epsilon \to 0 \) for the exponentially small gap width:

\[
\Delta \lambda \equiv \lambda_0 - \lambda_1 \sim 2\alpha^2 e^{-\epsilon^{-1} c_t/2} \left[ 1 + \frac{B^2}{16\alpha \epsilon^2} \right]^{1/2}, \quad B \equiv A_t e^{-\epsilon^{-1} c_t - \alpha/2} - A_r e^{-\epsilon^{-1} c_r - \alpha/2}.
\] (7.3)

The metastable time scale for internal layer motion near equilibrium is given by \((\Delta \lambda)^{-1}\). When \( B = 0 \), (7.3) was given in [27]. In Tables 9a, 9b we compare (7.3) with the corresponding numerical results for \( \Delta \lambda \) computed using the eigenvalue solver of [21]. From these tables we observe that (7.3) determines \( \Delta \lambda \) very closely and that the numerical method is able to calculate \( \Delta \lambda \) even when \( \Delta \lambda \approx 10^{-12} \).

7.2 Tunneling Through a High Barrier Square Well

The following exactly solvable tunnelling problem was studied in [12]:

\[
\psi''(x) + [\lambda - V_\epsilon(x)] \psi(x) = 0, \quad V_\epsilon(x) = \begin{cases} \epsilon & 0 \leq x < 1, \\ Fe^{-2} & 1 < x < \epsilon^{-1} - 1, \\ \epsilon^{-1} - 1 < x \leq \epsilon^{-1}, \end{cases}
\] (7.4)

with \( \psi(0) = \psi(\epsilon^{-1}) = 0 \). Here \( F \) is a positive constant. Let \( \lambda_0(\epsilon) \) and \( \lambda_1(\epsilon) \) be the first and second eigenvalues of (7.4). Then, as was shown in [12], the gap width between \( \lambda_0 \) and \( \lambda_1 \) is exponentially small and has following estimate for \( \epsilon \to 0 \):

\[
\Delta \lambda \equiv \lambda_1 - \lambda_0 = 4(G_\lambda[\lambda_0(\epsilon), \epsilon]^{-1} \exp \left[ -(Fe^{-2} - \lambda_0(\epsilon))^{1/2}(\epsilon^{-1} - 2) \right].
\] (7.5a)

Here \( G(\lambda, \epsilon) \) is defined by

\[
G(\lambda, \epsilon) = 1 + \tan[(\lambda - \epsilon)^{1/2}][\lambda - \epsilon]^{-1/2}(Fe^{-2} - \lambda)^{1/2},
\] (7.5b)
and \( \lambda_0(\epsilon) \) satisfies the transcendental equation \( G[\lambda_0(\epsilon), \epsilon] = 0 \), which can be solved numerically.

When \( F = 1 \), in Table 9c we compare (7.5a) with corresponding numerical results for \( \Delta \lambda \) computed using the eigenvalue solver of [21]. These comparisons show that, even in the presence of a strong discontinuity in the potential, the eigenvalue solver accurately computes the gap width. We remark that for the values of \( \epsilon \) given in Table 9c, the limiting form \( \Delta \lambda \sim 8\pi^2 \epsilon F^{-1/2} \exp [-F^{1/2} \epsilon^{-1}(\epsilon^{-1} - 2)] \), which can be obtained from (7.5a), agrees rather poorly with the numerical results.

7.3 The Forced Burgers Equation and the Double-Well Potential

Exponential eigenvalue degeneracy also occurs for the quantum anharmonic oscillator problem

\[
\psi''(x) + [\lambda - V_e(x)] \psi(x) = 0, \quad -\infty < x < \infty, \quad V_e(x) = x^2(1 - \epsilon x)^2, \tag{7.6}
\]

with \( \psi(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \). The potential \( V_e(x) \), has a symmetric double-well structure with wells of equal depth located at \( x = 0 \) and \( x = \epsilon^{-1} \). Let \( \lambda_0(\epsilon) \) and \( \lambda_1(\epsilon) \) be the first and second eigenvalues of (7.6), respectively. An asymptotic formula for the exponentially small gap width \( \Delta \lambda = \lambda_0 - \lambda_1 \) was proven in [12].

We now show that an eigenvalue problem, closely related to (7.6), is relevant to describing metastable internal layer motion for the following forced Burgers equation in the limit \( \epsilon \rightarrow 0 \):

\[
\begin{align*}
  u_t + uu_y &= \epsilon u_{yy} + \epsilon^2 f(y), \quad 0 \leq y \leq 1, \quad t > 0, \tag{7.7a} \\
  u(0, t) &= 0, \quad u(1, t) = 0. \tag{7.7b}
\end{align*}
\]

Here \( f(y) \) satisfies

\[
f(0) = f(1/2) = f(1) = 0, \quad f(y) = -f(1 - y), \quad f'(0) > 0, \quad f(y) > 0 \quad \text{for} \quad 0 < y < 1/2, \tag{7.8}
\]

so that \( \int_0^1 f(y) \, dy = 0 \). Replacing \( \epsilon^{-2} f(y) \) by \( f(y) \) in (7.7), the numerical study of [18] showed that metastable internal layer motion, similar to that for (6.1), occurs for this problem. We now calculate the metastable time scale for this motion.

Setting \( u(y, t) = -2\epsilon v_y(y, t)/v(y, t) \) and \( v(y, t) = \epsilon^{-\lambda \epsilon^{-1} t} \phi(y) \) in (7.7), we obtain the following eigenvalue problem on \( 0 < y < 1 \):

\[
\epsilon^2 \phi'' + [\lambda - \epsilon^{-2} F(y)] \phi = 0, \quad \phi'(0) = \phi'(1) = 0; \quad F(y) = \frac{1}{2} \int_0^y f(\eta) \, d\eta. \tag{7.9}
\]

Then, the solution to (7.7) is recovered from

\[
u(y, t) = -2\epsilon v_y(y, t)/v(y, t), \quad \text{where} \quad v(y, t) = \sum_{j=0}^{\infty} c_j e^{-\lambda_j \epsilon^{-1} t} \phi_j(y), \quad c_j = \int_0^1 v(y, 0) \phi_j(y) \, dy. \tag{7.10}
\]

Here \( \lambda_j \) and \( \phi_j \), for \( j \geq 0 \), are the eigenvalues and the normalized eigenfunctions of (7.9). For (7.9) it can be shown that \( \lambda_1 - \lambda_0 \) is exponentially small and that \( \lambda_1 - \lambda_0 = o(\lambda_2 - \lambda_1) \) as \( \epsilon \rightarrow 0 \). Thus, for \( t \gg 1 \), \( u(y, t) \) can be calculated asymptotically using the first two modes in (7.10). With
this approximation, we find that the slow motion shock layer trajectory \( y = y_0(t) \), defined by 
\( u[y_0(t), t] = 0 \), satisfies the transcendental equation

\[
\frac{\phi[y_0(t)]}{\phi'[y_0(t)']} \sim -\frac{c_1}{c_0} \exp[-t/t_c], \quad t_c = \epsilon (\lambda_1 - \lambda_0)^{-1}.
\] (7.11)

Here \( \phi_0, \phi_1, c_0, c_1 \) and \( \lambda_1 - \lambda_0 \) all depend on \( \epsilon \). Thus, the exponentially long time scale \( t_c \) is determined by the gap width \( \Delta \lambda = \lambda_1 - \lambda_0 \). Since \( t_c > 0 \) and \( \phi_0'(1/2) = 0 \), it is clear that 
\( y_0(t) \rightarrow 1/2 \) for \( t/t_c \rightarrow \infty \).

