Transition to Blow-up in a Reaction-Diffusion Model with Localized Spike Solutions

V. ROTTSCHÄFER, J. C. TZOU, M. J. WARD

Vivi Rottschäfer; Mathematical Institute, Leiden University, P.O. Box 9512, Leiden, the Netherlands.
Justin C. Tzou, Michael J. Ward; Dept. of Mathematics, University of British Columbia, Vancouver, British Columbia, V6T 1Z2, Canada.

(Received 8 July 2016)

For certain singularly perturbed two-component reaction-diffusion (RD) systems, the bifurcation diagram of steady-state spike solutions is characterized by a saddle-node behavior in terms of some parameter $\beta$ in the system. For some such systems, such as the Gray-Scott model, a spike self-replication behavior is observed as a parameter varies across the saddle-node point. We demonstrate and analyze a qualitatively new type of transition as a parameter is slowly decreased below the saddle node value, which is characterized by a finite-time blow-up of the spike solution. More specifically, we use a blend of asymptotic analysis, linear stability theory, and full numerical computations to analyze a wide variety of dynamical instabilities, and ultimately a finite-time blow-up behavior, for localized spike solutions that occur as a parameter $\beta$ is slowly ramped in time below various linear stability and existence thresholds associated with steady-state spike solutions. The transition or route to an ultimate finite-time blow-up can include spike nucleation, spike annihilation, or spike amplitude oscillation, depending on the specific parameter regime. Our detailed analysis of the existence and linear stability of multi-spike patterns, through the analysis of an explicitly solvable nonlocal eigenvalue problem, provides a theoretical guide for predicting which transition will be realized. Finally, we analyze the blow-up profile for a shadow limit of the RD system. For the resulting nonlocal scalar parabolic problem, we derive an explicit formula for the blow-up rate near the parameter range where blow-up is predicted. This blow-up rate is confirmed with full numerical simulations.

1 Introduction

In the singularly perturbed limit of small diffusivity, certain two-component reaction-diffusion (RD) systems in a one-dimensional spatial domain can support the existence of localized spike-type solutions, whereby one of the two components becomes localized around certain points in the domain. A prototypical such system admitting spike patterns is the Gierer-Meinhardt (GM) model (cf. [12]). There is now a rather well-developed theory for the existence, linear stability, and dynamics of localized spike patterns for many two-component RD systems in 1-D (see [4], [5], [6], [14], [13], [31] for some of the earlier studies). A much more extensive bibliography is given in [22].

For some such RD systems the bifurcation diagram of steady-state spike solutions is characterized by a saddle-node behavior in terms of some parameter $\beta$ in the system. In particular, for the well-known Gray-Scott (GS) RD model transitions in a feed-rate parameter past some critical saddle-node bifurcation point are known to lead to spike self-replication behavior, whereby a spike will dynamically split into two. Further transitions in the feed-rate parameter across other saddle-node points of multi-spike equilibria lead to the subsequent generation of additional spikes through a dynamical splitting phenomena (cf. [24], [4], [7], [16], [15]).

The goal of this paper is to analyze spike solutions for a RD model that exhibits a qualitatively new type of transition as a bifurcation parameter crosses a saddle-node value. More specifically, we will analyze the linear stability and transition
to finite-time blow-up behavior of localized spike solutions to the RD system

\[
v_t = \varepsilon^2 v_{xx} + v - v u + v^3, \quad |x| \leq 1, \quad t > 0; \quad v_x(\pm 1, t) = 0, \quad (1.1a)
\]

\[
\tau u_t = D u_{xx} - u + 2\varepsilon^{-1} \beta v^3, \quad |x| \leq 1, \quad t > 0; \quad u_x(\pm 1, t) = 0. \quad (1.1b)
\]

In (1.1), \(0 < \varepsilon \ll 1, D > 0, \beta > 0, \) and \(\tau \geq 0,\) are constants, with \(D = O(1)\) as \(\varepsilon \to 0.\) A very similar system, where \(\beta v^3\) is replaced by \(\beta v^2\) in (1.1b), was considered briefly in §5 of [16] where a blow-up behavior of localized spike solutions was conjectured. A related system, representing a variant of the GM model, was also conjectured in [6], based on numerical evidence, to lead to a finite-time blow-up of the spike amplitude.

In the shadow limit where \(D \gg 1\) and \(\tau = O(1),\) (1.1) reduces to the following PDE-ODE model for \(v(x,t)\) and \(u(t)\):

\[
v_t = \varepsilon^2 v_{xx} - (u - 1)v + v^3, \quad |x| \leq 1, \quad t > 0; \quad v_x(\pm 1, t) = 0, \quad (1.2a)
\]

\[
\tau \frac{d u}{d t} = -u + \varepsilon^{-1} \beta \int_{-1}^{1} v^3 \, dx. \quad (1.2b)
\]

In addition, if we assume that \(\tau \ll 1,\) then (1.2) reduces to the nonlocal Ginzburg-Landau (GL) model

\[
v_t = \varepsilon^2 v_{xx} - \left(\beta \varepsilon^{-1} \int_{-1}^{1} v^3 \, dx - 1\right) v + v^3, \quad |x| \leq 1, \quad t > 0; \quad v_x(\pm 1, t) = 0. \quad (1.3)
\]

A minor variant of (1.3), in which the nonlocal term has \(v^2\) rather than \(v^3\) in its integrand, arises in asymptotic theories of certain convection processes (cf. [20], [25]), and was studied in the context of localized steady-state spike solutions in [23]. The extension of the analysis of [23] to a nonlocal quintic GL model is given in [32].

For (1.1), (1.2), and (1.3), we will use a combination of asymptotic analysis, linear stability theory, and full numerical computations to investigate a wide variety of dynamical instabilities, and ultimately a finite-time blow-up behavior, for localized spike solutions that occur as \(\beta\) is slowly ramped in time below various linear stability and existence thresholds associated with steady-state spike solutions. Through full numerical simulations we will show that when \(\beta\) is slowly ramped there can be a dynamically intricate route, or transition, of either spike nucleation, spike annihilation, or spike amplitude oscillation that occurs before a spike ultimately undergoes a finite-time blow-up of its amplitude (see Experiments 1–4 in §4.3). The precise transition to blow-up that is observed depends on the parameter regime of \(D\) and \(\tau.\) Our detailed analysis of the existence and linear stability of multi-spike patterns will provide a theoretical guide for predicting which transition will be realized.

One common challenge in the analysis of the linear stability of multi-spike patterns for RD systems is that one must derive precise stability thresholds from the analysis of a nonlocal eigenvalue problem (NLEP). One key feature of our model (1.1) is that the underlying NLEP problem is “explicitly solvable” in the sense of [22], in that the problem of detecting any unstable eigenvalues of the NLEP is reduced to the highly tractable analytical problem of determining the roots to some explicit transcendental equations in the eigenvalue parameter. A related key feature of (1.1) is that we are also able to provide a detailed analysis of a delayed bifurcation behavior that occurs as \(\beta\) is slowly ramped below linear stability thresholds obtained from the NLEP analysis. Our analysis theoretically predicts the delayed value of \(\beta\) for which either a sign-fluctuating (competition) instability in the spike amplitude occurs for (1.1) or an oscillatory instability in the spike amplitude occurs for a one-spike solution of (1.2). Delayed bifurcation phenomena has been rather extensively analyzed in the context of ODEs. For some early studies of this type see [1], [8], and [19], and for a more recent overview see [18]. However, in the context of PDEs there have been relatively few detailed analytical studies of delayed bifurcation. One such study is [30] where a related explicitly solvable NLEP was key to analyzing delayed bifurcation behavior of spike solutions to a specific form of the GM model and a generalization of the GS model.
The outline of this paper is as follows.

In §2 the method of matched asymptotic expansions is used to construct \(k\)-spike steady-state solutions to (1.1) in the limit \(\varepsilon \to 0\). This analysis shows that there are critical existence thresholds \(\beta_k\), with \(\beta_{k+1} > \beta_k\), for which there is no \(k\)-spike steady-state if \(\beta < \beta_k\). This critical value of \(\beta\) corresponds to a saddle-node bifurcation point of spike equilibria.

In §3 we perform a linear stability analysis of such spike solutions. A novel feature in this analysis in comparison with similar studies for other models (cf. [4], [31]), is that we must consider both the stability of the background state in the outer region away from the spikes as well as the stability of the spike profile. In §3 we show that the background state is linearly stable only if \(D > D_b\), where \(D_b\) depends on \(\beta\) and the number of spikes. To analyze the linear stability of the spike profile, in §3 we first derive a nonlocal eigenvalue problem (NLEP) that characterizes any instabilities of the amplitudes of the spikes on an \(O(1)\) time-scale. With our modification of the exponent of the nonlinearity in (1.1) in comparison with the system studied in §5 of [16], we show that the NLEP is explicitly solvable and, hence, readily analyzed. For \(0 < \tau < \tau_{Hk}\) and for \(k \geq 2\), we show that a \(k\)-spike equilibrium is linearly stable only when \(\beta > \beta_{ck} > \beta_k\) for some thresholds \(\beta_{ck}\) and \(\tau_{Hk}\). The threshold \(\beta_{ck}\), referred to as the competition stability threshold, occurs as a result of a zero-eigenvalue crossing and leads to a sign fluctuating instability in the spike amplitudes. Moreover, the \(k\)-spike equilibria with \(k \geq 2\) is unstable for all \(\tau > 0\) on the range \(\beta_k < \beta < \beta_{ck}\). For \(k = 1\), we have \(\beta_{c1} = \beta_1\), and for \(\beta > \beta_1\) a one-spike steady-state is linearly stable only on the range \(0 < \tau < \tau_H\) and undergoes a Hopf bifurcation at some \(\tau = \tau_H\), which depends on \(\beta\) and \(D\). Asymptotic approximations for \(\tau_H\) in the limit \(D \gg 1\) and for \(D \ll 1\) are derived.

For two specific scenarios, in §4 we provide a detailed analysis of a delayed bifurcation behavior that occurs when \(\beta\) is slowly decreased. In §4.1 we study delayed bifurcation behavior for a two-spike steady-state of (1.1) with \(\tau = 0\) when \(\beta\) is slowly and linearly ramped in time below the competition threshold \(\beta_{c2}\). Our analysis predicts the existence of a critical value of \(\beta\), satisfying \(\beta_2 < \beta < \beta_{c2}\), at which the delayed competition instability is fully realized. For a fixed \(\tau > 0\), in §4.2 we analyze the delayed Hopf bifurcation that occurs for a one-spike solution to the nonlocal GL model (1.2) when \(\beta\) is slowly ramped below the Hopf bifurcation threshold associated with the NLEP. We show that the delayed oscillatory instability is fully realized before \(\beta\) reaches the existence threshold \(\beta_1\). In §4.3 we perform several numerical experiments on the RD system using FlexPDE6 [11] to confirm our theoretical predictions of delayed instabilities, and to exhibit several distinct dynamical routes, or transitions, that can occur leading to an ultimate finite-time blow-up of a spike amplitude.

In §5 we use a dynamical rescaling approach, based on a near-similarity group transformation, to analyze our numerically observed finite-time blow-up behavior for a solution to the nonlocal GL model (1.3) whenever \(\beta\) is below the saddle node point \(\beta_1 = 1/(\sqrt{2}\pi)\) associated with a one-spike steady-state. In our analysis we construct a solution to (1.3) that exhibits finite-time blow-up at \(x = 0\). Our analysis is related to the center-manifold approach developed in [10] (see also [9] for a related study of quenching) for the scalar model \(v_t = \Delta v + v^p\) for \(p > 1\) and to the more recent studies of [26] and [3] for finite-time blow-up in a complex Ginzburg-Landau equation. In contrast to the study of [10] of blow-up in the local model, our analysis of the nonlocal model (1.3) requires the numerical computation of a simple BVP characterizing the blow-up profile. We will show for (1.3) that the blow-up profile exists whenever \(\beta < \beta_1\). In the limit \(\beta \to \beta_1^−\), we construct the blow-up profile asymptotically to predict a precise blow-up rate.

