GREEN'S FUNCTIONS FOR TIME-DEPENDENT PROBLEMS

Lecture (a): Green's functions for time-dependent PDEs and free-space Green's functions

So far we have looked at space-dependent PDEs only. However, many problems in physics and engineering are governed by equations that depend both on space and time. Some of the most popular classical equations are the heat equation and the wave equation. We will revisit/extend the results from GREEN'S FUNCTIONS - Lectures (g)-(h) to these two problems. We start by the heat equation.

Green's function for the heat equation

Let $u(x,t)$ be the temperature at each point $x$ in a bounded region $V$ in $\mathbb{R}^n$ ($n=1,2,3$) and each time $t$. The region is subjected to heat sources $f(x,t)$ and at its boundary $\partial V$ the temperature is held at $g(x,t)$, $x \in \partial V$. The temperature $u(x,t)$ satisfies the following initial-boundary value problem for the heat equation:

\[
\begin{align*}
\text{Heat equation:} & \quad \frac{\partial u}{\partial t} - \Delta u = f(x,t), \quad x \in V, \quad t > 0 \\
\text{BC:} & \quad u(x,t) = g(x,t), \quad x \in \partial V, \quad t > 0 \\
\text{IC:} & \quad u(x,0) = \varphi(x), \quad x \in V
\end{align*}
\]

where $\varphi(x)$ is the temperature distribution in $V$ at time $t=0$ and $D > 0$ is the diffusivity.

Our goal is to write the solution of this problem using a Green's function.
The operator \( L \) acting on \( u \) is:

\[
L = \frac{\partial}{\partial t} - \Delta \tag{2}
\]

Let's first find the adjoint operator \( L^* \).

Before that, we fix a time interval \([0, T]\) \((T > 0)\) and denote \( C_T = V \times [0, T] \) the space-time cylinder.

Integrating over \( C_T \), we have:

\[
\int_{C_T} \nabla \cdot (u L u) \, dx \, dt = \int_{C_T} \nabla \cdot (\frac{2u}{\partial t} - D \Delta u) \, dx \, dt
\]

\[
= \left[ \nabla (u u) \right]_0^T \, dx - \int_{C_T} u \frac{\partial (\Delta u)}{\partial t} \, dx \, dt - \int_{0}^{T} \int_{V} u \cdot \nabla (D \Delta u) \, dx \, dt
\]

Green's second identity:

\[
\int_{V} u \, dx = \int_{V} u \, dx + \int_{\partial V} \left( u \frac{\partial u}{\partial n} - u \frac{\partial u}{\partial n} \right) \, ds
\]

\[
\Rightarrow \int_{C_T} \nabla \cdot (u L u) \, dx \, dt = \int_{C_T} u \left( -\frac{\partial u}{\partial t} - D \Delta u \right) \, dx \, dt + \left[ \nabla (u u) \right]_0^T \, dx
\]

\[
+ \int_{0}^{T} \int_{V} \left( \nabla \cdot D \Delta u \right) \, dx \, dt
\]

\[
= \int_{C_T} u \cdot L^* u + BT + IT
\]

\[
L^* = -\frac{\partial}{\partial t} - D \Delta \tag{3}
\]

and \( L \) is not self-adjoint.

Our Green's function should be, as before, a function of two sets of variables \((x, t), y, T\), \(x, y \in V\) and \(t, T \geq 0\).