To calculate \( \lambda_1 - \lambda_0 \), we let \( \psi(x) = \phi(\epsilon x) \) in (7.9) to obtain

\[
\begin{align*}
\psi''(x) + [\lambda - V_\epsilon(x)] \psi(x) &= 0, \quad 0 \leq x \leq \epsilon^{-1}, \\
V_\epsilon(x) &= \epsilon^{-2} F(\epsilon x), \\
\psi'(0) &= \psi'_{\epsilon}^{-1} = 0.
\end{align*}
\] (7.12a, 7.12b)

From (7.8) and the definition of \( F(y) \) in (7.9), it follows that \( V_\epsilon(x) \) is a symmetric double-well potential satisfying \( V_\epsilon(x) = V_\epsilon(1 - x) \) and

\[
V_\epsilon(0) = V_\epsilon'(0) = 0, \quad V_\epsilon''(0) = 2\beta, \quad V_\epsilon[(2\epsilon)^{-1}] = \epsilon^{-2} V_* = \max_{1 \leq x \leq \epsilon^{-1}} V_\epsilon(x) > 0.
\] (7.12c)

Here \( V_* \) and \( \beta \) are related to \( f(y) \) by \( V_* = \frac{1}{2} \int_0^{1/2} f(y) \, dy \) and \( \beta = \int f'(0) \). The main difference between (7.6) and (7.12) is that (7.12) is defined on a finite interval with no-flux boundary conditions imposed at the location of the minima of the potential. The gap width \( \Delta \lambda \) for (7.12) can be calculated asymptotically by formally extending the analysis of [12] to treat an arbitrary symmetric double well potential with boundary conditions (7.12b). This analysis, for \( \epsilon \to 0 \), yields

\[
\Delta \lambda \sim \lambda_1 - \lambda_0 \sim \frac{4}{\sqrt{\pi}} \beta^{1/2} \epsilon^{-1/2} \exp \left( -2 \int_{x_1(\epsilon)}^{(2\epsilon)^{-1}} \left[ V_\epsilon(x) - \beta^{1/2} \right]^{1/2} \, dx \right).
\] (7.13)

Here \( x_1(\epsilon) \) is the root of \( V_\epsilon(x) = \beta^{1/2} \) for which \( x_1(\epsilon) \to \beta^{-1/4} \) as \( \epsilon \to 0 \). Substituting (7.13) into (7.11) we obtain the exponentially long time scale \( t_c \).

As an example, let \( f(y) = \sin(2\pi y)/2 \). Then, the gap width satisfies (7.13), where

\[
\beta = \frac{\pi}{4}, \quad V_\epsilon(x) = \frac{1}{8\pi \epsilon^2} \left[ 1 - \cos(2\pi \epsilon x) \right].
\] (7.14)

In Table 9d we compare (7.13), (7.14) with the corresponding numerical results for \( \Delta \lambda \) computed using the eigenvalue solver of [21]. For this range of \( \epsilon \), the asymptotic result provides only a moderately good determination of the gap width. We note that our computations for the numerical gap width are limited to those values of \( \epsilon \) for which \( \Delta \lambda \) is greater than \( \Delta \lambda \approx O(10^{-13}) \). Thus, we can not verify (7.13), (7.14) numerically at very small values of \( \epsilon \). The pre-exponential factor in (7.13) represents the first term in an asymptotic expansion in powers of \( \epsilon \) of a more complicated, \( \epsilon \) dependent, pre-exponential factor. In analogy with the example in §7.2, it is our belief that if we were able to calculate higher order corrections for this pre-exponential factor, the resulting asymptotic gap width formula would agree more closely with the corresponding numerical result, at finite \( \epsilon \), than does (7.13).
Acknowledgements

J. Y. L. would like to thank Prof. L. Greengard for helpful discussions on the numerical computations. M. J. W. is grateful to Prof. U. Ascher for introducing us to the shape from shading problem and to Prof. J. B. Keller for a helpful correspondence on the double-well problem.

References

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\lambda_0$ (num.)</th>
<th>$\lambda_0$ (3.12a) 2 term</th>
<th>$\lambda_0$ (3.12a) 1 term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-2.8154935188744 × 10^{-1}</td>
<td>-2.7945 × 10^{-1}</td>
<td>-2.8846 × 10^{-1}</td>
</tr>
<tr>
<td>3.0</td>
<td>-1.6457072776 × 10^{-4}</td>
<td>-1.6586 × 10^{-4}</td>
<td>-1.6761 × 10^{-4}</td>
</tr>
<tr>
<td>5.0</td>
<td>-7.1939070 × 10^{-8}</td>
<td>-7.2134 × 10^{-8}</td>
<td>-7.2588 × 10^{-8}</td>
</tr>
<tr>
<td>7.0</td>
<td>-2.86443 × 10^{-11}</td>
<td>-2.8683 × 10^{-11}</td>
<td>-2.8812 × 10^{-10}</td>
</tr>
<tr>
<td>8.0</td>
<td>-5.6135 × 10^{-13}</td>
<td>-5.6194 × 10^{-13}</td>
<td>-5.6415 × 10^{-12}</td>
</tr>
<tr>
<td>9.0</td>
<td>-1.086 × 10^{-14}</td>
<td>-1.0921 × 10^{-14}</td>
<td>-1.0959 × 10^{-13}</td>
</tr>
<tr>
<td>9.5</td>
<td>-1.53 × 10^{-15}</td>
<td>-1.5188 × 10^{-15}</td>
<td>-1.5238 × 10^{-14}</td>
</tr>
<tr>
<td>10.0</td>
<td>-2.2 × 10^{-16}</td>
<td>-2.1093 × 10^{-16}</td>
<td>-2.1150 × 10^{-16}</td>
</tr>
</tbody>
</table>