Our construction of a single-point blow-up for (1.3), similar to that for scalar quasilinear heat equations, differs from that of [2] and [27] where a global blow-up behavior for \(u_t = \Delta u + u^p - |\Omega|^{-1}\int_{\Omega} u^p dx\) with \(\partial_n u = 0\) on \(\Omega\) was analyzed for various \(p > 1\). For \(p = 2\) and in 1-D, it was shown in [2] that \(u\) blows up at each point in \(\Omega\), but with a nonuniform blow-up rate depending on the point \(x \in \Omega\). Rigorous results characterizing a similar blow-up behavior for \(1 < p \leq 2\) in
higher dimensions were proved in [27]. A similar global blow-up behavior for the related model $u_t = \Delta u + \int_\Omega u^p\,dx$ was analyzed in [28]. An analysis of blowup behavior in a wide variety of other nonlocal models is given in [29].

The paper concludes with a brief discussion in §6.

2 Asymptotic Construction of Steady-State Spike Solutions

In this section we use the method of matched asymptotic expansions for $\varepsilon \to 0$ to construct a $k$-spike steady-state solution to (1.1), with evenly spaced interior spikes, on the interval $|x| \leq l$. To do so, we need only construct a one-spike solution centered at $x = 0$ on the interval $|x| \leq l$ and then set $l = 1/k$. From a periodic extension of this canonical one-spike solution, defined on $|x| \leq l = 1/k$, a $k$-spike steady-state solution to (1.1) is obtained.

We consider (1.1) on $|x| = l$, and construct a spike solution centered at $x = 0$. In the inner region near $x = 0$, we introduce the new variables $y$, $V(y)$, and $U(y)$, by

$$y = x/\varepsilon, \quad V(y) = v(\varepsilon y), \quad U(y) = u(\varepsilon y).$$

Upon introducing these variables into the steady-state problem for (1.1), we obtain to leading order that

$$\frac{\beta}{2} \varepsilon^{-1} y^3 \to 2 \beta \left(\int_{-\infty}^{0} V^3(\varepsilon y)\,dy\right) \delta(x) = 2 \beta (U_0 - 1)^{3/2} \left(\int_{-\infty}^{0} \left[w \left(\sqrt{U_0 - 1}\right)\right]^3\,dy\right) \delta(x) = 2\sqrt{2} \beta \pi (U_0 - 1) \delta(x).$$

Here we have used $\int_{-\infty}^{0} [w(z)]^3\,dz = \sqrt{2} \pi$. In this way, it follows from (1.1 b) and (2.3), that the steady-state outer solution for $u$ on $|x| \leq l$ is

$$Du_{xx} - u = -2\sqrt{2} \beta \pi (U_0 - 1) \delta(x), \quad |x| < l; \quad u_x(\pm l) = 0,$$

together with the matching condition that $u(0) = U_0$.

We represent the solution to (2.4) in terms of the Green’s function $G(x)$, satisfying

$$G_{xx} - \frac{1}{D} G = -\delta(x), \quad |x| \leq l; \quad G_x(\pm l) = 0,$$

which is given explicitly by

$$G(x) = \frac{\cosh[\omega_0(l - |x|)]}{2 \omega_0 \sinh(\omega_0 l)}, \quad \omega_0 = 1/\sqrt{D}.$$  

In terms of $G(x)$, the solution to (2.4) is

$$u(x) = 2\sqrt{2} \beta \pi \omega_0^2 (U_0 - 1) G(x).$$

Upon setting $u(0) = U_0$ in (2.6) and letting $l = 1/k$, we obtain for a $k$-spike steady-state solution on $|x| \leq 1$ that

$$\beta = \beta_k \left[1 + \frac{1}{U_0 - 1}\right], \quad \beta_k \equiv \frac{1}{\sqrt{2} \pi \omega_0} \tanh\left(\frac{\omega_0}{k}\right),$$

which determines $U_0$ in terms of $\beta$. Since we require $U_0 > 1$, we conclude that a $k$-interior-spike steady-state solution
exists only when $\beta > \beta_k$, where $\beta_k$ is the existence threshold for the $k$-spike solution. The limiting asymptotics of $\beta_k$ for both large and small diffusivity is

$$\beta_k \sim \frac{1}{\sqrt{2\pi k}}, \quad D \gg 1; \quad \beta_k \sim \frac{1}{\pi} \sqrt{\frac{D}{2}}, \quad D \ll 1.$$  \hfill (2.8)

For $D = 1$, in Fig. 1 we plot the solution branches consisting of $|v|_\infty = w(0)\sqrt{U_0 - 1}$, where $w(0) = \sqrt{2}$, versus $\beta$, as obtained from (2.8), for $k$-spike steady-state solutions to (1.1) for $k = 1, \ldots, 4$. In this figure, we also illustrate the stability results obtained in §3 from the NLEP associated with the spike profile.

Figure 1. The bifurcation diagram of $|v|_\infty = \sqrt{2(U_0 - 1)}$ versus $\beta$ when $D = 1$ for $k$-spike steady-state solutions of (1.1) for $k = 1, \ldots, 4$. The heavy solid portions of the curves are where the NLEP associated with the $k$-spike solution has stable eigenvalues when $0 \leq \tau < \tau_{Hk}$, where $\tau_{Hk}$ is a Hopf bifurcation threshold depending on $\beta$ and $D$. The dashed portions indicate where the NLEP has an unstable eigenvalue for any $\tau \geq 0$ due to a competition instability. The curves from top to bottom correspond to increasing values of $k$.

3 The Linear Stability Analysis

In this section we analyze the linear stability of the steady-state spike solutions constructed in §2. To do so, we let $v_e$ and $u_e$ denote the steady-state solution, and we introduce the perturbation

$$v = v_e + e^{\lambda t}\phi, \quad u = u_e + e^{\lambda t}\eta. \hfill (3.1)$$

Upon substituting (3.1) into (1.1) and linearizing, we obtain that $\phi$ and $\eta$ satisfy

$$\varepsilon^2 \phi_{xx} - (u_e - 1)\phi + 3v_e^2\phi - v_e \eta = \lambda \phi, \hfill (3.2a)$$
$$D\eta_{xx} - (1 + \tau \lambda)\eta = -6\varepsilon^{-1} \beta v_e^2 \phi. \hfill (3.2b)$$

The interval and boundary conditions for (3.2) used to analyze the linear stability of one-spike and multi-spike steady-state solutions are discussed below.

In contrast to similar stability analyses of $k$-spike steady-state solutions for the Gierer-Meinhardt and related models (see [31] and the references therein), we must, in addition to deriving a nonlocal eigenvalue problem associated with localized eigenfunctions near the spike locations, also consider the stability of the background outer solution $v_e = 0$.

More specifically, from (3.2a), we conclude for $\varepsilon \ll 1$ that the outer solution $v_e = 0$ is linearly stable for a $k$-spike pattern only when $u_e(l) - 1 > 0$ on $O(\varepsilon) < |x| \leq l$, with $l = 1/k$. Since $u_e$ is monotone decreasing on $0 < x < l$, it follows
that \( u_e(x) > 1 \) on \( 0 < x < l \) only when \( u_e(l) > 1 \). By evaluating the outer solution \( u_e(x) \), given by
\[
u_e(x) = U_0 G(x) / G(0),
\]
at \( x = l = 1/k \), where \( G(x) \) is given in (2.5b), we conclude that the background state is linearly stable if and only if \( U_0 \text{sech} (\omega_0/k) > 1 \). Upon setting \( U_0 \text{sech} (\omega_0/k) = 1 \), we obtain a curve in the \( D \) versus \( \beta \) parameter plane, parameterized by \( U_0 > 1 \), defined by
\[
D \equiv \frac{1}{k^2 \left( \log \left[ U_0 + \sqrt{U_0^2 - 1} \right] \right)^2}, \quad \beta = \frac{\sqrt{D}}{\sqrt{2\pi}} \tanh \left( \frac{1}{k \sqrt{D}} \right) \left( 1 + \frac{1}{U_0 - 1} \right).
\]
Upon eliminating \( U_0 \), we conclude that if \( D > D_b \), where \( D_b \) is the unique root of the transcendental equation
\[
\beta = \frac{1}{\pi} \sqrt{\frac{D}{2}} \coth \left( \frac{1}{k \sqrt{D}} \right),
\]
then the background state \( v_e = 0 \) is linearly stable. Otherwise, the background state is unstable. A plot of the stability boundary \( D_b \) versus \( \beta \) is shown in Fig. 2 for \( k = 1, \ldots, 4 \). Qualitatively, this result shows that only if \( D \) is sufficiently large, so that the inhibitor concentration does not become too small at the midpoint between adjacent spikes, will the background state \( v_e = 0 \) for the activator be linearly stable.

Figure 2. The stability threshold \( D_b \) versus \( \beta \), obtained from (3.3), characterizing the linear stability of the background state \( v_e = 0 \) for patterns with \( k = 1, \ldots, 4 \) steady-state spikes. The curves from top to bottom correspond to increasing values of \( k \). For each \( k \), the background state is linearly stable when \( D > D_b \), otherwise it is unstable.

In §3.1 we derive the NLEP associated with a one-spike solution, and we determine parameter ranges for which this NLEP has no unstable eigenvalues. The corresponding problem for multi-spike solutions is examined in §3.2.

### 3.1 The NLEP Problem for a One-Spike Solution

In this section we derive the NLEP associated with the stability of a one-spike solution centered at \( x = 0 \). As such, we consider (3.2) on \( |x| \leq 1 \) with Neumann conditions \( \eta_x(\pm 1) = \phi_x(\pm 1) = 0 \).

In the inner region near \( x = 0 \), we have that \( u_e \sim U_0 \) and \( v_e(x) \sim \sqrt{U_0 - 1} w \left( \sqrt{U_0 - 1} x / \varepsilon \right) \) from (2.1). Therefore, \( \phi \) will be localized near \( x = 0 \), while \( \eta = \eta_0 + o(1) \) for \( |x| \leq \mathcal{O}(\varepsilon) \) for some \( \eta_0 \) to be determined. More specifically, we look for a localized eigenfunction for \( \phi \) in the form
\[
\phi(x) = \Phi(z), \quad z \equiv \sqrt{U_0 - 1} \left( x / \varepsilon \right).
\]
Upon substituting (3.4) into (3.2 a), we obtain to leading-order that $\Phi(z)$ satisfies

$$L_0 \Phi - \frac{w(z)}{\sqrt{U_0 - 1}} \eta_0 = \frac{\lambda}{U_0 - 1} \Phi, \quad -\infty < z < \infty; \quad \Phi \to 0, \quad \text{as } |z| \to \infty. \quad (3.5 \text{ a})$$

Here $L_0$ is the local operator, defined in terms of the homoclinic $w(z) \equiv \sqrt{2} \text{sech}(z)$, by

$$L_0 \Phi \equiv \Phi_{zz} - \Phi + 3w^2 \Phi. \quad (3.5 \text{ b})$$

To determine $\eta_0$ in (3.5 a), we must first determine the limiting problem for the outer eigenfunction $\eta(x)$ from (3.2 b). Since $\nu_c$ is localized near $x = 0$, we calculate in the sense of distributions that

$$6\varepsilon^{-1} \beta w_0^2 \phi \to 6\beta \sqrt{U_0 - 1} \left( \int_{-\infty}^{\infty} w^2 \Phi \, dz \right) \delta(x). \quad (3.6)$$

In this way, we obtain that the outer approximation for $\eta$ satisfies

$$\eta_{xx} - \theta^2_\lambda \eta = -6\beta w_0^2 \sqrt{U_0 - 1} \left( \int_{-\infty}^{\infty} w^2 \Phi \, dz \right) \delta(x), \quad |x| \leq 1; \quad \eta_x(\pm 1) = 0, \quad (3.7)$$

subject to the matching condition that $\eta(0) = \eta_0$. In (3.7), we have defined $\theta_\lambda \equiv \sqrt{(1 + \tau \lambda)/D}$, where we have specified the principal value of the square root to ensure that $\eta(0)$ is analytic in $\text{Re}(\lambda) \geq 0$.