To determine the Green's function problem, we assume \( u \) solves problem (1) and redo the integration by parts above for \( u = 0 \) in variables \( y \) and \( T \). We get:

\[
\]
\[ \int_G G \cdot \mu \, dy \, dt = \int_C L \cdot \mu \, dy \, dt + \int_V \left( (G \mu_t)_{t=T} - (G \mu_t)_{t=0} \right) dy \]
\[ + \int_0^T \int_V D \left( \mu \frac{\partial G}{\partial x} - G \frac{\partial \mu}{\partial x} \right) ds(y) \, dt \]

If our Green's function \( g(x, t; y, \tau) \) solves:

\[ - \frac{\partial \phi}{\partial t} - D \frac{\partial \phi}{\partial x} = \delta(y-x) \, \delta(\tau-t), \quad y \in \Omega, \quad 0 < \tau \leq t \]
\[ g_{x \tau}(y, \tau) = 0, \quad y \in \Omega \]
\[ g_{x \tau}(y, \tau) = 0 \quad \text{for } \tau > t \quad (\text{"causality"}) \]

The solution formula for \( u(x, t), x \in \Omega, 0 \leq t \leq T \), reads

\[ u(x, t) = \int_0^T \int_V g(x, t; y, \tau) \, f(y, \tau) \, dy \, d\tau + \int_V g(x, t; y, 0) \, w(y) \, dy \]
\[ + \int_0^T \int_V D \frac{\partial g}{\partial x} \, q(y, \tau) \, ds(y) \, d\tau \]

Remarks:

1) Causality condition: it simply implies that the solution at time \( \tau \) should not depend on any of its values at any later time than \( \tau \) (quite rational!)

2) The POE that the Green's function \( G \) solves in problem (2) looks a bit odd due to the term \(- D \frac{\partial \phi}{\partial x}\) which is a backward in time derivative. As it is not very intuitive to solve backward in time (from \( \tau = t \) to \( \tau = 0 \)), it is convenient to apply the following change of variable: \( \tau = t - \tau \left( \frac{\partial}{\partial \tau} = - \frac{\partial}{\partial \tau} \right) \) and rewrite the problem in terms of \( \tau \) as the time variable. Doing this, we get:
\[
\frac{\partial \psi}{\partial t} - \Delta \psi = S(y-x) \delta(t), \quad y \in \mathbb{R}^n, \quad 0 \leq t < \tau
\]
\[
G(y, \tau) = 0, \quad y \in \mathbb{R}^n
\]
\[
G(y, \tau) = 0 \quad \text{for} \quad \tau < 0
\]  

(3)

Let's try to solve this problem in various situations. The easiest case is the one without any boundary: the free-space case.

- **Free-space Green's function for the heat equation**
  
  We want to solve problem (3) when \( V = \mathbb{R}^n \). To do that, we will use a Fourier transform in space of \( \mathbb{R}^n \).

Let's first define the definition and the major properties of Fourier transforms.

**Definition:** Let \( f \) be an integrable function on \( \mathbb{R}^n \) \( \left( \int_{\mathbb{R}^n} |f(x)| dx < \infty \right) \).

The Fourier transform \( \mathcal{F}(f) = \mathcal{F}(f) \) is defined by

\[
\mathcal{F}(f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iwx} f(x) \, dx, \quad w \in \mathbb{R}^n
\]

Let \( f \) and \( g \) be smooth functions with rapid decay at \( \infty \).

**Properties of Fourier transform (FT)**

a) **Linearity:** \( \mathcal{F}(af + bg) = a \mathcal{F}(f) + b \mathcal{F}(g) \), \( a, b \in \mathbb{R} \).

b) **Inverse Fourier transform:** \( \mathcal{F}^{-1}(H)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iwx} H(w) \, dw \)

If \( H(w) = \mathcal{F}(f) \), we have: \( \mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x) \)

c) **Parseval's formula:**

\[
\int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx = \int_{\mathbb{R}^n} \mathcal{F}(f) \overline{\mathcal{F}(g)} \, dw, \quad \text{where} \quad \overline{\cdot} = \text{conjugate}
\]
that leads to Plancherel’s theorem:

\[ \int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |F_\mathcal{F}(w)|^2 dw \]

FT preserves the \( L^2 \)-norm.