**Table 1a:** Comparison of asymptotic and numerical values for $\lambda_0$ of (3.2) with $p(x) = 2\pi x^{-1} \sin(\pi x/2)$, $m = 0$, and $a = b = 1$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\lambda_0$ (num.)</th>
<th>$\lambda_0$ (3.12b) 2 term</th>
<th>$\lambda_0$ (3.12b) 1 term</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.0</td>
<td>-6.1848298938 × 10^{-5}</td>
<td>-6.5684 × 10^{-5}</td>
<td>-6.8344 × 10^{-5}</td>
</tr>
<tr>
<td>20.0</td>
<td>-3.2582667 × 10^{-8}</td>
<td>-3.3695 × 10^{-8}</td>
<td>-3.4743 × 10^{-8}</td>
</tr>
<tr>
<td>28.0</td>
<td>-1.60965 × 10^{-11}</td>
<td>-1.6477 × 10^{-11}</td>
<td>-1.6908 × 10^{-11}</td>
</tr>
<tr>
<td>32.0</td>
<td>-3.537 × 10^{-13}</td>
<td>-3.6096 × 10^{-13}</td>
<td>-3.6978 × 10^{-13}</td>
</tr>
<tr>
<td>36.0</td>
<td>-7.7 × 10^{-15}</td>
<td>-7.8742 × 10^{-15}</td>
<td>-8.0552 × 10^{-15}</td>
</tr>
</tbody>
</table>

**Table 1b:** Comparison of asymptotic and numerical values for $\lambda_0$ of (3.2) with $p(x) = 2\pi(1 + x^2)^{-1}$, $m = 1$, and $a = b = 1$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\lambda_0$ (num.)</th>
<th>$\lambda_0$ (3.12c) 2 term</th>
<th>$\lambda_0$ (3.12c) 1 term</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>-3.98321716946 × 10^{-3}</td>
<td>-4.0748 × 10^{-3}</td>
<td>-4.5049 × 10^{-2}</td>
</tr>
<tr>
<td>10.0</td>
<td>-1.97255241 × 10^{-6}</td>
<td>-1.9804 × 10^{-6}</td>
<td>-2.0797 × 10^{-5}</td>
</tr>
<tr>
<td>12.0</td>
<td>-9.008550 × 10^{-8}</td>
<td>-9.0322 × 10^{-8}</td>
<td>-9.4065 × 10^{-7}</td>
</tr>
<tr>
<td>14.0</td>
<td>-4.07296 × 10^{-9}</td>
<td>-4.0805 × 10^{-9}</td>
<td>-4.2246 × 10^{-8}</td>
</tr>
<tr>
<td>16.0</td>
<td>-1.8287 × 10^{-10}</td>
<td>-1.8313 × 10^{-10}</td>
<td>-1.8876 × 10^{-10}</td>
</tr>
<tr>
<td>18.0</td>
<td>-8.1769 × 10^{-12}</td>
<td>-8.1779 × 10^{-12}</td>
<td>-8.4008 × 10^{-12}</td>
</tr>
<tr>
<td>20.0</td>
<td>-3.635 × 10^{-13}</td>
<td>-3.6382 × 10^{-13}</td>
<td>-3.7272 × 10^{-13}</td>
</tr>
<tr>
<td>22.0</td>
<td>-1.61 × 10^{-14}</td>
<td>-1.6137 × 10^{-14}</td>
<td>-1.6495 × 10^{-14}</td>
</tr>
</tbody>
</table>

**Table 1c:** Comparison of asymptotic and numerical values for $\lambda_0$ of (3.2) with $p(x) = 2\pi$, $m = 1$, and $a = b = 1$. 
<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$C_0$ (num.)</th>
<th>$C_0 - 1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>1.499999999999740</td>
<td>$-2.600 \times 10^{-13}$</td>
</tr>
<tr>
<td>15.0</td>
<td>1.499999999219</td>
<td>$-7.812 \times 10^{-9}$</td>
</tr>
<tr>
<td>19.0</td>
<td>1.4999891</td>
<td>$-1.093 \times 10^{-6}$</td>
</tr>
<tr>
<td>23.0</td>
<td>1.50145</td>
<td>$1.446 \times 10^{-3}$</td>
</tr>
<tr>
<td>25.0</td>
<td>1.5106</td>
<td>$1.064 \times 10^{-2}$</td>
</tr>
<tr>
<td>26.0</td>
<td>1.625</td>
<td>0.125</td>
</tr>
</tbody>
</table>

**Table 2a: Ex. 3.1:** Comparison of the numerical value for $C_0$ with the exact value $C_0 = 1.5$ at different $\epsilon$. The parameter values in (3.24) are $b = 1$, $s = 0$, $\mu = 0$.

<table>
<thead>
<tr>
<th>$z \equiv \epsilon^{-1}(b - 1)$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (3.25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.00</td>
<td>1.99998234</td>
<td>1.99998238</td>
</tr>
<tr>
<td>-0.50</td>
<td>1.82036</td>
<td>1.82055</td>
</tr>
<tr>
<td>-0.20</td>
<td>1.64914</td>
<td>1.64870</td>
</tr>
<tr>
<td>0.00</td>
<td>1.50030</td>
<td>1.50000</td>
</tr>
<tr>
<td>0.10</td>
<td>1.42250</td>
<td>1.42328</td>
</tr>
<tr>
<td>0.50</td>
<td>1.17248</td>
<td>1.17302</td>
</tr>
<tr>
<td>1.00</td>
<td>1.04185</td>
<td>1.04019</td>
</tr>
</tbody>
</table>

**Table 2b: Ex. 3.1:** Comparison of the $C_0$ values as $b$ is varied.

In (3.24) we set $s = 0$, $\mu = 1$, and $\epsilon = 1/36$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (3.26b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>1.76707</td>
<td>1.76492</td>
</tr>
<tr>
<td>11.0</td>
<td>1.75670</td>
<td>1.75574</td>
</tr>
<tr>
<td>15.0</td>
<td>1.74945</td>
<td>1.74899</td>
</tr>
<tr>
<td>18.0</td>
<td>1.74608</td>
<td>1.74578</td>
</tr>
<tr>
<td>20.0</td>
<td>1.74432</td>
<td>1.74413</td>
</tr>
<tr>
<td>23.0</td>
<td>1.74302</td>
<td>1.74214</td>
</tr>
</tbody>
</table>

**Table 2c: Ex. 3.1:** Comparison of the $C_0$ values as $\epsilon$ is varied.

In (3.24) we set $s = 1$, $\mu = 0$, $b = 1$, and $d = \pi/2$.  

