Lecture (c): Euler-Lagrange equations

We have already shown in different examples the equivalence between the minimization of a functional and the solution of a BVP.

In other words, given a functional $I(u)$ on $D$ with $u$ sufficiently smooth and satisfying BCs on $\partial D$, we have formally

\[ u^* \text{ minimizes } I(u) \iff u^* \text{ is the solution of a BVP} \]

The equation that $u^*$ solves in $D$ is the Euler-Lagrange equation.

The principle to derive the EL equation is always the same:

1) Suppose that $u^*$ is a minimizer of $I(u)$ and construct the following function $u = u^* + \varepsilon w$.

2) If $u^*$ is indeed a minimizer, we have $\left( \frac{d}{d\varepsilon} I(u^* + \varepsilon w) \right)_{\varepsilon=0} = 0$.

3) Calculating $\left( \frac{d}{d\varepsilon} I(u^* + \varepsilon w) \right)_{\varepsilon=0}$ supplies the EL equation.

Depending on the type of BCs, either the trial functions have to satisfied them (Dirichlet BC) or they do not have to and the minimizer will automatically satisfy them (Neumann or mixed BCs).

As a generalization of Example 1 at the start of VH-Lecture (a), we have:

\[
\text{Min } I(u) = \int_D \left( \frac{1}{2} \nabla u^2 + q(x)u \right) dx \iff \begin{cases} \Delta u = q(x) \text{ in } D \\ u = g \text{ on } \partial D \end{cases}
\]

where $D \subset \mathbb{R}^n \ (n=2, 3, 4)$.
Similarly, we can generalize Example 2 from $D \subset \mathbb{R}^2$ to $D \subset \mathbb{R}^n$. Let's find the EL equation for the minimization problem

$$\min_{u \in H} I(u) = \int_D F(x, u, \nabla u) \, dx, \quad H = \{u \in C^2(D), u=g \text{ on } \partial D\}$$

If $u^*$ is a minimizer, introducing $u = u^* + \varepsilon w$ ($w = 0$ on $\partial D$), we have

$$\left(\frac{d}{d\varepsilon} I(u^* + \varepsilon w)\right)_{\varepsilon=0} = 0$$

$$\Rightarrow 0 = \left(\frac{d}{d\varepsilon} \left(\int_D F(x, u^* + \varepsilon w, \nabla u^* + \varepsilon \nabla w) \, dx\right)\right)_{\varepsilon=0}$$

$$= \int_D \left( F_u(x, u^*, \nabla u^*) w + F_{uu}(x, u^*, \nabla u^*) \nabla w \right) \, dx$$

Using the divergence theorem and $w = 0$ on $\partial D$, we get:

$$0 = \int_D \left( -\nabla \cdot F_u(x, u^*, \nabla u^*) + F_u(x, u^*, \nabla u^*) \right) \, dx$$

and the EL equation is:

$$\nabla \cdot F_u(x, u, \nabla u) = F_u(x, u, \nabla u)$$

We can give other examples.

**Example 1:** Higher derivatives case

Let $q(x)$ be a given function on $D$. Find the EL equation for the minimization problem

$$\min_{u \in H} I(u) = \int_D \left( \frac{1}{2} (\Delta u)^2 - q(x) u \right) \, dx, \quad H = \{u \in C^4(D), u=g \text{ on } \partial D\}$$

Let's construct $u = u^* + \varepsilon w$ ($w = \frac{\partial w}{\partial n} = 0$ on $\partial D$)

If $u^*$ is a minimizer, we have

$$\left(\frac{d}{d\varepsilon} I(u^* + \varepsilon w)\right)_{\varepsilon=0} = 0$$
\[ 0 = \left( \frac{d}{dx} \left( \frac{1}{2} (u' + 3w)' \right)^2 - q(u' + 3w) \right) dx \]

Using Green's second identity
\[ \int_D (\Delta u + \Delta w - w \Delta u) \, dx = \int_{\partial D} (\frac{\partial u}{\partial n} + \frac{\partial w}{\partial n} - w \frac{\partial}{\partial n} \Delta u) \, dS(x) = 0 \]

So we get:
\[ 0 = \int_D (\Delta u + \Delta w - w \Delta u) \, dx \]

The EL equation is \( \Delta u = q \) (biharmonic equation)
and the BVP that the minimizer solves is:
\[ \begin{cases} \Delta \Delta u = q & \text{in } D \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \end{cases} \]