We represent the solution to (3.7) in terms of the $\lambda$-dependent Green’s function $G_\lambda(x)$ satisfying

$$G_{\lambda xx} - \theta^2_\lambda G_\lambda = -\delta(x), \quad |x| \leq 1; \quad G_{\lambda x}(\pm 1) = 0, \quad (3.8 \text{ a})$$

which has the explicit solution

$$G_\lambda(x) = \frac{\cosh \left( \theta_\lambda (1 - |x|) \right)}{2 \theta_\lambda \sinh(\theta_\lambda)}, \quad \theta_\lambda \equiv \sqrt{\frac{1 + \tau \lambda}{D}}. \quad (3.8 \text{ b})$$

In terms of $G_\lambda(x)$, the solution to (3.7) is

$$\eta(x) = 6\beta w_0^2 \sqrt{U_0 - 1} \left( \int_{-\infty}^{\infty} w^2 \Phi \, dz \right) G_\lambda(x). \quad (3.9)$$

We then set $x = 0$ in (3.9) and solve for $\eta(0) \equiv \eta_0$. Finally, upon substituting the resulting expression for $\eta_0$ into (3.5 a), we obtain the following NLEP for $\Phi(z)$:

**Principal Result 3.1** For $0 < \varepsilon \ll 1$, the NLEP associated with the stability of a one-spike steady-state solution for (1.1) is

$$L_0 \Phi - \chi_0 w \left( \int_{-\infty}^{\infty} w^2 \Phi \, dz \right) = \frac{\lambda}{U_0 - 1} \Phi, \quad -\infty < z < \infty; \quad \Phi \to 0, \quad \text{as } |z| \to \infty. \quad (3.10 \text{ a})$$

Here $L_0$ is defined in (3.5 b), and the multiplier $\chi_0 = \chi_0(\tau \lambda)$ of the nonlocal term is

$$\chi_0(\tau \lambda) \equiv \frac{3\omega_0 \tanh \omega_0}{\theta_\lambda \tanh(\theta_\lambda)} \left( 1 + \frac{1}{U_0 - 1} \right), \quad (3.10 \text{ b})$$

where $\omega_0 \equiv 1/\sqrt{D}$ and $\theta_\lambda$ is defined in (3.8 b). Here $U_0$ is defined in terms of $\beta$ and $D$ by (2.7) with $k = 1$.

We now show that the problem of determining unstable discrete eigenvalues of (3.10) in $\text{Re}(\lambda) > 0$ can be reduced to the simpler problem of determining the roots to a simple explicit transcendental equation in the eigenvalue parameter.
Lemma 3.1 Any unstable eigenvalue of (3.10) in \( \text{Re}(\lambda) > 0 \) must be a root of \( F_0(\lambda) = 0 \), where

\[
F_0(\lambda) \equiv \chi_0(\tau \lambda) + \frac{\lambda}{U_0 - 1} - 3. \tag{3.11}
\]

Proof To prove this we consider only the region \( \text{Re}(\lambda) > -(U_0 - 1) \), where we can guarantee that \( |\Phi| \to 0 \) exponentially as \( |z| \to \infty \). The continuous spectrum for (3.10) is \( \lambda < -(U_0 - 1) < 0 \), with \( \lambda \) real. To establish (3.11) we use Green’s identity on \( w^2 \) and \( \Phi \), which is written as \( \int_{-\infty}^{\infty} (w^2 L_0 \Phi - \Phi L_0 w^2) \, dy = 0 \). Upon using (3.10a), together with the key identity that \( L_0 w^2 = 3w^2 \) (see Lemma 2.3 of [22]), the expression above from Green’s identity reduces to

\[
\int_{-\infty}^{\infty} w^2 \Phi \, dz \left( \chi_0 + \frac{\lambda}{U_0 - 1} - 3 \right) = 0.
\]

Therefore, for eigenfunctions for which \( \int_{-\infty}^{\infty} w^2 \Phi \, dz \neq 0 \), (3.11) readily follows. In contrast, for eigenfunctions for which \( \int_{-\infty}^{\infty} w^2 \Phi \, dz = 0 \), then (3.10) reduces to the local eigenvalue problem \( L_0 \Phi = \nu \Phi \), where \( \nu \equiv \lambda/(U_0 - 1) \). It was proved in Proposition 5.6 of [5] that the point spectrum of this local eigenvalue problem consists only of \( \nu = 3 \), for which \( \Phi_0 > 0 \) and consequently \( \int_{-\infty}^{\infty} w^2 \Phi \, dz \neq 0 \), together with translation mode \( \nu = 0 \), associated with the odd eigenfunction \( \Phi_1 = w' \).

As such, any unstable eigenvalue of the NLEP (3.10) must be a root of \( F_0(\lambda) = 0 \).

We remark that our modification of the nonlinearity in (1.1 b) to \( 2\varepsilon^{-1}\beta v^3 \) rather than \( 2\varepsilon^{-1}\beta v^2 \), as considered previously in [16], has resulted in an NLEP for which the problem of determining the unstable spectrum is reduced to the simpler problem of determining the roots to an explicit transcendental equation. More generally, further examples of nonlinear kinetics that lead to such “explicitly solvable” NLEP problems are discussed in [22] and [21].

We now analyze the roots of \( F_0(\lambda) = 0 \) in \( \text{Re}(\lambda) \geq 0 \). Upon substituting (3.10 b) for \( \chi_0 \) into (3.11), we conclude that any unstable eigenvalue of the NLEP (3.10) is a root of

\[
2\sqrt{1 + \tau \lambda} \left( \frac{\tanh \theta_\lambda}{\tanh \omega_0} \right) = -\frac{d_1}{b - \lambda}, \tag{3.12 a}
\]

where \( \omega_0 = 1/\sqrt{D} \). Here \( d_1 \) and \( b \) are defined by

\[
d_1 \equiv -6U_0, \quad b \equiv 3(U_0 - 1), \quad \text{where} \quad \beta = \beta_1 \left( 1 + \frac{1}{U_0 - 1} \right), \quad \beta_1 = \frac{\tanh \omega_0}{\sqrt{2\pi \omega_0}}. \tag{3.12 b}
\]

By using a winding number approach, similar to that done in [22], one can prove that (3.12) has a Hopf bifurcation at some critical value of \( \tau \), depending on \( \beta \) and \( D \).

For \( \beta > \beta_1 \), in Fig. 3 we show numerical results for the unique value \( \tau = \tau_H > 0 \) for which (3.12) admits a purely complex conjugate pair of eigenvalues \( \lambda = \pm i\lambda_H \), with \( \lambda_H > 0 \). We have \( \text{Re}(\lambda) > 0 \) when \( \tau > \tau_H \), and \( \text{Re}(\lambda) < 0 \) when \( 0 < \tau < \tau_H \). In the limit \( D \gg 1 \) or \( D \ll 1 \), we now show that the Hopf bifurcation value \( \tau_H \) can be calculated analytically.

For \( D \gg 1 \), for which \( \omega_0 \ll 1 \), (3.12) reduces to

\[
2(1 + \tau \lambda) = -\frac{d_1}{b - \lambda} \equiv G(\lambda), \tag{3.13}
\]

where \( d_1 \) and \( b \) are given in (3.12 b). This problem was studied in [22]. Since \( d_1 < 0 \) and \( G(0) > 2 \) for any \( U_0 > 1 \), Principal Result 2.5 of [22] applies and we obtain for \( D \gg 1 \) and \( \beta > \beta_1 \) that

\[
\tau_H \sim \frac{1}{3} \left( \frac{\beta}{\beta_1} - 1 \right), \quad \lambda_H \sim 3 \left( \frac{\beta}{\beta_1} - 1 \right)^{-1/2}, \quad \text{where} \quad \beta_1 \equiv \frac{1}{\sqrt{2\pi}}. \tag{3.14}
\]
Figure 3. Hopf bifurcation threshold $\tau_H$ versus $\beta$ for a one-spike solution. Left panel: $\tau_H$ for $D = 1$ (top solid curve) and for $D = 10$ (bottom solid curve), as computed from (3.12). The dashed curve is the large $D$ approximation (3.14). Right panel: $\tau_H$ for $D = 0.2$ (heavy solid curve), computed from (3.12), and compared with the small $D$ approximation (3.16) (dashed curve).

Alternatively, for $D \ll 1$, for which $\omega_0 \gg 1$, (3.12) reduces to the following problem, which was also studied in [22]:

$$2\sqrt{1 + \tau \lambda} = -\frac{d_1}{b - \lambda} \equiv G(\lambda).$$

From Principal Result 3.8 of [22], we conclude for $D \ll 1$ and $\beta > \beta_1 \sim \pi^{-1}\sqrt{D/2}$ that

$$\tau_H \sim \frac{1}{3} \left( \frac{\beta}{\beta_1} - 1 \right) \left[ 1 + \frac{\beta^2}{4\beta_1^2} + \frac{\beta}{\sqrt{2}\beta_1} \right], \quad \lambda_H \sim 3 \left( \frac{\beta}{\beta_1} - 1 \right)^{-1} \sqrt{1 - \frac{2}{3\tau_H} \left( \frac{\beta}{\beta_1} - 1 \right)}.$$  (3.16)

The limiting expressions for $\tau_H$ for small $D$ in (3.16) and for large $D$ in (3.14) are shown in Fig. 3 to be rather accurate even when $D = 10$ and $D = 0.2$, respectively, when compared with the Hopf threshold computed from (3.12).

In Fig. 4 we re-parametrize the Hopf bifurcation curve by plotting a Hopf bifurcation threshold value $\beta_H$, versus $D$ for various values of $\tau > 0$. This plot shows that $\beta_H > \beta_1$ for $\tau > 0$, and so as $\beta$ is adiabatically decreased starting from a value above $\beta_H$ a Hopf bifurcation of the spike amplitude will occur before we reach the steady-state spike-existence threshold $\beta_1$.

### 3.2 Linear Stability of Multi-Spike Solutions

To analyze the linear stability of the multi-spike steady-state solution $v_e$ and $u_e$, we will use a Floquet approach in the same spirit as done in [17]. On the interval $|x| \leq l$, we begin by introducing the perturbation (3.1) into (1.1), to obtain the linearized problem

$$\varepsilon^2 \phi_{xx} - (u_e - 1) \phi + 3v_e^2 \phi - v_e \eta = \lambda \phi, \quad |x| \leq l, \quad (3.17a)$$

$$D \eta_{xx} - (1 + \tau \lambda) \eta = -6\varepsilon^{-1} \beta v_e^2 \phi, \quad |x| \leq l, \quad (3.17b)$$

where, in terms of a complex parameter $\xi$, we impose the Floquet-type boundary conditions

$$\phi(l) = \xi \phi(-l), \quad \phi_x(l) = \xi \phi_x(-l), \quad \eta(l) = \xi \eta(-l), \quad \eta_x(l) = \xi \eta_x(-l).$$  (3.17c)

After deriving the NLEP associated with solving (3.17) for arbitrary $\xi$, we must determine $\xi$ so that our NLEP problem applies to a $K$-spike pattern on the domain $[-l, (2K - 1)l]$ with periodic boundary conditions. This is done by translating
Figure 4. Hopf bifurcation thresholds $\beta_H$ versus $D$ for various $\tau$. The one-spike steady-state is linearly stable (unstable) when $\beta < \beta_H$ ($\beta > \beta_H$). The values of $\tau$ are as follows: heavy dashed ($\tau = 0.1$), dashed-dotted ($\tau = 1$), dotted ($\tau = 2$). The heavy solid curve is the existence threshold $\beta_1 = \beta_1(D)$, while the light dashed curve indicates the limiting value $\beta_1 \rightarrow 1/(\sqrt{2\pi})$ as $D \rightarrow \infty$. For any $\tau > 0$, the Hopf threshold always occurs before the existence threshold as $\beta$ is decreased.
We then evaluate (3.21 b) at \( x = 0 \) to determine \( \eta_0 \equiv \eta(0) \), which is needed in (3.19).