32
\begin{table}[h]
\begin{center}
\begin{tabular}{|l|c|c|}
\hline
$z \equiv \epsilon^{-1}(d - \omega_1)$ & $C_0$ (num.) & $C_0$ (3.26b) \\
\hline
-4.00 & -0.22561 & -0.22583 \\
-2.25 & -4.55763 & -4.57322 \\
-1.75 & 7.73796 & 7.71092 \\
-1.00 & 2.42249 & 2.42122 \\
0.00 & 1.74432 & 1.74413 \\
1.00 & 1.58150 & 1.58143 \\
2.00 & 1.52900 & 1.52896 \\
4.00 & 1.50382 & 1.50386 \\
\hline
\end{tabular}
\end{center}
\end{table}

**Table 2d: Ex. 3.1:** Comparison of the $C_0$ values as $d$ is varied.
In (3.24) we set $s = 1, \mu = 0, b = 1$, and $\epsilon = 1/20$.

\begin{table}[h]
\begin{center}
\begin{tabular}{|l|c|c|}
\hline
$z \equiv \epsilon^{-1}(d - \omega_b)$ & $C_0$ (num.) & $C_0$ (3.26a) \\
\hline
-4.00 & -0.05636 & -0.05653 \\
-1.00 & -2.44963 & -2.46578 \\
-0.85 & -3.54966 & -3.57891 \\
-0.70 & -5.78603 & -5.85314 \\
-0.20 & 10.88045 & 10.74001 \\
0.00 & 6.02836 & 5.99282 \\
0.20 & 4.41603 & 4.40038 \\
1.00 & 2.65192 & 2.64938 \\
4.00 & 2.02478 & 2.02471 \\
\hline
\end{tabular}
\end{center}
\end{table}

**Table 2e: Ex. 3.1:** Comparison of the $C_0$ values as $d$ is varied.
In (3.24) we set $s = 1, \mu = 0, b = 3/4$, and $\epsilon = 1/36$.

\begin{table}[h]
\begin{center}
\begin{tabular}{|l|c|c|}
\hline
$1/\epsilon$ & $C_0$ (num.) & $C_0$ (3.29) \\
\hline
11.0 & 3.16129 & 2.83985 \\
18.0 & 2.75742 & 2.59345 \\
20.0 & 2.69740 & 2.55946 \\
22.0 & 2.64942 & 2.53248 \\
26.0 & 2.57865 & 2.49235 \\
28.0 & 2.54913 & 2.47702 \\
\hline
\end{tabular}
\end{center}
\end{table}

**Table 3a: Ex. 3.2:** Comparison of the $C_0$ values as $\epsilon$ is varied.
In (3.28) we set $b = 2$ and $d = 4/3$. 

\[
\begin{array}{|c|c|c|}
\hline
z \equiv \epsilon^{-1}(d - \omega_b) & C_0 \text{ (num.)} & C_0 \text{ (3.30)} \\
\hline
-3.0 & -0.00876 & -0.00826 \\
0.0 & -0.18445 & -0.18003 \\
1.0 & -0.59487 & -0.57891 \\
2.0 & -3.30333 & -3.13076 \\
3.0 & 4.88651 & 5.03637 \\
4.0 & 2.55533 & 2.57000 \\
6.0 & 2.06061 & 2.06189 \\
\hline
\end{array}
\]

Table 3b: Ex. 3.2: Comparison of the \(C_0\) values as \(d\) is varied.

In (3.28) we set \(b = 1\) and \(\epsilon = 1/60\).

\[
\begin{array}{|c|c|c|c|}
\hline
1/\epsilon & \lambda_N \text{ (num.)} & \lambda_N \text{ (4.13) 2 term} & \lambda_N \text{ (4.13) 1 term} \\
\hline
15.0 & -1.991931677513 \times 10^{-2} & -2.0510 \times 10^{-2} & -2.5637 \times 10^{-2} \\
25.0 & -3.218742898 \times 10^{-4} & -3.2708 \times 10^{-4} & -3.7168 \times 10^{-4} \\
35.0 & -3.76665870 \times 10^{-6} & -3.7929 \times 10^{-6} & -4.1485 \times 10^{-6} \\
45.0 & -3.788847 \times 10^{-8} & -3.8034 \times 10^{-8} & -4.0751 \times 10^{-8} \\
55.0 & -3.4992 \times 10^{-10} & -3.5077 \times 10^{-10} & -3.7101 \times 10^{-10} \\
65.0 & -3.061 \times 10^{-12} & -3.0635 \times 10^{-12} & -3.2117 \times 10^{-12} \\
70.0 & -2.813 \times 10^{-13} & -2.8201 \times 10^{-13} & -2.9463 \times 10^{-13} \\
75.0 & -2.8 \times 10^{-14} & -2.5749 \times 10^{-14} & -2.6822 \times 10^{-14} \\
\hline
\end{array}
\]

Table 4a: Comparison of asymptotic and numerical values for \(\lambda_N\)
of the Hermite operator with \(N = 1\) and \(a = b = 1\).

\[
\begin{array}{|c|c|c|c|}
\hline
1/\epsilon & \lambda_N \text{ (num.)} & \lambda_N \text{ (4.13) 2 term} & \lambda_N \text{ (4.13) 1 term} \\
\hline
15.0 & -1.0949321155467 \times 10^{-1} & -1.0255 \times 10^{-1} & -1.9228 \times 10^{-1} \\
25.0 & -3.28048865792 \times 10^{-3} & -3.3451 \times 10^{-3} & -4.6460 \times 10^{-3} \\
35.0 & -5.765094351 \times 10^{-5} & -5.8079 \times 10^{-5} & -7.2598 \times 10^{-5} \\
45.0 & -7.7162465 \times 10^{-7} & -7.7426 \times 10^{-7} & -9.1689 \times 10^{-7} \\
55.0 & -8.88700 \times 10^{-9} & -8.9042 \times 10^{-9} & -1.0203 \times 10^{-8} \\
65.0 & -9.3026 \times 10^{-11} & -9.3140 \times 10^{-11} & -1.0438 \times 10^{-10} \\
70.0 & -9.272 \times 10^{-12} & -9.2809 \times 10^{-12} & -1.0312 \times 10^{-11} \\
75.0 & -9.10 \times 10^{-13} & -9.1194 \times 10^{-13} & -1.0058 \times 10^{-12} \\
\hline
\end{array}
\]

Table 4b: Comparison of asymptotic and numerical values for \(\lambda_N\)
of the Hermite operator with \(N = 2\) and \(a = b = 1\).
\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$z \equiv \epsilon^{-1}(b-1)$ & $C_0$ (num.) & $C_0$ (4.26) \\
\hline
-20.0 & 2.999999155 & 2.999999165 \\
-10.0 & 2.3993701 & 2.3993718 \\
-4.0 & 2.06334 & 2.06339 \\
-2.0 & 1.65588 & 1.65602 \\
-0.5 & 0.85775 & 0.85787 \\
0.0 & 0.49993 & 0.50000 \\
0.5 & 0.14319 & 0.14303 \\
1.0 & -0.17670 & -0.17668 \\
3.0 & -0.85139 & -0.85188 \\
10.0 & -0.99989 & -0.99991 \\
\hline
\end{tabular}
\end{center}
\caption{Table 5a: Ex. 4.1: Comparison of the $C_0$ values as $b$ is varied.}
\end{table}

In (4.24) we set $s = 0$ and $\epsilon = 1/60$.