**Example 2**: Dirichlet versus Neumann (natural) BCS

For a function \( f(x) \) on \([-1, 1]\), we define the functional \( I(f) \) as
\[ I(f) = \int_{-1}^{1} \left( \frac{d}{dx} \left( \frac{1}{2} (f'(x))^2 + x f(x) \right) \right) dx \]

We will find the minimizer of \( I(f) \) among smooth functions:
(i) satisfying Dirichlet (zero) BCS: \( f(-1) = f(1) = 0 \)
(ii) all smooth functions

The first step is to write the EL equation. Applying (2) in \( D \), i.e., \( \frac{d}{dx} (Fp'(x,f,f')) = Fp(x,f,f') \), we get:
\[ f''(x) = x \]

Hence \( f'(x) = \frac{1}{2} x^2 + A \) and \( f(x) = \frac{x^3}{6} + Ax + B \)
(i) Applying BCs \( f(-2) = f(2) = 0 \), we get:

\[
\begin{align*}
4a + b + c &= 0 \\
-4a - b + c &= 0
\end{align*}
\]

\[\Rightarrow c = 0 \quad \text{and} \quad b = -\frac{4}{9} \]

so \( f(x) = \frac{4}{9} (x^3 - x) = f_{\text{neu}}(x) \)

(ii) Applying no condition on the trial functions, the minimizer will naturally satisfy homogeneous Neumann BCs

\[\Rightarrow f'(-4) = f'(4) = 0 \]

The EL equation is the same but applying the BCs, we get

\[
\begin{align*}
\frac{4}{2} + a &= 0 \\
\frac{4}{2} + a &= 0
\end{align*}
\]

\[\Rightarrow a = -\frac{4}{2} \]

and we have one free parameter \( b \).

so \( f(x) = \frac{x^3}{6} - \frac{x}{2} + b = f_{\text{neu}}(x) \)

Let's calculate the value of \( I(f) \) for these 2 functions

\[
I(f_{\text{neu}}) = \int_{-4}^{4} \left( \frac{4}{2} \cdot \frac{4}{30} \left( 3x^2 - t \right)^2 + x \frac{4}{6} (x^3 - x) \right) \, dx
\]

\[
= \int_{-4}^{4} \left( \frac{4}{72} \left( 9x^4 - 6x^2 + 4 \right) + \frac{7}{6} \left( x^4 - x^2 \right) \right) \, dx
\]

\[
= \frac{7}{72} \left[ \frac{9}{5} x^5 - 2x^3 + x \right]_{-4}^{4} + \frac{7}{6} \left[ \frac{x^5}{5} - \frac{x^3}{3} \right]_{-4}^{4}
\]

\[
= \frac{4}{36} \left( \frac{9}{5} - 2 + 4 \right) + \frac{4}{3} \left( \frac{4}{5} - \frac{1}{3} \right) = -\frac{1}{45}
\]

\[
I(f_{\text{neu}}) = \int_{-4}^{4} \left( \frac{4}{2} \cdot \frac{4}{4} \left( x^2 - t \right)^2 + x \left( \frac{4}{6} x^3 - \frac{2}{2} + b \right) \right) \, dx
\]

\[
= \int_{-4}^{4} \left( \frac{4}{8} \left( x^4 - 2x^2 + 1 \right) + \frac{x^4}{6} - \frac{x^2}{2} + bx \right) \, dx
\]

\[
= \frac{4}{8} \left[ \frac{x^5}{5} - \frac{2}{3} x^3 + x \right]_{-4}^{4} + \left[ \frac{4}{30} x^5 - \frac{4}{6} x^3 + \frac{b}{2} x^2 \right]_{-4}^{4}
\]
\[
= \frac{4}{4} \left( \frac{4}{5} - \frac{2}{3} + \frac{1}{2} \right) + \frac{4}{45} - \frac{1}{3} = -\frac{2}{45}
\]

So we have

\[
I(\text{green}) = -\frac{2}{45} \leq I(\text{rain}) = -\frac{4}{45}
\]

As expected, minimizing over a smaller set of functions (Dirichlet BCs) gives a higher minimum.

Finally, let's show that in both cases (Dirichlet and Neumann BCs, the solution is indeed a minimizer.