To calculate \( \eta_0 \) we must determine \( G_q(0) \). To do so, we solve (3.21 a) on \(-l < x < 0\) and on \(0 < x < l\), and impose the Floquet boundary conditions to obtain

\[
G_q(x) = \begin{cases} 
A \cosh(\theta_L(x + l)) + B \sinh(\theta_L(x + l)), & -l < x < 0 \\
\xi A \cosh(\theta_L(x - l)) + \xi B \sinh(\theta_L(x - l)), & 0 < x < l.
\end{cases}
\]

Upon imposing \( G_q(0^+) = G_q(0^-) \) and \( G_{qx}(0^+) - G_{qx}(0^-) = -1 \), we obtain the linear system for \( A \) and \( B \) given by

\[
\begin{pmatrix}
(1 - \xi) \cosh(\theta_L l), & (1 + \xi) \sinh(\theta_L l) \\
\theta_L (1 + \xi) \sinh(\theta_L l), & \theta_L (1 - \xi) \cosh(\theta_L l)
\end{pmatrix}
\begin{pmatrix}
A \\ B
\end{pmatrix} = \begin{pmatrix}
0 \\ 1
\end{pmatrix}.
\]

We then solve for \( A \) and \( B \) from (3.23), to calculate \( G_q(0) \) as

\[
G_q(0) = A \cosh(\theta_L l) + B \sinh(\theta_L l) = \frac{1}{\theta_L} \left[ \cosh(\theta_L l) \sinh(\theta_L l) \left( \frac{\cosh(\theta_L l) \sinh(\theta_L l)}{\cosh^2(\theta_L l) + \sinh^2(\theta_L l) - \frac{1}{2}(\xi + \xi^{-1})} \right) \right].
\]

Next, we set \( \xi = e^{2\pi i j/K} \) in (3.24), and upon re-arranging the resulting expression, we obtain that

\[
G_q(0) = \frac{1}{2\theta_L} \left[ \tanh(\theta_L l) + \frac{(1 - \cos(2\pi j/K))}{\sinh(2\theta_L l)} \right]^{-1}.
\]

Finally, by using (3.25) to calculate \( \eta_0 \), (3.19) is the NLEP corresponding to a \( K \)-spike steady-state solution on a domain of length \( 2Kl \) subject to periodic boundary conditions.

The final step in the analysis is extract the NLEP for the Neumann problem from the NLEP for the periodic problem. More specifically, the stability thresholds for a \( k \)-spike solution with Neumann boundary conditions can be obtained from the thresholds for a \( 2k \)-spike solution with periodic boundary conditions on a domain of twice the length. To see this, suppose that \( \phi \) is a Neumann eigenfunction on the interval \([0, a]\). If we extend it by an even reflection about the origin to the interval \([-a, a]\), then such an extension satisfies periodic boundary conditions on \([-a, a]\). Alternatively, if \( \phi(x) \) is an eigenfunction with periodic boundary conditions at the edge of the interval \([-a, a]\), then if we define \( \hat{\phi}(x) = \phi(x) + \phi(-x) \) it follows that \( \hat{\phi} \) is an eigenfunction for the Neumann problem on \([0, a]\).

Therefore, to obtain the NLEP governing the stability of a steady-state \( k \)-spike pattern on an interval of length 2 subject to Neumann boundary conditions, we simply replace \( \cos(2\pi j/K) \) with \( \cos(\pi j/k) \) in (3.25) and then set \( l = 1/k \). In this way, we formulate our NLEP for the linear stability of a \( k \)-spike steady-state solution as follows:

**Principal Result 3.2** Consider a \( k \)-spike steady-state solution to (1.1) on an interval of length 2 subject to Neumann boundary conditions. Then, the NLEP characterizing \( O(1) \) time-scale instabilities of the pattern is

\[
L_0 \Phi - \chi_j w \left( \int_{-\infty}^{\infty} w^2 \Phi \, dz \right) = \frac{\lambda}{U_0 - 1}, \quad -\infty < z < \infty; \quad \Phi \to 0, \quad \text{as} \quad |z| \to \infty,
\]

where \( L_0 \) is defined in (3.5 b), and where the multipliers \( \chi_j = \chi_{qj}(\tau \lambda), \) for \( j = 0, \ldots, k - 1, \) of the nonlocal term are defined by

\[
\chi_j = 6\sqrt{2} \beta \pi \omega_0^2 G_q(0), \quad G_q(0) = \frac{1}{2\theta_L} \left[ \tanh \left( \theta_L/k \right) + \frac{(1 - \cos(\pi j/k))}{\sinh(2\theta_L/k)} \right]^{-1}, \quad j = 0, \ldots, k - 1.
\]

Here \( \theta_L \) is defined in (3.8 b), and \( w(z) = \sqrt{z} \text{sech} z \) is the homoclinic solution satisfying (2.2). Any unstable discrete eigenvalue of the NLEP must be a root of one of the \( k \)-transcendental equations \( F_j(\lambda) = 0 \), defined by

\[
F_j(\lambda) \equiv \chi_j(\tau \lambda) + \frac{\lambda}{U_0 - 1} - 3, \quad j = 0, \ldots, k - 1.
\]
As derived in Proposition 5.1 of \[ \lambda \] Rottschäfer, J. Tzou, M. J. Ward

Moreover, the right-hand side \(< \lambda <\)

Here \(\omega\)

V. Rottschäfer, J. Tzou, M. J. Ward

terminology in \[ \tau \]

exists a positive minimal value of \(\tau\)

there are no zero-eigenvalue crossings, the

eigenvalues

trigger a finite-time blow-up behavior of the spike pattern. Both transitions involve slowly decreasing

\(\tau\)

bifurcation as \(\beta = \beta(0) - \sigma \tau\), where \(\sigma \ll 1\). We assume that \(O(\epsilon^2) \ll \sigma \ll O(1)\) so that the transition is slow with respect to either the \(O(1)\) time-scale of the destabilization of the background state or of the spike amplitudes, as characterized by the NLEP, but is fast with respect to the usual \(O(\epsilon^2)\) speed characteristic of slowing drifting spikes (cf. [14]).

4 Slow Transition Across Stability Thresholds: Triggering a Finite-Time Blow-up

In this section we examine two scenarios involving slow passage through an instability threshold, which can ultimately trigger a finite-time blow-up behavior of the spike pattern. Both transitions involve slowly decreasing \(\beta\) in time as \(\beta = \beta(0) - \sigma \tau\), where \(\sigma \ll 1\). We assume that \(O(\epsilon^2) \ll \sigma \ll O(1)\) so that the transition is slow with respect to either the \(O(1)\) time-scale of the destabilization of the background state or of the spike amplitudes, as characterized by the NLEP, but is fast with respect to the usual \(O(\epsilon^2)\) speed characteristic of slowing drifting spikes (cf. [14]).

4.1 Slow Passage Through a Competition Instability

Our first scenario is to consider a two-spike steady-state solution when \(\tau = 0\), and to study the effect of a slow transition in \(\beta\) below the competition stability threshold \(\beta_{c2}\) of the NLEP, which triggers a sign-fluctuating instability of the spike amplitudes. To analyze this transition, we proceed as in [1], [8], [18], and [30], by introducing the WKB-type perturbation

\[ v = v_c + \phi(x)e^{\sigma^{-1}\Lambda(\sigma \tau)}, \quad u = u_c + \eta(x)e^{\sigma^{-1}\Lambda(\sigma \tau)}, \]
with \( \Lambda(0) = 0 \) and \( \phi \ll 1, \psi \ll 1 \). Here \( v_e, u_e \) is the two-spike steady-state solution. Upon substituting (4.1) into (1.1), and linearizing the resulting system, we obtain that \( \phi \) and \( \eta \) satisfy

\[
\begin{align*}
\varepsilon^2 \phi_{xx} - (u_e - 1) \phi + 3v_e^2 \phi - v_e \eta &= \Lambda'(T) \phi, \quad |x| \leq 1; \quad \phi_x(\pm 1) = 0, \quad (4.2a) \\
D \eta_{xx} - (1 + \tau \Lambda'(T)) \eta &= -6\varepsilon^{-1} \beta(T)v_e^2 \phi, \quad |x| \leq 1; \quad \eta_x(\pm 1) = 0, \quad (4.2b)
\end{align*}
\]

where \( \beta(T) = \beta(0) - T \) and \( T = \sigma t \).

Upon comparing (4.2) with (3.2), we observe that the eigenvalue parameter \( \lambda \) in (3.2) is replaced by \( \Lambda'(T) \). As such, when \( \tau = 0 \), we conclude from the results (3.27) and (3.26) of the NLEP theory with \( k = 2, j = 1 \), and \( \tau = 0 \), that \( \Lambda'(T) \) satisfies

\[
\frac{\Lambda'}{U_0 - 1} = 3 - \chi_1(0), \quad \chi_1(0) = 3\sqrt{2}\pi \omega_0 \beta(T) \left[ \tanh \left( \frac{\omega_0}{2} \right) + \frac{1}{\sinh \omega_0} \right]^{-1},
\]

where \( \omega_0 = 1/\sqrt{D} \). To simplify (4.3), we first use (2.7) with \( k = 2 \), written as

\[
U_0 - 1 = \left( \frac{\beta}{\beta_2} - 1 \right)^{-1}, \quad \beta_2 = \frac{1}{\sqrt{2}\pi \omega_0} \tanh (\omega_0/2),
\]

to reduce (4.3) to

\[
\Lambda' = 3 \left( \frac{\beta}{\beta_2} - 1 \right)^{-1} \left[ 1 - \frac{\beta}{\beta_2} \left( \frac{\tanh(\omega_0/2)}{\tanh(\omega_0/2) + \csc \omega_0} \right) \right].
\]

Then, we use the identity

\[
\frac{\tanh(\omega_0/2)}{\tanh(\omega_0/2) + \csc \omega_0} = 1 - \gamma, \quad \text{where} \quad \gamma \equiv \frac{1}{1 + 2 \sinh^2 (\omega_0/2)} < 1,
\]

which expresses (4.4) in compact form as

\[
\Lambda'(T) = -3 + \frac{3\gamma [\beta(T)/\beta_2]}{(\beta(T)/\beta_2) - 1}.
\]

We recall from (2.7) that there is no two-spike steady-state solution when \( \beta < \beta_2 \). Moreover, from (3.29) with \( k = 2 \), the two-spike steady-state is linearly unstable to a competition instability on the range \( \beta_2 < \beta < \beta_{c2} \) when \( \tau = 0 \). Upon setting \( k = 2 \) in (3.29), we can readily write the competition instability threshold \( \beta_{c2} \) in terms of \( \gamma \), defined in (4.5), as

\[
\beta_{c2} = \frac{\beta_2}{1 - \gamma}.
\]

Our result for \( \Lambda(T) \) is as follows.