\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$1/\epsilon$ & $C_0$ (num.) & $C_0$ (4.29) \\
\hline
15.0 & 0.30022 & 0.30769 \\
25.0 & 0.31690 & 0.31884 \\
35.0 & 0.32244 & 0.32323 \\
45.0 & 0.32515 & 0.32558 \\
55.0 & 0.32677 & 0.32704 \\
60.0 & 0.32737 & 0.32759 \\
\hline
\end{tabular}
\end{center}
\caption{Table 5b: Ex. 4.1: Comparison of the $C_0$ values as $\epsilon$ is varied.}
\end{table}

In (4.24) we set $s = 1$, $b = 1$, and $d = 1/2$.

\begin{table}
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$1/\epsilon$ & $C_0$ (num.) & $C_0$ (4.28) \\
\hline
30.0 & 0.65386 & 0.68677 \\
50.0 & 0.72339 & 0.72991 \\
70.0 & 0.74502 & 0.74783 \\
90.0 & 0.75606 & 0.75764 \\
100.0 & 0.75980 & 0.76105 \\
110.0 & 0.76330 & 0.76384 \\
\hline
\end{tabular}
\end{center}
\caption{Table 5c: Ex. 4.1: Comparison of the $C_0$ values as $\epsilon$ is varied.}
\end{table}

In (4.24) we set $s = 1$, $b = 3/4$, and $d = b^2/2$. 

35
\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$z \equiv \epsilon^{-1} (d - 1/2)$ & $C_0$ (num.) & $C_0$ (4.29) \\
\hline
-5.0 & 0.00631 & 0.00632 \\
-2.0 & 0.10210 & 0.10227 \\
-1.0 & 0.20542 & 0.20570 \\
-0.5 & 0.26740 & 0.26770 \\
0.0 & 0.32737 & 0.32759 \\
0.5 & 0.37889 & 0.37901 \\
1.0 & 0.41873 & 0.41889 \\
2.0 & 0.46661 & 0.46675 \\
5.0 & 0.49813 & 0.49823 \\
\hline
\end{tabular}
\end{center}

**Table 5d: Ex. 4.1:** Comparison of the $C_0$ values as $d$ is varied.

In (4.24) we set $s = 1$, $b = 1$, and $\epsilon = 1/60$.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$z \equiv \epsilon^{-1} (d - b^2/2)$ & $C_0$ (num.) & $C_0$ (4.28) \\
\hline
-5.0 & 0.00714 & 0.00716 \\
-2.0 & 0.13644 & 0.13674 \\
-1.0 & 0.34091 & 0.34160 \\
-0.5 & 5.19023 & 5.19999 \\
0.0 & 7.59801 & 7.6105 \\
0.5 & 1.05726 & 1.05875 \\
1.0 & 1.38657 & 1.38807 \\
2.0 & 1.99057 & 1.99173 \\
3.0 & 2.37047 & 2.37108 \\
5.0 & 2.62235 & 2.62242 \\
\hline
\end{tabular}
\end{center}

**Table 5e: Ex. 4.1:** Comparison of the $C_0$ values as $d$ is varied.

In (4.24) we set $s = 1$, $b = 3/4$, and $\epsilon = 1/100$. 

36
<table>
<thead>
<tr>
<th>$z \equiv \epsilon^{-1}(b - 1)$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (4.31)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10.0</td>
<td>3.22489</td>
<td>3.24908</td>
</tr>
<tr>
<td>-5.0</td>
<td>2.50892</td>
<td>2.52004</td>
</tr>
<tr>
<td>-3.0</td>
<td>2.22446</td>
<td>2.23261</td>
</tr>
<tr>
<td>-1.0</td>
<td>1.80284</td>
<td>1.80833</td>
</tr>
<tr>
<td>0.0</td>
<td>1.53061</td>
<td>1.53488</td>
</tr>
<tr>
<td>0.5</td>
<td>1.40188</td>
<td>1.40567</td>
</tr>
<tr>
<td>1.0</td>
<td>1.29115</td>
<td>1.29457</td>
</tr>
<tr>
<td>3.0</td>
<td>1.06665</td>
<td>1.06953</td>
</tr>
<tr>
<td>10.0</td>
<td>1.02043</td>
<td>1.02327</td>
</tr>
</tbody>
</table>

**Table 6a: Ex. 4.2: Comparison of the $C_0$ values as $b$ is varied.**
In (4.30) we set $s = 0$ and $\epsilon = 1/60$.  

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (4.33)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0</td>
<td>11.26078</td>
<td>11.72932</td>
</tr>
<tr>
<td>30.0</td>
<td>5.29738</td>
<td>5.28736</td>
</tr>
<tr>
<td>40.0</td>
<td>4.30928</td>
<td>4.32079</td>
</tr>
<tr>
<td>50.0</td>
<td>3.9205</td>
<td>3.92933</td>
</tr>
<tr>
<td>60.0</td>
<td>3.71039</td>
<td>3.71712</td>
</tr>
<tr>
<td>70.0</td>
<td>3.57790</td>
<td>3.58391</td>
</tr>
</tbody>
</table>

**Table 6b: Ex. 4.2: Comparison of the $C_0$ values as $\epsilon$ is varied.**
In (4.30) we set $s = 1$, $b = 1$, and $d = 1/2$.  

<table>
<thead>
<tr>
<th>$z \equiv \epsilon^{-1}(d - 1/2)$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (4.33)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6.0</td>
<td>-0.00645</td>
<td>-0.00644</td>
</tr>
<tr>
<td>-1.0</td>
<td>-2.53919</td>
<td>-2.53073</td>
</tr>
<tr>
<td>-0.8</td>
<td>-4.91151</td>
<td>-4.88559</td>
</tr>
<tr>
<td>-0.7</td>
<td>-8.20719</td>
<td>-8.14192</td>
</tr>
<tr>
<td>-0.4</td>
<td>12.55606</td>
<td>12.67107</td>
</tr>
<tr>
<td>-0.3</td>
<td>7.43801</td>
<td>7.47417</td>
</tr>
<tr>
<td>-0.2</td>
<td>5.43367</td>
<td>5.45119</td>
</tr>
<tr>
<td>0.0</td>
<td>3.71039</td>
<td>3.71712</td>
</tr>
<tr>
<td>1.0</td>
<td>1.94728</td>
<td>1.94795</td>
</tr>
<tr>
<td>6.0</td>
<td>1.52765</td>
<td>1.52765</td>
</tr>
</tbody>
</table>

**Table 6c: Ex. 4.2: Comparison of the $C_0$ values as $d$ is varied.**
In (4.30) we set $s = 1$, $b = 1$, and $\epsilon = 1/60$.  