Let's denote \( u^+ \) the minimizer. For any function \( u \) on \([-1,1]\), we have:

\[
I(u) - I(u^+) = \int_{-1}^{1} \left( \frac{4}{5} |u|^2 - \frac{4}{5} |u^+|^2 + x(u - u^+) \right) dx
\]

and

\[
|u|^2 - |u^+|^2 = (u - u^+)^2 + 2u^+u - 2(u^+)\]

\[
= (u - u^+)^2 + 2u^+(u - u^+)
\]

\[
\Rightarrow I(u) - I(u^+) = \int_{-1}^{1} \left( \frac{4}{5} (u - u^+)^2 + \int_{-1}^{1} \left( u^+ |u - u^+| + x(u - u^+) \right) dx
\]

\[
= \int_{-1}^{1} \left( u^+ |u - u^+| + x(u - u^+) \right) dx
\]

\[
= \left[ u^+ (u - u^+) \right]_{-1}^{1} - \int_{-1}^{1} u^+ (u - u^+) dx
\]

So we have:

\[
I(u) - I(u^+) \geq \int_{-1}^{1} (-u^+ + x)(u - u^+) dx + \left[ u^+ (u - u^+) \right]_{-1}^{1} = 0
\]

in case of Dirichlet BCs

\[
I(u^+) \leq I(u)
\]

and \( u^+ \) is indeed the minimizer.
Example 3: Brachistochrone problem
A bead slides (frictionlessly) down a curve from a point P to a lower point Q by gravity. What curve gives the shortest travel time?

Let's point P at the origin of the xy-plane, y and the gravity g pointing downwards, put Q lower than P at coordinates (x_Q, y_Q) and describe possible curves C from P to Q as functions x=f(y) (with y_t=f(x_t)).

Let's denote s the arc length along the curve C and t the time. Then the speed of the bead along the curve is u=ds/dt (always tangent to the curve), and the travel time is

\[ T = \int_C dt = \int_C \frac{ds}{u} \]

The arc length differential is given by

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (f'(x)dx)^2} = \sqrt{1+(f'(x))^2} \, dx \]

The velocity u is determined by conservation of energy:

energy = kinetic energy + potential energy = constant

\[ \frac{1}{2} mv^2 - mg y = \text{constant} \]  with m the mass of the bead

A t=0, v=0 and y=0 \implies \text{constant} = 0

and \[ v = \sqrt{2gy} = \sqrt{2g f(x)} \]
So now the travel time can be written:

\[ T = \int_C \frac{ds}{v} = \int_0^{x_2} \sqrt{\frac{4 + (f(x))^2}{2g f(x)}} \, dx = \frac{1}{2g} \int_0^{x_2} \sqrt{\frac{4 + (f(x))^2}{f(x)}} \, dx \]

and we can formulate the following minimization problem: among all smooth functions \( f(x) \) on \([0,x_2]\) that satisfy Dirichlet BCs \( f(0) = 0 \) and \( f(x_2) = y_2 \), find the one that minimizes the functional

\[ I(f) = \int_0^{x_2} F(x, f(x), f'(x)) \, dx \quad \text{with} \quad F(x, f, f') = \sqrt{\frac{4 + (f')^2}{f}} \]

In other words, we want to find \( f \) smooth on \([0,x_2]\) solution of

\[
\begin{align*}
\min_{f \text{ on } [0,x_2]} & \quad I(f) = \int_0^{x_2} F(x, f(x), f'(x)) \, dx \\
\text{with} & \quad F(x, f, f') = \sqrt{\frac{4 + (f')^2}{f}}
\end{align*}
\]

Let’s determine the EL equation for this problem

\[
\begin{align*}
F_{f} & = -\frac{1}{2} f^{-3/2} (1 + (f')^2)^{3/2} \\
F_{f'} & = f^{-1/2} \cdot 2 f' \cdot (1 + (f')^2)^{-1/2} = f' f^{-1/2} (1 + (f')^2)^{-1/2} + \frac{f''}{f (1 + (f')^2)^{3/2}}
\end{align*}
\]

\[
\frac{d}{dx} (F_{f}) = \frac{6' f^{-3/2} (1 + (f')^2)^{1/2} - f' \cdot \frac{1}{2} (f' (1 + (f')^2) + 2 f f'' f'') (f (1 + (f')^2)^{-3/2}}{f (1 + (f')^2)}
\]

such that

\[
0 = F_{f} - \frac{d}{dx} (F_{f'}) = \frac{1}{2 f^{-3/2} (1 + (f')^2)^{3/2}} \left( - (1 + (f')^2)^2 - 2 f (1 + (f')^2) f'' + (f')^2 (1 + (f')^2) + 2 f f'' f' \right) = - \frac{1}{2 f^{-3/2} (1 + (f')^2)^{3/2}} \left( 1 + (f')^2 + 2 f f'' \right)
\]