**Lemma 4.1** Let \( \beta(T) = \beta(0) - T \) where \( \beta(0) \) is any value satisfying \( \beta(0) > \beta_{c2} \). Then, we have

\[
(i) \quad \Lambda'(0) < 0,
(ii) \quad \Lambda' = 0, \quad \text{when} \quad \beta = \beta_{c2},
(iii) \quad \exists \text{a unique} \beta = \beta^*, \quad \text{with} \quad \beta_2 < \beta^* < \beta_{c2}, \quad \text{at which} \quad \Lambda = 0.
\]
Here $\beta^* \equiv \beta(T^*) = \beta(0) - T^*$, relates $\beta^*$ to the time $T^*$ at which the competition instability is finally triggered.

**Proof** To prove (i), we use $\beta(0) > \beta_{c2} = \beta_2/(1 - \gamma)$ from (4.7) to conclude from (4.6) that

$$
\Lambda'(0) = \frac{3 + 3(\gamma - 1)(\beta(0)/\beta_2)}{(\beta(0)/\beta_2) - 1} < \frac{3 + 3(\gamma - 1)/(1 - \gamma)}{(\beta(0)/\beta_2) - 1} = 0,
$$

since $\gamma < 1$. To establish (ii), we set $\Lambda' = 0$ to obtain the unique root $\gamma \beta = \beta - \beta_2$, which yields $\beta = \beta_{c2} \equiv \beta_2/(1 - \gamma)$. This establishes that $\Lambda(0) = 0$, $\Lambda'(T) < 0$ for $0 \leq T < T_c$, $\Lambda'(T_c) = 0$ where $\beta(T_c) = \beta_{c2}$, and $\Lambda'(T) > 0$ for $T > T_c$. To prove (iii) it suffices to show that $\Lambda(T) \to +\infty$ as $\beta \to \beta_2^+$. To establish such a result, we use $\beta(T) = \beta(0) - T$ to get $d\Lambda/dT = -d\Lambda/d\beta$, so that (4.6) becomes

$$
\frac{d\Lambda}{d\beta} = 3 - \frac{3\gamma(\beta/\beta_2)}{(\beta/\beta_2) - 1}.
$$

Upon integrating this separable ODE, and imposing $\Lambda(0) = 0$ when $\beta = \beta(0)$, we readily obtain that

$$
\Lambda(T) = -3T(1 - \gamma) - 3\beta_2\gamma \log \left(1 - \frac{T}{\beta(0)/\beta_2} \right),
$$

where $0 < \gamma < 1$ is defined in (4.5). Therefore, $\Lambda \to +\infty$ as $T$ approaches $\beta(0) - \beta_2$ from below, corresponding to $\beta(T) \to \beta_2^+$. \qed

Figure 5. Left panel: $\Lambda(T)$ versus $T$, computed from (4.10), for a two-spike steady-state solution with $D = 1$ and $\tau = 0$ as $\beta = 0.5 - T$ is slowly swept below the competition instability threshold $\beta_{c2} \approx 0.296$. The instability is triggered when $\Lambda(T^*) = 0$, which yields $T^* \approx 0.323$ and $\beta(T^*) \approx 0.1768$. Right panel: $|v|_\infty = \sqrt{2(U_0 - 1)}$ versus $\beta$ for a two-spike steady-state with $D = 1$. The heavy solid portion is linearly stable for $\tau = 0$, while the dashed portion is unstable. The transition occurs at the competition threshold $\beta_{c2} \approx 0.296$. The open circle indicates the starting point for the slow sweep, while the bullet indicates where the delayed competition instability is finally triggered.

In the left panel of Fig. 5 we use (4.10) to plot $\Lambda(T)$ versus $T$ for $D = 1$ and $\beta(0) = 0.5$. For this example, we calculate that $\beta_2 \approx 0.104$, $\beta_{c2} \approx 0.296$, and $\beta^* \approx 0.176$, corresponding to $T^* \approx 0.323$. In the right panel of Fig. 5 we plot the bifurcation diagram of the two-spike steady-state showing the initial point for the slow sweep and the point at which we predict that the competition instability is triggered. From the criterion (3.3) for the stability of the background state, we conclude from Fig. 2 that, for this parameter set, the background state is linearly stable throughout the slow sweep. As shown in the full numerical computations in Experiment 4 of §4.3, this competition instability first leads to the annihilation of one of the two spikes and can lead to a subsequent finite-time blow-up of the remaining spike.
4.2 Slow Passage Through a Hopf Bifurcation

Next, we consider a one-spike steady-state solution in the shadow limit $D \gg 1$ for a fixed $\tau > 0$. We analyze the effect of decreasing $\beta$ slowly in time towards the existence threshold $\beta_1$, so that $\beta$ that must cross below the Hopf bifurcation threshold $\beta_H = \beta_1(1 + 3\tau)$, as obtained from (3.14) with $\beta_1 = 1/(\sqrt{2\pi})$. From the numerical results shown in Experiment 3 of §4.3, this slow sweep initially triggers a time-periodic oscillation of the spike amplitude, which is followed by a finite-time blow-up.

To analyze this slow passage problem, we proceed as in §4.1 and introduce the perturbation (4.1) to obtain the linearized problem (4.2). For $D \gg 1$, the spectral problem associated with the NLEP is (3.13), and with the identification that $\lambda$ in (3.13) is to be replaced by $\Lambda'(T)$, we conclude that $\Lambda'(T)$ must satisfy

$$(1 + \tau \Lambda') = \frac{3U_0}{3(U_0 - 1) - \Lambda'},$$

From (2.7) we have $U_0 - 1 = (\beta/\beta_1 - 1)^{-1}$ where $\beta_1 = 1/(\sqrt{2\pi})$. Upon using this expression for $U_0$, we solve the quadratic equation above for $\Lambda'$ to obtain

$$\Lambda' = \frac{1}{2\tau} \left[ - \left( 1 - \frac{3\tau}{(\beta/\beta_1 - 1)} \right) \pm \sqrt{\left( 1 - \frac{3\tau}{(\beta/\beta_1 - 1)} \right)^2 - 12\tau} \right],$$

(4.11)

where $\beta(T) = \beta(0) - T$. We assume that $\beta(0) > \beta_H = \beta_1(1 + 3\tau)$, and where $\beta(0)$ is chosen so that the discriminant in (4.11) is negative. This latter condition holds when

$$\beta(0) > \beta_H, \quad \text{for } \tau \geq 1/12; \quad \beta_H < \beta(0) < \beta_1 \left[ 1 + \frac{3\tau}{(1 - 2\sqrt{3}\tau)} \right], \quad \text{for } 0 < \tau < 1/12.$$  

(4.12)

Assuming that the condition (4.12) on $\beta(0)$ holds, we separate $\Lambda(T)$ into real and imaginary parts, as $\Lambda = \Lambda_R + i\Lambda_I$, to obtain from (4.11) that

$$\Lambda'_R(T) = \frac{1}{2\tau} \left( \frac{3\tau \beta_1}{\beta - \beta_1} - 1 \right).$$

(4.13)

The criterion for the triggering of an instability is that there exists a unique $T^*$ such that $\Lambda_R(T^*) = 0$, with $\Lambda_R(T) < 0$ on $0 < T < T^*$. The following result guarantees the existence of such a $T^*$.

Lemma 4.2 Let $\beta(T) = \beta(0) - T$ where $\beta(0)$ satisfies condition (4.12). Then, for $\text{Re}(\Lambda(T)) = \Lambda_R(T)$, we have

(i) $\Lambda'_R(0) < 0$,

(ii) $\Lambda_R = 0$, when $\beta = \beta_H = \beta_1(1 + 3\tau)$,

(iii) $\exists$ a unique $\beta = \beta^*$, with $\frac{1}{\sqrt{2\pi}} \equiv \beta_1 < \beta^* < \beta_H$, at which $\Lambda_R = 0$.

Here $\beta^* = \beta(T^*) = \beta(0) - T^*$, relates $\beta^*$ to the time $T^*$ at which the oscillatory instability due to the Hopf bifurcation is finally triggered.

The proof of this result is very similar to that in Lemma 4.1 and is omitted. In place of (4.10), and with $\Lambda_R(0) = 0$ when $\beta = \beta(0)$, we readily derive from (4.13) that

$$\Lambda_R(T) = -\frac{T}{2\tau} - \frac{3\beta_1}{2} \log \left( 1 - \frac{T}{\beta(0) - \beta_1} \right), \quad \beta_1 \equiv \frac{1}{\sqrt{2\pi}}.$$  

(4.15)

For $\tau = 1.0$ and $\beta(0) = 1$, in the left panel of Fig. 6 we use (4.15) to plot $\Lambda_R(T)$ versus $T$. For this example, we
Experiment 1

β predictions and, in particular, our predictions of triggered competition or oscillatory instabilities under a slow sweep in D at which we predict that the Hopf bifurcation will be triggered. Since this slow passage problem is for the shadow limit the bifurcation diagram of the one-spike steady-state solution showing the initial point for the slow sweep and the point T which yields D ≫ 1, we conclude from Fig. 2 that the trivial background state for the activator is always linearly stable.

Figure 6. Left panel: Λ_R(T) versus T, computed from (4.15), for a one-spike steady-state solution with D ≫ 1 and τ = 1.0 as β = 1 − T is slowly swept below the Hopf instability threshold β_H ≈ 0.90. The instability is triggered when Λ_R(T^*) = 0, which yields T^* ≈ 0.191 and β(T^*) ≈ 0.809. Right panel: |v|_∞ = \sqrt{2(U_0 - 1)} versus β for a one-spike steady-state with D ≫ 1 and τ = 1. The heavy solid portion is linearly stable for τ = 1, while the dashed portion for β < β_H = β_1(1 + 3τ) with β_1 ≈ 0.225 is unstable due to a Hopf bifurcation. The open circle indicates the starting point for the slow sweep, while the bullet indicates where we predict that the delayed oscillatory instability is finally triggered.

calculate that β_1 ≈ 0.225, β_H ≈ 0.90, and β^* ≈ 0.809, corresponding to T^* ≈ 0.191. In the right panel of Fig. 6 we plot the bifurcation diagram of the one-spike steady-state solution showing the initial point for the slow sweep and the point at which we predict that the Hopf bifurcation will be triggered. Since this slow passage problem is for the shadow limit D ≫ 1, we conclude from Fig. 2 that the trivial background state for the activator is always linearly stable.

4.3 Full Numerical Results Under a Slow Sweep

Next, we show results from full numerical computations on (1.1) using FlexPDE6 [11] that validate our linear stability predictions and, in particular, our predictions of triggered competition or oscillatory instabilities under a slow sweep in β. For one particular example we will consider the effect of an instability of the trivial background state for the activator.

Experiment 1: We first illustrate an apparent finite-time blow-up of a single spike solution in the shadow limit as β is slowly swept below the existence threshold β_1. We take τ = 0, D = 100 ≫ 1, and ε = 0.01 in (1.1) and construct a one-spike equilibrium solution as in §2. With D ≫ 1 and k = 1, the result (2.8) states that the one-spike steady-state exists for β > 1/(\sqrt{2π}) ≈ 0.225. For β we consider the slow ramp β = max(0.3 - σt, β_f), where σ = 0.01. For β_f, we consider the two scenarios given by β_f = β_1 ± 0.01, which are depicted by the heavy curves in Fig. 7 (left). In the first scenario where β_f > β_1, the system reaches an equilibrium with a spike amplitude close to that predicted by the steady state theory of §2 (dashed horizontal). In the second scenario, where β_f < β_1, Fig. 7 shows a rapid increase in the spike amplitude beginning when β is ramped below β_1. The heavy solid (heavy dotted) portion indicates the time during which β > β_1 (β < β_1). In Fig. 7 (right), we show the spike profile at various times over the course of the slow ramp of β for the second scenario. Notice that as the spike amplitude grows, the width narrows due to the spatial scaling in (2.1).