37
<table>
<thead>
<tr>
<th>$z \equiv \epsilon^{-1} (d - b^2/2)$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (4.32)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.40</td>
<td>-0.01348</td>
<td>-0.01349</td>
</tr>
<tr>
<td>-1.0</td>
<td>-0.29143</td>
<td>-0.29169</td>
</tr>
<tr>
<td>0.0</td>
<td>-0.91938</td>
<td>-0.92033</td>
</tr>
<tr>
<td>0.5</td>
<td>-1.81480</td>
<td>-1.81707</td>
</tr>
<tr>
<td>1.0</td>
<td>-4.43427</td>
<td>-4.44246</td>
</tr>
<tr>
<td>2.0</td>
<td>10.90977</td>
<td>10.89166</td>
</tr>
<tr>
<td>2.5</td>
<td>6.08675</td>
<td>6.08333</td>
</tr>
<tr>
<td>3.0</td>
<td>4.79979</td>
<td>4.79847</td>
</tr>
<tr>
<td>4.0</td>
<td>3.97981</td>
<td>3.97948</td>
</tr>
<tr>
<td>10.0</td>
<td>3.62072</td>
<td>3.62072</td>
</tr>
</tbody>
</table>

**Table 6d: Ex. 4.2:** Comparison of the $C_0$ values as $d$ is varied.

In (4.30) we set $s = 1$, $b = 3/4$, and $\epsilon = 1/100$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$C_0$ (num.)</th>
<th>$C_0$ (5.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.0</td>
<td>2.60703</td>
<td>2.66667</td>
</tr>
<tr>
<td>30.0</td>
<td>2.65035</td>
<td>2.60870</td>
</tr>
<tr>
<td>40.0</td>
<td>2.60159</td>
<td>2.57576</td>
</tr>
<tr>
<td>50.0</td>
<td>2.57422</td>
<td>2.55814</td>
</tr>
<tr>
<td>60.0</td>
<td>2.55818</td>
<td>2.54717</td>
</tr>
<tr>
<td>75.0</td>
<td>2.54413</td>
<td>2.53676</td>
</tr>
</tbody>
</table>

**Table 7a: Ex. 5.1:** Comparison of the $C_0$ values as $\epsilon$ is varied.

In (5.6) we set $b = 1$ and $d = 1/2$.  

38
\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$z \equiv \epsilon^{-1}(d - 1/2)$ & $C_0$ (num.) & $C_0$ (5.6) \\
\hline
-2.0 & 9.15647 & 9.05677 \\
-1.5 & 6.15388 & 6.09455 \\
-1.0 & 4.33274 & 4.29787 \\
-0.5 & 3.22815 & 3.20813 \\
0.0 & 2.55818 & 2.54717 \\
1.0 & 1.90537 & 1.90312 \\
2.0 & 1.66521 & 1.66619 \\
3.0 & 1.57685 & 1.57903 \\
4.0 & 1.54435 & 1.54696 \\
\hline
\end{tabular}
\caption{Ex. 5.1: Comparison of the $C_0$ values as $d$ is varied. In (5.6) we set $b = 1$ and $\epsilon = 1/60$.}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$z \equiv \epsilon^{-1}(d - 4)$ & $C_0$ (num.) & $C_0$ (5.15b) \\
\hline
-4.0 & 8.52468 & 8.68957 \\
-3.0 & 3.13605 & 3.19671 \\
-2.0 & 1.15369 & 1.17600 \\
-1.0 & 0.42442 & 0.43263 \\
0.0 & 0.15613 & 0.15915 \\
0.5 & $9.4701 \times 10^{-2}$ & $9.6532 \times 10^{-2}$ \\
1.0 & $5.7439 \times 10^{-2}$ & $5.8550 \times 10^{-2}$ \\
2.0 & $2.1131 \times 10^{-2}$ & $2.1539 \times 10^{-2}$ \\
3.0 & $7.7735 \times 10^{-3}$ & $7.9239 \times 10^{-3}$ \\
4.0 & $2.8597 \times 10^{-3}$ & $2.9150 \times 10^{-3}$ \\
\hline
\end{tabular}
\caption{Ex. in §5.2: For (5.14), (5.15b) we compare $C_0$ values as $d$ is varied near $d = \gamma = 4$. In (5.14), (5.15b), $g(x) = 1$, $\nu = 0$, $\epsilon = 1/8$, and $y_0(x)$ is of type A.}
\end{table}
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$1/\epsilon$ & $\lambda_0$ (num.) & $\lambda_0$ (asy.) (6.8) \\
\hline
10.0 & $-6.771209410691 \times 10^{-2}$ & $-6.737946999085 \times 10^{-2}$ \\
20.0 & $-9.0801947630 \times 10^{-4}$ & $-9.0799859525 \times 10^{-4}$ \\
30.0 & $-9.1770710 \times 10^{-6}$ & $-9.17706962 \times 10^{-6}$ \\
40.0 & $-8.244615 \times 10^{-8}$ & $-8.244615 \times 10^{-8}$ \\
50.0 & $-6.94399 \times 10^{-10}$ & $-6.94397 \times 10^{-10}$ \\
60.0 & $-5.633 \times 10^{-12}$ & $-5.615 \times 10^{-12}$ \\
\hline
\end{tabular}
\caption{Comparison of asymptotic and numerical values for $\lambda_0$ of (6.3) for Burgers equation with $\alpha = 1$, $\kappa_r = \kappa_l = 2.0$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$1/\epsilon$ & $\lambda_0$ (num.) & $\lambda_0$ (asy.) (6.9) \\
\hline
12.0 & $-2.9203067484 \times 10^{-4}$ & $-2.9492219296 \times 10^{-4}$ \\
18.0 & $-1.09601753 \times 10^{-6}$ & $-1.09655854 \times 10^{-6}$ \\
24.0 & $-3.62404 \times 10^{-9}$ & $-3.62413 \times 10^{-9}$ \\
26.0 & $-5.3134 \times 10^{-10}$ & $-5.3135 \times 10^{-10}$ \\
28.0 & $-7.7438 \times 10^{-11}$ & $-7.7441 \times 10^{-11}$ \\
30.0 & $-1.1222 \times 10^{-11}$ & $-1.1229 \times 10^{-11}$ \\
34.0 & $-2.41 \times 10^{-13}$ & $-2.33 \times 10^{-13}$ \\
\hline
\end{tabular}
\caption{Comparison of asymptotic and numerical values for $\lambda_0$ of (6.3) for Burgers equation with $\alpha = 1$, $\kappa_r = \kappa_l = 1.0$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$1/\epsilon$ & $\lambda_0$ (num.) & $\lambda_0$ (asy.) (6.10) \\
\hline
15.0 & $-1.17355083586999 \times 10^{-2}$ & $-1.17326912609947 \times 10^{-2}$ \\
35.0 & $-1.24288119 \times 10^{-6}$ & $-1.24288117 \times 10^{-6}$ \\
45.0 & $-1.076715 \times 10^{-8}$ & $-1.076717 \times 10^{-8}$ \\
50.0 & $-9.8212 \times 10^{-10}$ & $-9.8203 \times 10^{-10}$ \\
55.0 & $-8.865 \times 10^{-11}$ & $-8.867 \times 10^{-11}$ \\
60.0 & $-7.96 \times 10^{-12}$ & $-7.94 \times 10^{-12}$ \\
\hline
\end{tabular}
\caption{Comparison of asymptotic and numerical values for $\lambda_0$ of (6.3) for Burgers equation with $\alpha = 1$, $\kappa_l = \infty$, $\kappa_r = 2.0$.}
\end{table}
Table 8d: Comparison of asymptotic and numerical values for $\lambda_0$ of (6.3) for Burgers equation with $\alpha = 1$, $\kappa_l = \infty$, $\kappa_r = 1.0$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\lambda_0$ (num.)</th>
<th>$\lambda_0$ (asy.) (6.11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0</td>
<td>-9.697029317 $\times 10^{-5}$</td>
<td>-9.717580754 $\times 10^{-5}$</td>
</tr>
<tr>
<td>25.0</td>
<td>-4.33157634 $\times 10^{-6}$</td>
<td>-4.3331139 $\times 10^{-6}$</td>
</tr>
<tr>
<td>30.0</td>
<td>-1.8548979 $\times 10^{-7}$</td>
<td>-1.8550383 $\times 10^{-7}$</td>
</tr>
<tr>
<td>35.0</td>
<td>-7.72049 $\times 10^{-9}$</td>
<td>-7.72061 $\times 10^{-9}$</td>
</tr>
<tr>
<td>40.0</td>
<td>-3.14769 $\times 10^{-10}$</td>
<td>-3.14771 $\times 10^{-10}$</td>
</tr>
<tr>
<td>45.0</td>
<td>-1.2629 $\times 10^{-11}$</td>
<td>-1.2633 $\times 10^{-11}$</td>
</tr>
<tr>
<td>50.0</td>
<td>-4.9 $\times 10^{-13}$</td>
<td>-5.0 $\times 10^{-13}$</td>
</tr>
</tbody>
</table>