So the minimizing function solves the following non-linear ODE:

\[ 2 f f'' + (f')^2 = -1 \]

with BCS \( f(0) = 0 \) and \( f(x_2) = y_2 \)
The solution to the PDE is a cycloid, a parametric representation of which is

\[ x(\theta) = R(\theta - \sin \theta) \]
\[ y(\theta) = R(1 - \cos \theta) \]

where \( \theta \in [0, \theta_2] \) when \( x \in [0, x_2] \).

The two parameters \( R \) and \( \theta_2 \) are determined such that the BC \( y_2 = f(x_2) \) is satisfied, i.e.

\[ x_2 = R(\theta_2 - \sin \theta_2) \]
\[ y_2 = R(1 - \cos \theta_2) \]

A cycloid is the curve traced by a point on a circle rolling along a line.

Let's verify that the cycloid indeed solves the PDE:

\[ f' = \frac{dy}{dx} = \frac{R \sin \theta}{R(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta} \]

\[ f'' = \frac{d^2y}{dx^2} = \frac{1}{R(1 - \cos \theta)} \frac{d}{d\theta} \left( \frac{\sin \theta}{1 - \cos \theta} \right) = \frac{1}{R(1 - \cos \theta)} \cdot \frac{\cos \theta(1 - \cos \theta) - \sin^2 \theta}{(1 - \cos \theta)^2} = -\frac{1}{R(1 - \cos \theta)^2} \]

\[ 2f'' + (f')^2 + 1 = 2R'(1 - \cos \theta) \left( -\frac{1}{R(1 - \cos \theta)^2} \right) + \frac{\sin^2 \theta}{(1 - \cos \theta)^2} + 1 \]

\[ = \frac{-2 + 2 \cos \theta + \sin^2 \theta + 1 - 2 \cos \theta + \cos^2 \theta}{(1 - \cos \theta)^2} \]

\[ = \frac{-1 + \sin^2 \theta + \cos^2 \theta}{(1 - \cos \theta)^2} = 0 \]
Among the various properties of the solution, note that
\[
\lim_{\theta \to 0} f' = \lim_{\theta \to 0} \frac{\sin \theta}{\sin^2 \theta} = +\infty
\]

(Rayleigh series at \( \theta \to 0 \):
\[
\frac{\sin \theta}{\sin^2 \theta} \approx \frac{\theta}{\theta^2} = \frac{1}{\theta}
\]

so at the start, the tangent to the curve is always vertical.

**Example 4: Shortest curve between two points**

Suppose we want to connect \( P(0,0) \) to \( Q(x_2, y_2) \) by a curve of least possible length.

In other words, we want to find \( y = f(x) \) such that
\[
L = \int_{x_0}^{x_2} \sqrt{1 + (f')^2} \, dx
\]
is minimum, with \( f(0) = 0 \) and \( f(x_2) = y_2 \).

So here we have
\[
F(x, f, f') = \sqrt{1 + (f')^2}
\]

Let's determine the EL equation for this problem.

\[
F_f = 0
\]

\[
F_{f'} = \frac{f'}{(1 + (f')^2)^{3/2}}
\]

\[
\frac{d}{dx} F_{f'} = \frac{f''(1 + (f')^2)^{4/2} - f'f'f''(1 + (f')^2)^{-1/2}}{(1 + (f')^2)^{4/2}} = \frac{f''}{(1 + (f')^2)^{3/2}}
\]

The minimizing function solves \( f'' = 0 \Rightarrow f = Ax + B \)

Now using the BCs: \( f(0) = 0 \Rightarrow B = 0 \)
\( f(x_2) = y_2 \Rightarrow A = \frac{y_2}{x_2} \)

Finally: \( f(x) = \frac{y_2}{x_2} \cdot x \) : the solution is a line! Amazing!!