To support our conjecture of a finite-time blow-up in the second scenario, in Fig. 8 we plot the numerically computed norms \|u\|_\infty and \|v\|_\infty near the apparent singularity time T ≈ 8.1197. The log-log plot in the right panel of Fig. 8 supports the conjectured scaling law \|u\|_\infty \sim O((T - t)^{-1}) and \|v\|_\infty \sim O((T - t)^{-1/2}) as t \to T^-.

Experiment 2: Next, we investigate the effects of an instability of the trivial background state for the activator on the dynamics of a one-spike pattern subject to a slow sweep of β. Here, we take D = 0.1, τ = 0, and ε = 0.02. For this example
Transition to Finite-Time Blow-up in a Reaction-Diffusion Model with Localized Spike Solutions

Figure 7. Left panel: The evolution of the spike amplitude as $\beta$ is slowly ramped according to $\beta = \max(0.3 - \sigma t, \beta_f)$. When $\beta_f > \beta_1$ (heavy dashed), the spike amplitude reaches an equilibrium value close to that predicted by the steady state theory of §2 (dashed horizontal). When $\beta_f < \beta_1$, an apparent finite-time blow-up of the amplitude occurs. The heavy solid (heavy dotted) portion indicates the time during which $\beta > \beta_1$ ($\beta < \beta_1$). Right panel: The corresponding spike profiles at times $t = 0$ (solid), $t = 3.8$ (dashed), and $t = 7.6$ (dotted). In both figures, $D = 100$, $\tau = 0$, $\varepsilon = 0.01$, and $\sigma = 0.01$.

Figure 8. Left panel: Numerical results computed from the PDE model (1.1) supporting the apparent finite-time blow-up conjecture for $|u|_\infty$ (heavy solid) and $|v|_\infty$ (heavy dashed) when $\beta = \max(0.3 - \sigma t, \beta_f)$, with $\beta_f = \beta_1 - 0.01$, is slowly ramped below the existence threshold $\beta_1$. The parameters are $D = 100$, $\varepsilon = 0.01$, $\sigma = 0.01$, and $\tau = 0$. The apparent blow-up time (light solid vertical line) is $T \approx 8.1197$. Right panel: The corresponding log-log plot where the solid (dashed) is a plot of $(T - t)^{-1}$ ($(T - t)^{-1/2}$), while the solid (open) circles are (renormalized) data for $|u|_\infty$ ($|v|_\infty$) from the left panel.

we also need to calculate the competition stability thresholds $\beta_{c2}$ and $\beta_{c3}$ for a two- and a three-spike steady-state, and the threshold $\beta_h$, obtained from (3.3), at which the background state goes unstable. These thresholds are plotted for a range of $D$ in Fig. 9. In particular, for $D = 0.1$, we calculate that $\beta_{c3} \approx 0.082$, $\beta_h \approx 0.077$, and $\beta_{c2} = 0.071$. For $D = 0.1$, the existence thresholds given in (2.7) for a $k$-spike steady-state with $k = 1, 2, 3$ are $\beta_1 \approx 0.079$, $\beta_2 \approx 0.0654$, and $\beta_3 \approx 0.0588$.

We will show that the instability of the background state can lead initially to the formation of additional spikes that have the effect of “absorbing” extra energy when the condition $\beta < \beta_1$ would otherwise lead to a finite-time blow-up of a one-spike pattern. However, since $\beta_1$ is below the competition thresholds for multi-spike patterns, this absorption is only temporary. The competition instabilities lead to elimination of all but one spike, which is then assured to blow-up when $\beta$ is below the instability threshold of the background state. We illustrate this phenomenon below.
Figure 9. Plot of the competition instability thresholds \( \beta_{c3} \) and \( \beta_{c3} \) for two-spike and three-spike steady-state solutions versus \( D \), together with the instability threshold \( \beta_b \) of the trivial background state for a one-spike solution. The one-spike existence threshold \( \beta_1 \) is always below the two competition thresholds \( \beta_{ck} \) for \( k = 2, 3 \). Therefore, if \( \beta \) is slowly ramped to below \( \beta_1 \), a finite-time blow-up of a one-spike pattern is assured, no matter how slowly the sweep is performed.

Starting from a one-spike steady-state solution, as shown in the left panel of Fig. 10 as constructed in §2, we perform a slow sweep of \( \beta \) according to \( \beta(t) = \max(0.15 - \sigma t, 0.065) \) for two values of \( \sigma \). Because \( \beta(0) > \beta_b \), we observe two qualitatively different evolutions depending on \( \sigma \). For \( \sigma \) only moderately small, the instability of the background state does not have enough time to develop before \( \beta \) is ramped to below \( \beta_b \). The only pattern that exists during the sweep in \( \beta \) is a single spike centered at \( x = 0 \). In this scenario, because \( \min(\beta(t)) < \beta_1 \), we observe a finite-time blow-up of the one-spike pattern in the same manner as in Experiment 1 (not shown).

However, for \( \sigma \) sufficiently small, the instability of the background state does have time to grow into a three-spike pattern as shown in the center and right panels of Fig. 10 before \( \beta \) crosses below the background instability threshold \( \beta_b \). However, once \( \beta \) is ramped below \( \beta_{c3} \), the competition instability eliminates the middle spike, leaving a two-spike quasi-equilibrium pattern as shown in the left panel of Fig. 11. A further decrease in \( \beta \) triggers another competition instability, leaving a one-spike quasi-equilibrium pattern as shown in the center panel of Fig. 11. The process by which this off-centered quasi-equilibrium spike then apparently undergoes a finite-time blow-up is the same as shown below in Experiment 4.

Figure 10. Left panel \((t = 0, \beta = 0.15)\): A one-spike equilibrium initial condition. Center panel \((t = 20, \beta = 0.13)\): The instability of the background state results in the growth of two bumps on either side of the spike. Right panel \((t = 40, \beta = 0.11)\): The two small bumps develop into spikes located at equilibrium locations \( x = \pm 2/3 \). In all figures, \( \beta \) is above the instability threshold \( \beta_b = 0.077 \) of the background state. Here, \( D = 0.1, \tau = 0, \varepsilon = 0.02, \) and \( \sigma = 0.001 \). Solid: \( v \), heavy solid: \( u \).
Transition to Finite-Time Blow-up in a Reaction-Diffusion Model with Localized Spike Solutions

Figure 11. Continuation of the evolution begun in Fig. 10 (left). Left panel ($t = 77.5$, $\beta = 0.0725$): Two-spike quasi-equilibrium pattern after the middle spike in Fig. 10 (right) has been eliminated due to a competition instability. Center panel ($t = 105$, $\beta = 0.045$): A one-spike quasi-equilibrium solution with an off-centered spike after the right spike in the left panel has been eliminated. Right panel: Evolution of $|v|_{\infty}$ in time. Spike elimination events cause the heights of the remaining spikes to increase. Rapid increase of the amplitude of the final remaining spike suggests a finite time blow-up. The background state is stable in the left and center figures. Here, $D = 0.1$, $\tau = 0$, $\varepsilon = 0.02$, and $\sigma = 0.001$. Solid: $v$, heavy solid: $u$.

This experiment should be compared to Experiment 5.2(a) of [16], where $\beta$ was also ramped past the one-spike existence threshold. In the process of the sweep, a dynamic similar to Fig. 10 (center) developed, where two bumps were formed on either side of the original spike. However, instead of evolving to close to a three-spike equilibrium, the pattern evolved to a constant steady state. This is in contrast to the process described above, where a series of unstable intermediate states gives way to a one-spike pattern that has an apparent finite-time blow-up.

**Experiment 3:** In this experiment, we consider a one-spike pattern with $\tau = 1$, $\varepsilon = 0.01$, and with $D \gg 1$ in (1.1). We aim to illustrate the theory of §4.2, and in particular the example of Fig. 6, for slow passage through a Hopf bifurcation point. We begin with a slight perturbation of a one-spike steady-state as constructed in §2. With $\tau = 1$ and $D \gg 1$, we obtain from (3.14) that the Hopf bifurcation threshold for $\beta$ is $\beta_H \approx 0.9$. To illustrate the theory presented in Fig. 6, we perform the slow ramp $\beta(T) = 1 - T$, where $T = 0.002t$. The resulting deviation of the spike amplitude from its steady-state value is shown by the heavy solid curve against $\beta(T)$ in the left panel of Fig. 12. For comparison against the theory, in this figure we also plot in light solid the quantity $\pm C\exp(\Lambda/0.002)$ for some chosen $C$, where $\Lambda$ is calculated from (4.15).

The one-spike steady-state solution is stable during the time that $\beta > \beta_H$. This is consistent with Fig. 12, where we initially observe decaying oscillations in the spike amplitude. However, once $\beta$ is slowly decreased below $\beta_H \approx 0.9$ (dashed vertical), the Hopf bifurcation threshold for the unramped problem is crossed and the amplitude oscillations begin to increase. The delay in the onset of the Hopf bifurcation results from the time needed in order for the oscillations to recover their strength lost during the stable phase of the ramp. We observe that the value of $\beta$ at which the amplitude of oscillations returns to its original value (dotted horizontal) is $\beta(T^*) \approx 0.788$. This is in excellent agreement with the NLEP theory of §4.2, which predicts a value of $\beta(T^*) \approx 0.809$. In the right panel of Fig. 12, we show that the delayed Hopf bifurcation results in fully nonlinear oscillations, leading to an apparent finite-time blow-up in the spike amplitude before $\beta$ reaches the one-spike existence threshold $\beta_1 \approx 0.225$. We note that the right panel of Fig. 12 is a plot of the actual spike amplitude; no steady-state value has been subtracted from it.

**Experiment 4:** For our final experiment we consider a two-spike solution with $\tau = 0$, $D = 1$, and $\varepsilon = 0.002$ in (1.1). We aim to illustrate the theory of §4.1 and in particular the example of Fig. 5 for slow passage through a competition instability threshold. We begin with an odd perturbation of a two-spike equilibrium solution with spikes centered at
Figure 12. Left panel: The deviation of the spike amplitude from its steady-state value (heavy solid curve) as $\beta$ is slowly ramped according to $\beta(\sigma t) = 1 - \sigma t$. The light solid curve is the quantity $\pm C \exp(\Lambda/\sigma)$ for some chosen $C$, with $\Lambda$ given by (4.15). When $\beta$ is above the Hopf bifurcation threshold $\beta_H$ (dashed vertical line), the one-spike steady-state is stable, causing oscillations in the deviation to decay. When $\beta < \beta_H$, the Hopf mode is unstable so that the oscillations grow. The bifurcation is fully triggered when the oscillations return to their original magnitude (dotted horizontal line). This occurs at $\beta(T^*) \approx 0.788$, in excellent agreement with the theoretical value of $\beta(T^*) \approx 0.809$. Right panel: The ensuing fully nonlinear oscillations of the spike amplitude leading to an apparent finite-time blow-up before $\beta$ reaches the existence threshold $\beta_1 \approx 0.225$. In both figures, $D = 100$, $\tau = 1$, $\varepsilon = 0.01$, and $\sigma = 0.002$.

$x = \pm 0.5$ as shown in the left panel of Fig. 13. For $D = 1$, we calculate from the result of the NLEP analysis (3.29) that the competition instability threshold is $\beta_c^2 \approx 0.296$. To best illustrate the theory, we choose the perturbation to be the eigenfunction corresponding to a competition instability for $\beta = 0.28 < \beta_c^2$. We begin with $\beta = 0.5 > \beta_c^2$ so that the two-spike pattern is initially stable. We then slowly ramp $\beta$ according to $\beta = 0.5 - T$, where $T = 0.02t$. For the time during which $\beta > \beta_c^2$, the pattern is stable, leading to a decay of the odd perturbation. This is observed in the center panel of Fig. 13, where we plot as a function of $\beta(T)$ the difference in spike amplitudes (circles) normalized to have an initial value of unity. For comparison, the solid curve in the center panel of Fig. 13 is the quantity $\exp(\Lambda(\beta)/0.02)$, with $\Lambda(\beta)$ as evolved according to the dynamics of (4.9). We observe excellent agreement in the rate of the initial decay.