d) Interchange of differentiation and coordinate multiplication

\[ \frac{\partial}{\partial x_j} F_\mathcal{F}(\omega) = i \omega_j F_\mathcal{F}(\omega) \quad \text{and} \quad F_\mathcal{F}(f(x)) = i \frac{\partial}{\partial x_j} F_\mathcal{F}(w) \]

e) Interchange of convolution and multiplication

\[ F_\mathcal{F}(f * g)(\omega) = (2\pi)^n F_\mathcal{F}(f)(\omega) F_\mathcal{F}(g)(\omega) \] with \( (f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy \)

f) Interchange of coordinate translation and multiplication by an exponential

\[ F_\mathcal{F}(e^{i\omega \cdot x})(\omega) = e^{-i\omega \cdot x} F_\mathcal{F}(f)(\omega) \quad \text{and} \quad a \in \mathbb{R}^n \]

g) Mapping of Gaussians to Gaussians

\[ F_\mathcal{F}(e^{-\frac{1}{2} \omega \cdot x^2})(\omega) = a^{-\frac{n}{2}} e^{-\frac{1}{2a} \omega \cdot \omega} , \quad a \in \mathbb{R}^n \]

Let’s prove (or at least give sketches of proof) some of these properties

+ Linearity is obvious

\[ F_\mathcal{F}(\alpha f + \beta g)(\omega) = \frac{4}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\omega \cdot (\alpha f(x) + \beta g(x))} dx \]

\[ = \alpha \cdot \frac{4}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\omega \cdot f(x)} dx + \beta \cdot \frac{4}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\omega \cdot g(x)} dx = \alpha F_\mathcal{F}(f)(\omega) + \beta F_\mathcal{F}(g)(\omega) \]

+ Proving \( F_\mathcal{F}^{-1}(F_\mathcal{F}(f)(x)) = f(x) \) is quite difficult (see an analysis textbook)

We can give a “loose” proof using one of the definitions of the delta Dirac function as a limit: \( \delta(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \cdot \sin \left( \frac{x}{\epsilon} \right) \)
Let's show how to proceed in \( \mathbb{R}^n \) \((n=1)\)

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega x} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(y) e^{-i\omega y} dy e^{i\omega x} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(y) \frac{e^{i\omega(x-y)}}{\cos(\omega(x-y)) + i\sin(\omega(x-y))} dy
\]

\[
\int_{-\infty}^{+\infty} \cos(\omega(x-y)) d\omega = \lim_{L \to \infty} \int_{-L}^{L} \cos(\omega(x-y)) d\omega = \lim_{L \to \infty} \left[ \frac{\sin(L(x-y))}{(x-y)} \right]_{-L}^{L} = \lim_{L \to \infty} \frac{2}{(x-y)} \sin(L(x-y))
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sin\left(\frac{x-y}{\varepsilon}\right)
\]

\[
= 2\pi \lim_{\varepsilon \to 0} \frac{1}{\pi} \sin\left(\frac{x-y}{\varepsilon}\right) = 2\pi \delta(x-y)
\]

\[
\int_{-\infty}^{+\infty} \sin(\omega(x-y)) d\omega = \lim_{L \to \infty} \int_{-L}^{L} \sin(\omega(x-y)) d\omega = \lim_{L \to \infty} \left[ -\frac{1}{(x-y)} \cos(\omega(x-y)) \right]_{-L}^{L} = 0
\]

So

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi \delta(x-y) \hat{f}(y) dy = \hat{f}(x)
\]

* Parseval's formula

\[
g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(w) e^{i\omega x} dw = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(w) e^{i\omega x} dw
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(w) e^{-i\omega x} dw
\]

\[
\int_{\mathbb{R}^n} \hat{f}(w) \hat{g}(w) dw = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(w) \overline{\hat{g}(w)} e^{-i\omega x} dw dw
\]

\[
= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(w) e^{-i\omega x} dw \cdot \overline{\hat{g}(w)} dw = \int_{\mathbb{R}^n} \overline{\hat{f}(w)} \hat{g}(w) dw
\]

\[
= F_g(w)
\]
Taking \( g = f \), we get Plancherel’s theorem as \( \overline{z} z = |z|^2 \).