Table 8e: Comparison of asymptotic and numerical values for $\lambda_0$ of (6.14). The asymptotic result is $\lambda_0 \sim 96 \epsilon^{-2/5}$.

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\lambda_0$ (num.)</th>
<th>$\lambda_0$ (asy.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.0</td>
<td>7.952873335 $\times 10^{-5}$</td>
<td>7.982675703 $\times 10^{-5}$</td>
</tr>
<tr>
<td>9.0</td>
<td>1.46135344 $\times 10^{-6}$</td>
<td>1.46207805 $\times 10^{-6}$</td>
</tr>
<tr>
<td>11.0</td>
<td>2.677719 $\times 10^{-8}$</td>
<td>2.677889 $\times 10^{-8}$</td>
</tr>
<tr>
<td>13.0</td>
<td>4.9043 $\times 10^{-10}$</td>
<td>4.9047 $\times 10^{-10}$</td>
</tr>
<tr>
<td>14.0</td>
<td>6.634 $\times 10^{-12}$</td>
<td>6.638 $\times 10^{-11}$</td>
</tr>
<tr>
<td>15.0</td>
<td>9.02 $\times 10^{-12}$</td>
<td>8.98 $\times 10^{-12}$</td>
</tr>
<tr>
<td>16.0</td>
<td>1.18 $\times 10^{-12}$</td>
<td>1.22 $\times 10^{-12}$</td>
</tr>
</tbody>
</table>

Table 9a: Comparison of asymptotic and numerical values for $\Delta \lambda = \lambda_0 - \lambda_1$ of (7.2) with $\alpha = 1$, $A_l = A_r = 0$. The asymptotic result is (7.3).

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\Delta \lambda$ (num.)</th>
<th>$\Delta \lambda$ (asy.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.0</td>
<td>1.659387897453 $\times 10^{-2}$</td>
<td>1.659253110444 $\times 10^{-2}$</td>
</tr>
<tr>
<td>25.0</td>
<td>1.8633266072 $\times 10^{-4}$</td>
<td>1.8633265860 $\times 10^{-4}$</td>
</tr>
<tr>
<td>35.0</td>
<td>1.75769941 $\times 10^{-6}$</td>
<td>1.75769941 $\times 10^{-6}$</td>
</tr>
<tr>
<td>45.0</td>
<td>1.522708 $\times 10^{-8}$</td>
<td>1.522708 $\times 10^{-8}$</td>
</tr>
<tr>
<td>55.0</td>
<td>1.253989 $\times 10^{-10}$</td>
<td>1.253991 $\times 10^{-10}$</td>
</tr>
<tr>
<td>65.0</td>
<td>9.991 $\times 10^{-13}$</td>
<td>9.986 $\times 10^{-13}$</td>
</tr>
<tr>
<td>$1/\epsilon$</td>
<td>$\Delta \lambda$ (num.)</td>
<td>$\Delta \lambda$ (asy.)</td>
</tr>
<tr>
<td>-------------</td>
<td>-------------------------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>15.0</td>
<td>$1.712877958476 \times 10^{-2}$</td>
<td>$1.710318958498 \times 10^{-2}$</td>
</tr>
<tr>
<td>25.0</td>
<td>$1.9207082608 \times 10^{-4}$</td>
<td>$1.9206730823 \times 10^{-4}$</td>
</tr>
<tr>
<td>35.0</td>
<td>$1.81179541 \times 10^{-8}$</td>
<td>$1.81179508 \times 10^{-8}$</td>
</tr>
<tr>
<td>45.0</td>
<td>$1.569572 \times 10^{-8}$</td>
<td>$1.569572 \times 10^{-6}$</td>
</tr>
<tr>
<td>55.0</td>
<td>$1.29257 \times 10^{-10}$</td>
<td>$1.29258 \times 10^{-10}$</td>
</tr>
<tr>
<td>65.0</td>
<td>$1.0289 \times 10^{-12}$</td>
<td>$1.0293 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

**Table 9b:** Comparison of asymptotic and numerical values for $\Delta \lambda = \lambda_0 - \lambda_1$ of (7.2) with $\alpha = A_l = 1$, $A_r = 0$, $c_l = 1/2$. The asymptotic result is (7.3).