Figure 13. Left panel: A two-spike steady-state solution. Center panel: The normalized difference in spike amplitudes as $\beta$ is slowly ramped according to $\beta(\sigma t) = 0.5 - \sigma t$. The light solid is the quantity $\exp(\Lambda/\sigma)$, where $\Lambda$ is obtained from (4.9). When $\beta$ is above the two-spike competition threshold $\beta_{c2}$ (dashed vertical), the two-spike pattern is stable, resulting in a decay in the amplitude difference. When $\beta$ is ramped below $\beta_{c2}$, the amplitude difference begins to grow. The competition instability is fully realized when the normalized amplitude difference returns to 1, occurring at $\beta(T^*) \approx 0.187$, which is close to the theoretically predicted value of $\beta(T^*) \approx 0.176$ given in Fig. 5. Right panel: The resulting quasi-equilibrium one-spike pattern after the left spike has been eliminated. In all figures, $D = 1$, $\sigma = 0.02$, $\tau = 0$, and $\varepsilon = 0.002$. Solid: $v$, heavy solid: $u$. 
When $\beta$ is ramped to $\beta_{c2} \approx 0.296$, the amplitude difference begins to increase as the competition threshold is slowly passed. We again observe excellent agreement between when this occurs numerically and the corresponding NLEP prediction. For all $\beta < \beta_{c2}^2 \approx 0.187$, the two-spike equilibrium is unstable to an $O(1)$ competition instability. This is reflected by the increasing normalized amplitude difference in the center panel of Fig. 13. By the convention of the delayed bifurcation analysis in §4.1, we consider the competition instability fully triggered when the normalized amplitude difference reaches its initial value of 1. This occurs when $\beta \approx 0.187$, as compared to the theoretically predicted value of $\beta \approx 0.176$. We conjecture that this slight discrepancy is due to the accumulation of numerical errors, as we observe full agreement in both the initial decay (Fig. 13 (center panel)) as well as initial growth rates when $\beta < \beta_{c2}^2$ (not shown).

The growth of the competition instability for $\beta < \beta_f$ does not saturate, but instead continues until only one spike remains in a quasi-equilibrium state, as shown in the right panel of Fig. 13. This remaining spike then drifts on an $O(\varepsilon^2)$ time-scale toward its equilibrium location $x = 0$ while its amplitude evolves accordingly. The differential-algebraic system governing these dynamics has been derived in many past works through application of a Fredholm alternative (e.g., [14]), and will not be repeated here. We instead focus on the possibility of a finite time blow-up of the amplitude of the remaining spike triggered by the slow drift. To illustrate this phenomenon, we follow the procedure of §2 to obtain that a single quasi-equilibrium spike centered at $x = x_0$ exists only if $\beta > \beta_{q1}$, where $\beta_{q1}$ is given by

$$\beta_{q1} = \frac{\tanh(\omega_0(1 + x_0)) + \tanh(\omega_0(1 - x_0))}{2\sqrt{2\pi\omega_0}}, \quad \omega_0 = 1/\sqrt{D}.$$  \hspace{1cm} (4.16)

For $D = 1$ a plot of (4.16) versus $x_0$ is shown in Fig. 14, and this result agrees with the existence threshold $\beta_1$, defined in (2.7) with $k = 1$, when $x_0 = 0$.

Because $\beta_{q1}$ is a decreasing function of $x_0$, there are three different scenarios for the fate of the remaining spike. If the ramp of $\beta$ is terminated at $\beta_f > \beta_1$ (i.e., $\beta = \max(\beta_0 - T, \beta_f)$), the spike will simply drift to $x = 0$ and remain there for all time. If $\beta_f < \beta_{q1}(0.5)$, the amplitude of the remaining spike will blow-up in finite time before any substantial drift in location occurs. If $\beta_{q1}(0.5) < \beta_f < \beta_1$, such as shown in Fig. 14, the spike drifts toward $x = 0$ until the drift triggers a finite time blow-up. We illustrate this phenomenon in Fig. 15, where we set $\beta_f = 0.1589$ so that $x_0^* \approx 0.42$. In the left panel of Fig. 15 we plot the location $x_0$ of the right spike as a function of $T$. During the time over which the competition instability eliminates the other spike located at $x = -0.5$, the rightmost spike is essentially stationary. In the right panel of Fig. 15, we show the corresponding evolution of the logarithm of the amplitude of the right spike. As
the competition mode grows, so does the amplitude of the right spike. Once the left spike is eliminated, the right spike drifts slowly towards $x = 0$. The amplitude grows correspondingly during this time. However, once the spike location reaches near $x_0 = x_0^* \approx 0.42$, we observe a rapid increase in the amplitude of the spike, indicative of a finite-time blow-up. Quasi-equilibrium theory is no longer valid in this regime, and the spike ceases its slow drift.

Figure 15. Left panel: Location of the right spike as a function of $T = \sigma t$. It initially stays at $x_0 = 0.5$ as the competition instability develops. Once the left spike is eliminated, the right spike begins to drift slowly toward $x = 0$. During this drift, the quasi-equilibrium spike-existence threshold is reached near $x_0 \approx 0.42$. Right panel: The corresponding evolution of the logarithm of the spike amplitude. The rapid increase near $T \approx 2.84$ suggests that the finite time blow-up coincides with the spike’s approach to $x = 0.42$. Here, $D = 1$, $\sigma = 0.004$, $\tau = 0$, and $\varepsilon = 0.002$.

5 The Blow-up Profile for the Nonlocal Ginzburg-Landau Model

In this section we examine the blow-up profile for a solution to the nonlocal GL model (1.3) to have a finite-time singularity at $x = 0$ at some $t = T < \infty$. We will show that this blow-up profile exists only if $\beta < \beta_1$, where $\beta_1 = 1/(\sqrt{2\pi})$ is the existence threshold of a one-spike steady-state solution.

To study the local behavior of the blow-up profile near a singularity, we use the near similarity group dynamical re-scaling variables used in [10] for the local model $u_t = \Delta u + u^3$, defined by

$$y \equiv \frac{x}{\varepsilon L(t)}, \quad s = -\log(T - t), \quad L(t) \equiv \sqrt{T - t}, \quad v(x,t) = \frac{1}{L(t)} w\left[ x/\varepsilon L(t), s \right].$$

In terms of these local variables, (1.3) becomes

$$w_s = w_{yy} - \frac{1}{2} (w + y w_y) + e^{-s} w + w^3 - \beta w I(w), \quad I(w) \equiv \int_{-\infty}^{\infty} w^3 dy.$$  \hfill (5.2)

Since $t \to T^-$ corresponds to $s \to \infty$, we will first determine the limiting behavior of the solution to (5.2) as $s \to \infty$. This limiting steady-state solution $W(y)$ satisfies

$$W_{yy} - \frac{1}{2} (y W)_y + W^3 - \beta W I(W) = 0.$$  \hfill (5.3)

We will look for a solution to (5.3) for which $W$ is even, i.e. $W_y = 0$ at $y = 0$, and $W \to 0$ as $y \to \infty$. By integrating (5.3) over $-\infty < y < \infty$, we readily derive the integral identity that

$$\int_{-\infty}^{\infty} W(y) \, dy = \frac{1}{\beta}.$$  \hfill (5.4)
We conveniently parametrize the solution branch of (5.3) by seeking a solution $W(y;c)$ to
\begin{equation}
W_{yy} - \frac{1}{2} (yW)_y - cW + W^3 = 0,
\end{equation}
which satisfies $W_y = 0$ at $y = 0$ and $W \to 0$ as $y \to \infty$. In terms of this solution, $\beta = \beta(c)$ is given by
\begin{equation}
\beta(c) \equiv \frac{c}{\int_{-\infty}^{\infty} [W(y;c)]^3 dy}.
\end{equation}
By solving (5.5a) numerically as $c$ is varied, and by using (5.5b) to obtain $\beta$, we obtain the solution branch $W(0)$ versus $\beta$ as shown in the left panel of Fig. 16.

Figure 16. Left panel: $W(0)$ versus $\beta$ as computed numerically (solid curve) from (5.5) in terms of the parameter $c$. The discrete points are the asymptotic result $W(0) = \sqrt{2c}$ versus $\beta = \beta_1 (1 - 1/\pi)$. As $c \to \infty$, we have $\beta \to \beta_1 = 1/(\sqrt{2\pi})$ from below, where $\beta_1$ is the existence threshold for a one-spike solution (vertical dashed line). Right panel: validation of the asymptotics $\beta \sim \beta_1 (1 - 1/(3c))$ as $c \to +\infty$. We plot $c(\beta/\beta_1 - 1)$ versus $c$, where $\beta$ is computed numerically from (5.5). This quantity tends to the theoretically predicted value of $-1/3$ as $c \to \infty$ given by the horizontal dashed line.

We now claim that $\beta$ tends from below to the existence threshold $\beta_1 = 1/(\sqrt{2\pi})$ of a one-spike solution as $c \to +\infty$ in (5.5a). This will explain the vertical asymptote at $\beta_1$ in the left panel of Fig. 16. As expected, this implies that there is no solution to the blow-up profile BVP (5.5) on the range of $\beta$ where a one-spike steady-state solution exists.

To establish this claim we analyze (5.5) for $c \gg 1$. We rescale $y$ and $W$ by introducing the local variables $z$ and $W$ by
\begin{equation}
z = \sqrt{cy}, \quad W(z) = c^{-1/2} W \left( z/\sqrt{c} \right),
\end{equation}
so that when $|y| = O(c^{-1/2}) \ll 1$, we have $|z| = O(1)$. We refer to this region as the “bump” region. Upon substituting (5.6) into (5.5a), we obtain that $W(z)$ satisfies
\begin{equation}
W_{zz} - \frac{1}{2c} [W + zW_z] + W - W^3 = 0.
\end{equation}
We then expand
\begin{equation}
W = W_0 + \frac{1}{c} W_1 + \cdots,
\end{equation}
to obtain to leading-order that $W_{0zz} - W_0 + W_0^3 = 0$, so that $W_0 = \sqrt{2}\text{sech} z$. At next order, we obtain that $W_1$ satisfies
\begin{equation}
L_0 W_1 = W_{1zz} - W_1 + 3W_0^2 W_1 = \frac{1}{2} [W_0 + zW_0].
\end{equation}
Although our two-term approximation (5.8) does not provide a uniformly valid characterization of $W(y)$ for $y \gg 1$, we show below in our WKB analysis of the far-field of the blow-up profile that our two-term result (5.8) is sufficient to
calculate a two-term approximation of the integral in (5.5 b) defining $\beta$. Upon using (5.8) and (5.6), we calculate that

$$
\int_{-\infty}^{\infty} W^3 \, dy = c \int_{-\infty}^{\infty} (W_0 + c^{-1}W_1 + \cdots)^3 \, dz \sim c \left[ \int_{-\infty}^{\infty} W_0^3 \, dz + 3c^{-1} \int_{-\infty}^{\infty} W_0^2 W_1 \, dz \right]. \tag{5.10}
$$

Now since $\int_{-\infty}^{\infty} W_0^3 \, dz = \sqrt{2} \pi$, we obtain from (5.5 b) that, for $c \gg 1$, we have

$$
\beta \sim \frac{c}{c\sqrt{2\pi} + 3 \int_{-\infty}^{\infty} W_0^2 W_1 \, dz} \sim \frac{1}{\sqrt{2\pi} \int_{-\infty}^{\infty} W_0^2 W_1 \, dz} \tag{5.11}
$$