For any complex number \( z \in C \).

Taking \( g = F \hat{h} \) (with \( h \) being a function of \( \omega \) here), we get:

\[
(F \hat{g}, \hat{h}) = (g, \hat{F}^\dagger \hat{h})
\]

relation between adjoint and inverse operators.

The differentiation property (property d) is very useful for solving differential equations as it turns PDEs into ODEs and ODEs into algebraic equations. It is very easy to prove by simple integration by parts:

\[
\begin{align*}
\mathcal{F} \frac{\partial f}{\partial \omega}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial}{\partial \omega} e^{i \omega \cdot x} \, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[ f e^{i \omega \cdot x} \right]_{-\infty}^{+\infty} \, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-i \omega \cdot \nabla) e^{i \omega \cdot x} f(x) \, dx \\
&= i \omega \cdot \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i \omega \cdot x} f(x) \, dx = i \omega \cdot \mathcal{F} f(\omega)
\end{align*}
\]

Hence Gaussians to Gaussians

We will prove this result in \( \mathbb{R}^n (n=1) \), the generalization to higher dimensions \( (n > 1) \) being similar. Also, in \( \mathbb{R}^1 \), we can show the result in two different ways:

1) Using Cauchy’s theorem (integration over a closed contour in the complex plane).

2) By deriving an 1st order ODE for \( \mathcal{F} e^{-a \frac{x^2}{2}}(\omega) \)

We have \( f(x) = e^{-a \frac{x^2}{2}} \) and \( \mathcal{F} f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i \omega x} e^{-a \frac{x^2}{2}} \, dx 

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i \omega x - a \frac{x^2}{2}} \, dx
\]
Recognizing that: 
\[-ixw - \frac{a}{2}x^2 = -(ixw + \frac{x^2}{2\sqrt{2}} - \frac{w^2}{2a}) - \frac{w^2}{2a}\]
\[= -\frac{a}{2} \left( x^2 - \frac{w^2}{a^2} + 2i\frac{xw}{a} \right) - \frac{w^2}{2a}\]
\[= -\frac{a}{2} \left( x + i\frac{w}{a} \right)^2 - \frac{w^2}{2a}\]

we get: 
\[F(z) = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx = I\]

I is the integral of the complex function \(R(z) = e^{-\frac{z^2}{2}}\) along the contour \(z = x + i\frac{w}{a}, -\infty < x < \infty\) in the complex plane.

To compute \(I\), we use Cauchy's theorem over the following close contour \(\gamma\):

\[
\oint \gamma R(z) dz = 0 \quad \text{from Cauchy's theorem}
\]

And \(\oint \gamma R(z) dz = \int_{-R}^{R} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx + \int_{-\infty}^{0} e^{-\frac{a}{2}(R^2 - y^2)} dy + \int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy + \int_{-\infty}^{0} e^{-\frac{a}{2}(R^2 - y^2)} dy\)

\[= \int_{-R}^{R} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx + \int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy\]

\[= \int_{-R}^{R} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx + \int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy\]

\[\leq \int_{-R}^{R} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx + \int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy\]

\[\leq \int_{-R}^{R} e^{-\frac{a}{2}(x + i\frac{w}{a})^2} dx + \int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy\]

And so \(\int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy \leq \frac{\omega^2}{a} e^{-\frac{a}{2}(R^2 - \frac{\omega^2}{a})} = e^{-\frac{a}{2}\frac{R^2}{a} - \frac{\omega^2}{2a}} \rightarrow 0\)

We can do the same with \(\int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy\) and show that \(\int_{0}^{\infty} e^{-\frac{a}{2}(R^2 - y^2)} dy \rightarrow 0\)
Hence we get: \[ \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x + i \frac{w}{a})^2} \, dx = \int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} \, dx \]

To compute the last integral, it is easier to compute its square:

\[ (\int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} \, dx)^2 = \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} \, dx \cdot \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} \, dx \]