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\Delta \lambda$ (num.)</th>
<th>$\Delta \lambda$ (asy.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>$-1.4586846337 \times 10^{-2}$</td>
<td>$-1.4585545566 \times 10^{-2}$</td>
</tr>
<tr>
<td>4.5</td>
<td>$-7.43390203 \times 10^{-4}$</td>
<td>$-7.433900041 \times 10^{-4}$</td>
</tr>
<tr>
<td>5.0</td>
<td>$-2.16014875 \times 10^{-5}$</td>
<td>$-2.16014875 \times 10^{-4}$</td>
</tr>
<tr>
<td>5.5</td>
<td>$-3.646034 \times 10^{-7}$</td>
<td>$-3.646032 \times 10^{-7}$</td>
</tr>
<tr>
<td>6.0</td>
<td>$-3.61599 \times 10^{-9}$</td>
<td>$-3.61596 \times 10^{-9}$</td>
</tr>
<tr>
<td>6.5</td>
<td>$-2.14 \times 10^{-11}$</td>
<td>$-2.12 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

**Table 9c:** Comparison of asymptotic and numerical values for $\Delta \lambda = \lambda_1 - \lambda_0$ of (7.4) with $F = 1$. The asymptotic result is (7.5).

<table>
<thead>
<tr>
<th>$1/\epsilon$</th>
<th>$\Delta \lambda$ (num.)</th>
<th>$\Delta \lambda$ (asy.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.0</td>
<td>$1.798097341 \times 10^{-4}$</td>
<td>$2.02453234 \times 10^{-4}$</td>
</tr>
<tr>
<td>9.0</td>
<td>$9.7353845 \times 10^{-6}$</td>
<td>$1.0695192 \times 10^{-5}$</td>
</tr>
<tr>
<td>10.0</td>
<td>$3.613396 \times 10^{-7}$</td>
<td>$3.901407 \times 10^{-7}$</td>
</tr>
<tr>
<td>11.0</td>
<td>$9.2370 \times 10^{-9}$</td>
<td>$9.8467 \times 10^{-9}$</td>
</tr>
<tr>
<td>12.0</td>
<td>$1.631 \times 10^{-10}$</td>
<td>$1.722 \times 10^{-10}$</td>
</tr>
<tr>
<td>13.0</td>
<td>$1.99 \times 10^{-12}$</td>
<td>$2.09 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

**Table 9d:** Comparison of asymptotic and numerical values for $\Delta \lambda = \lambda_1 - \lambda_0$ of (7.12) with $V_\epsilon$ given in (7.14). The asymptotic result is (7.13).
Figure 1: (Example 3.1 with $s = 0, b = 1$) The accuracy loss of $C_0$ as $\lambda_0 \to 0$ is plotted. The solid lines represent the computed $\lambda_0(\epsilon)$ and the dotted lines show the numerical computation error, $C_0(\epsilon) - 3/2$. The dashed lines at the bottom of the figures show the numerical error of $C_0$ multiplied by $\lambda_0$. Numerical values for $C_0 - 3/2$ for the case $\mu = 0$ are given in Table 2a. Numerical values for $\lambda_0$ are given in Table 1b ($\mu = 1$) and Table 1c ($\mu = 0$).
Figure 2: (Example 3.1 with $s = 1, \mu = 0, b = 1, \omega_1 = \pi/2, \epsilon = 1/20$) The solid curve in the leftmost figure shows the asymptotic result for $C_0 = C_0(d)$ obtained from (3.26b). The dotted points in this figure are the numerically computed values of $C_0(d)$ (see also Table 2d). Numerically computed solutions to (3.24) at a few selected values of $d$ are plotted in the rightmost figure. The two * marks in the rightmost figure denote the boundary data.
Figure 3: (Example 3.3 with \( \alpha = -1, \beta = 1, \epsilon = 1/60 \)) The shock layer solution to (3.32) is plotted for several values of \( x(1) \) for \( x(1) \) near one. Notice that \( O(\epsilon) \) changes in \( x(1) \) causes the shock layer location to move by \( O(1) \).
Figure 4: (Example 4.1 with s = 0) The solids curves show $C_0 = C_0(b)$, obtained from the asymptotic result (4.26), for three different values of $\epsilon$. The dotted points are the numerically computed values of $C_0$ at a few values of $b$ for $\epsilon = 1/60$ (see also Table 5a).
Figure 5: (Example 4.1 with $s = 1, b = 1, \omega_1 = 1/2, \epsilon = 1/60$) The solid curve in the leftmost figure shows the asymptotic result for $C_0 = C_0(d)$ obtained from (4.29). The dotted points in this figure are the numerically computed values of $C_0(d)$ (see also Table 5d). Numerically computed solutions to (4.24) at a few selected values of $d$ are plotted in the rightmost figure. The two * marks in the rightmost figure denote the boundary data.
Figure 6: (Example 4.2 with \( s = 1, b = 3/4, \omega_{3/4} = 9/32, \epsilon = 1/100 \)) The solid curve in the leftmost figure shows the asymptotic result for \( C_0 = C_0(d) \) obtained from (4.32). The dotted points in this figure are the numerically computed values of \( C_0(d) \) (see also Table 6d). Numerically computed solutions to (4.30) at a few selected values of \( d \) are plotted in the rightmost figure. The two * marks in the rightmost figure denote the boundary data.
Figure 7: (Example 5.1 with $a = 1, b = 1, \nu = -5/2, \epsilon = 1/60$) The solid curve in the leftmost figure shows the asymptotic result for $C_0 = C_0(d)$ obtained from (5.6). The dotted points in this figure are the numerically computed values of $C_0(d)$ (see also Table 7b). Numerically computed solutions to (5.1) at a few selected values of $d$ are plotted in the rightmost figure. The two * marks in the rightmost figure denote the boundary data.
Figure 8: (Example in Section 5.2 with \( \nu = 0, g(x) = 1, \epsilon = 1/8 \)) The solid curve in the leftmost figure shows the asymptotic result for \( C_0 = C_0(d) \) obtained from (5.15b). The dotted points in this figure are the numerically computed values of \( C_0(d) \) (see also Table 7c). Numerically computed solutions \( u(x) \) to (5.14) (dashed curves) and \( v(x) = y - y_0 = u(x)w(x) \) (solid curves) at a few selected values of \( d \) are plotted in the rightmost figure. The two * marks in the rightmost figure denote the boundary data of \( u(x) \).