Next, we show how to calculate the integral in the last expression of (5.11). We use the identity $L_0 W_0^2 = 3 W_0^2$ (see Lemma 2.3 of [22]), together with integration by parts and the decay of $W_0$ at infinity, to obtain

$$
J \equiv \int_{-\infty}^{\infty} W_0^2 W_1 \, dz = \frac{1}{3} \int_{-\infty}^{\infty} (L_0 W_0^2) L_0^{-1} \left[ \frac{1}{2} (W_0 + z W_{0z}) \right] \, dz,
$$

$$
= \frac{1}{6} \int_{-\infty}^{\infty} W_0^2 (W_0 + z W_{0z}) \, dz,
$$

$$
= \frac{1}{6} \int_{-\infty}^{\infty} \left( W_0^3 + \frac{1}{3} (W_0^3)_z \right) \, dz = \frac{1}{9} \int_{-\infty}^{\infty} W_0^3 \, dz = \frac{\sqrt{2}\pi}{9}.
$$

Upon substituting this final result for $J$ into (5.11) we conclude that

$$
\beta \sim \beta_1 \left( 1 - \frac{1}{3c} \right), \quad \text{as} \quad c \to \infty, \tag{5.12}
$$

where $\beta_1 = 1/(\sqrt{2}\pi)$ is the existence threshold of a one-spike steady-state solution. In the right panel of Fig. 16 we plot $c(\beta/\beta_1 - 1)$ versus $c$, where $\beta$ is computed numerically from (5.5). This plot confirms the asymptotic prediction of (5.12) that $c(\beta/\beta_1 - 1) \to -1/3$ as $c \to \infty$. In the left panel of Fig. 16 we show that the curve $W(0)$ versus $\beta$ parameterized by $c$ as $W(0; c) = \sqrt{2c}$ and $\beta = \beta_1 (1 - \frac{1}{3c})$, indicated by the discrete points, agrees rather well with the full numerical result even when $\beta$ is not too close to the existence threshold.

![Figure 17](image)

Figure 17. The solid curves are the full numerical solution to (5.5 a) for $c = 2.0455$ ($\beta \approx 0.1899$) and for $c = 7.558$ ($\beta \approx 0.21508$). The curve with the larger $W(0)$ value is $c = 7.558$. The dashed curves are the leading-order bump solution (5.13). This bump solution is essentially indistinguishable from the numerical solution for $c = 7.558$, and for $c = 2.0455$ differs from the numerical solution only away from the peak of the bump profile.

Therefore, for $|y| = O(c^{-1/2}) \ll 1$, we have as $c \to \infty$, or equivalently as $\beta \to \beta_1^-$, that the leading order solution in the bump region is

$$
W(y) \sim \sqrt{2c} \text{sech}(\sqrt{c}y), \quad \text{where} \quad c \sim \left[ 3 \left( 1 - \frac{\beta}{\beta_1} \right) \right]^{-1}. \tag{5.13}
$$
In Fig. 17 the solid curves show the full numerical solution to (5.5 a) for \( c = 2.0455 \) (\( \beta \approx 0.1899 \)) and for \( c = 7.558 \) (\( \beta \approx 0.21508 \)). In this figure we also plot the leading order bump solution (5.13) by the dashed curves. We observe that the bump solution is essentially indistinguishable from the full numerical result for \( c = 7.558 \). For \( c = 2.0455 \), the bump solution still provides a decent approximation of the true solution for small \( y \), but differs from the numerical solution in the far-field. We remark that the value \( \beta \approx 0.21508 \) for \( c = 7.558 \) corresponds to the terminal value of \( \beta \) used in the slow sweep as shown in the right panel of Fig. 7.

Next, we use a WKB analysis to analyze (5.5 a) for \( |y| \gg O(c^{-1/2}) \). By symmetry we need only consider the right-side \( y > 0 \) of the maximum of \( W \). For this range of \( y \) where \( W \ll 1 \), we can neglect the \( W^3 \) term in (5.5 a) to obtain the linearized problem

\[
W_{yy} - \frac{y}{2} W_y - \left( c + \frac{1}{2} \right) W = 0. \tag{5.14}
\]

Upon introducing the Liouville transformation

\[
W = \exp \left( \frac{y^2}{8} \right) U(y), \tag{5.15}
\]

we obtain that \( U(y) \) satisfies

\[
U_{yy} - \left[ c + \frac{1}{4} + \frac{y^2}{16} \right] U = 0, \tag{5.16}
\]

Assuming that \( y \ll O(c^{1/2}) \) so that \( c \gg y^2/16 \), we can readily derive a two-term approximate WKB solution for \( U \). Upon using (5.15) to recover \( W \), we obtain in terms of some unknown constant \( b_0 \) that

\[
W_{\text{wkb}} \sim b_0 \exp \left( -\sqrt{c} y + \frac{y^2}{8} - \frac{1}{8\sqrt{c}} \left( y + \frac{y^3}{12} \right) \right). \tag{5.17}
\]

To determine the constant \( b_0 \), we let \( y \to 0 \) in (5.17) and asymptotically match to the far-field behavior \( W \sim \sqrt{2\sqrt{c} e^{-\sqrt{c}y}}/2 \) of the leading-order bump solution \( W(y) = \sqrt{2\sqrt{c} \text{sech}(\sqrt{c}y)} \). This yields that \( b_0 = \sqrt{2\sqrt{c}}/2 \). As such, we have that

\[
W \sim \begin{cases} \sqrt{2\sqrt{c} \text{sech}(\sqrt{c}y)}, & 0 \leq y \leq O(c^{-1/2}), \\
\frac{y^2}{2c} \exp \left( -\sqrt{c} y + \frac{y^2}{8} - \frac{1}{8\sqrt{c}} \left( y + \frac{y^3}{12} \right) \right), & O(c^{-1/2}) \leq y \ll O(c^{1/2}). \end{cases} \tag{5.18}
\]

Next, we observe that the second expression in (5.18) is no longer valid when \( y = O(c^{1/2}) \) as the \(-\sqrt{c}y, y^2/16 \) and \( y^3/\sqrt{c} \) terms are all of comparable order. For this regime where \( y = O(c^{1/2}) \), we return to (5.16) and re-derive the leading-order WKB solution

\[
U \sim \frac{d_0}{[q_0(y)]^{1/4}} e^{-f y^{3/2}} q_0(y), \quad q_0(y) \equiv c + \frac{1}{4} \frac{y^2}{16}. \tag{5.19}
\]

Upon evaluating the integral in (5.19), and using \( W = U e^{y^2/8} \) to recover \( W \), we get for \( y = O(c^{1/2}) \) that

\[
W_{\text{trans}} \sim d_0 \left( c + \frac{y^2}{16} \right)^{-1/4} \left[ \frac{y}{4\sqrt{c}} + \sqrt{1 + \frac{y^2}{16c}} \right]^{-2c-1} \exp \left( \frac{y^2}{8} - \frac{y}{2\sqrt{c} \left( c + \frac{1}{4} + \frac{y^2}{16} \right)^{1/2}} \right). \tag{5.20}
\]

Upon matching this solution to the WKB solution in (5.18), which is valid for \( O(c^{-1/2}) \ll y \ll O(c^{1/2}) \), we identify that \( d_0 = c^{3/4}/\sqrt{2} \). Then, upon letting \( y \gg O(c^{1/2}) \), we obtain from (5.20) that in this far-field region we have algebraic decay for \( W \) as

\[
W_{\text{ff}} \sim \sqrt{2d_0} c^{-1/4} \left( \frac{y}{2\sqrt{c}} \right)^{-2c-1} e^{-c-1/4} = \sqrt{c} \left( \frac{y}{2\sqrt{c}} \right)^{-2c-1} e^{-c-1/4}, \quad y \gg O(\sqrt{c}). \tag{5.21}
\]

This WKB analysis has shown how the exponential decay of the bump solution makes a transition to an algebraic decay in the far-field where \( y \gg O(c^{1/2}) \).
Finally, we observe from (5.18) that, to leading-order, we can use the far-field behavior \( W \sim e^{-\sqrt{2\pi} y/2} \) of the bump solution on the entire range \( O(e^{-1/2}) \ll y \ll O(e^{1/2}) \). Therefore, the integral \( \int_{-\infty}^{\infty} W^3 \, dy \) can be evaluated asymptotically up to negligible exponentially small terms in \( c \) by using only the approximate solution in the bump region. This justifies the calculation in (5.10)–(5.12).

In summary, upon returning to the dynamical re-scaling variables (5.1), it follows that for \( t \to T^- \) and for \( \beta \to \beta_1^- \), where \( \beta_1 \equiv 1/(\sqrt{2\pi}) \), the blow-up solution to (1.3) with a blow-up at \( x = 0 \) has the local behavior

\[
v \sim \frac{1}{\sqrt{T-t}} W \left[ \frac{x}{\sqrt{\varepsilon \sqrt{T-t}}} \right], \quad W(y) \sim \sqrt{\frac{2}{3(1-\beta/\beta_1)}} \text{sech} \left( \sqrt{\frac{1}{3(1-\beta/\beta_1)}} y \right),
\]

when \( |y| = O\left( \sqrt{1-\beta/\beta_1} \right) \ll 1 \). We conclude that the maximum of the profile has the scaling behavior

\[
v_{\text{max}} \sim \frac{1}{\sqrt{T-t}} \sqrt{\frac{2}{3(1-\beta/\beta_1)}}, \quad \beta_1 \equiv \frac{1}{\sqrt{2\pi}}.
\]

In Fig. 18 we show that (5.22 b) compares very favorably with full numerical results computed from (1.3) using FlexPDE6 [11] when \( \beta = \beta_1 - 0.01 \).

6 Discussion

We have used a combination of asymptotic analysis, linear stability theory, and full numerical simulations to investigate intricate dynamical behavior of spike-type solutions to a RD system that leads to an ultimate finite-time blow-up behavior of a localized solution. By slowly ramping a bifurcation parameter linearly in time through various linear stability and existence thresholds associated with multi-spike steady-states, we show that there can be a dynamically intricate route, or transition, of either spike nucleation, spike annihilation, or spike amplitude temporal oscillation that precedes an ultimate finite-time blow-up of a spike amplitude. Our analysis of this new type of spike behavior was motivated by some numerical observations of [16] and [6].

From a mathematical viewpoint, our analysis, together with that in [30], provides one of the first detailed analyses of
delayed bifurcation behavior for localized structures in PDEs. Our study also hints at the rather wide variety of routes to finite-time blow-up in PDE RD systems, as opposed to that which occurs for standard well-studied scalar models, such as those associated with quasilinear heat equations.

A key open theoretical question is to provide a theoretical global existence analysis of solutions to (1.1). For the nonlocal GL model (1.3) we have used a combination of asymptotic analysis and numerical methods to suggest that finite-time blow-up at \( x = 0 \) will occur when \( \beta \) is below the existence threshold \( \beta_1 \) of a one-spike steady-state solution. It would be interesting to provide a rigorous proof of this conjecture, and to analyze the stability of the blow-up profile. In addition, for those parameter ranges for the full RD system (1.1) where finite-time blow-up is observed, it would be interesting to extend the center-manifold approach of [10] to provide a detailed characterization of the blow-up profile through the use of a near-similarity group transformation. In this direction, it would be interesting to characterize analytically the novel route to blow-up observed numerically in Fig. 12, whereby the spike amplitude develops increasingly large oscillations, before apparently bouncing off to infinity.

Acknowledgements

M. J. Ward was supported by an NSERC Discovery Grant 81541 (Canada). J. C. Tzou was supported by a PIMS postdoctoral fellowship.

References