Changing to cylindrical coordinates: \( r = \sqrt{x^2 + \frac{w^2}{a}} \)

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{a}{2}(x^2 + \frac{w^2}{a})} \, dx \, dr \]

\[ = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{a}{2}r^2} \, r \, dr \, d\theta \]

\[ = 2\pi \left[-\frac{1}{a} e^{-\frac{a}{2}r^2}\right]_{0}^{\infty} = \frac{2\pi}{a} \]

Finally, we have:

\[ F_{\phi}(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2a}} \sqrt{\frac{2\pi}{a}} = \frac{1}{\sqrt{a}} e^{-\frac{w^2}{2a}} \]

2) Let's calculate \( \frac{dF_{\phi}(w)}{dw} \) and integrate by parts:

\[ \frac{dF_{\phi}(w)}{dw} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -ix e^{-\frac{a}{2}x^2} e^{-iwx} \, dx \]

\[ = \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{a} e^{-\frac{a}{2}x^2} e^{-iwx}\right]_{-\infty}^{+\infty} - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{a}{2} e^{-\frac{a}{2}x^2} (-iw) e^{-iwx} \, dx \]

\[ = -\frac{1}{\sqrt{2\pi}} \frac{w}{a} \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} e^{-iwx} \, dx = \left(\frac{w}{a}\right) F_{\phi}(w) \]

So \( F_{\phi}(w) \) satisfies the following ODE:

\[ \frac{dF_{\phi}(w)}{dw} + \frac{w}{a} F_{\phi}(w) = 0 \]

\[ \Rightarrow F_{\phi}(w) = F_{\phi}(0) e^{-\frac{w^2}{2a}} \quad \text{with} \quad F_{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} \, dx = \sqrt{\frac{2\pi}{a}} \]

From above, we already know that \( \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} \, dx = \sqrt{\frac{2\pi}{a}} \)

and we obtain again \( F_{\phi}(w) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{a}} \cdot e^{-\frac{w^2}{2a}} = \frac{1}{\sqrt{a}} e^{-\frac{w^2}{2a}} \)
Back to the problem of finding $G_{x}(y, T)$, let's apply a Fourier transform in space $y \rightarrow w$ to the PDE in problem (3). Property d) turns $\Delta$ in $y$ to a multiplication by $-|w|^2$ in $w$.

Also, the Fourier transform of $S_{x}(y)$ is

$$F_{S_{x}}(w) = \frac{1}{(2\pi)^{n}} \int_{R^{n}} e^{-iwy} S(y-x) dy = \frac{4}{(2\pi)^{n}} e^{-iwx}$$

So we get:

$$\frac{2}{\partial T} F_{\tau} + D |w|^2 F_{\tau} = \frac{4}{(2\pi)^{n}} e^{-iwx} S(T)$$

This is an ODE in $\tau$ that we can easily solve for $T > 0$.

For $T > 0$, we actually solve:

$$\frac{2}{\partial T} F_{\tau} + D |w|^2 F_{\tau} = 0 \text{ for all } w,$$

that gives $F_{\tau} = Ce^{-D|w|^2\tau}$, with $C = C(x_{1}, t, \omega)$.

We determine $C$ by a jump condition in time:

$$\frac{4}{(2\pi)^{n}} \int_{-\varepsilon}^{+\varepsilon} e^{-iwx} S(T) dT = \int_{-\varepsilon}^{+\varepsilon} \left( \frac{\partial F_{\tau}}{\partial T} + D |w|^2 F_{\tau} \right) dT$$

$$= \left[ F_{\tau} \right]_{-\varepsilon}^{+\varepsilon} + \int_{-\varepsilon}^{+\varepsilon} D |w|^2 F_{\tau} dT$$

$$\xrightarrow{\varepsilon \to 0} 0 \text{ as } F_{\tau} \text{ is bounded}$$

$$= F_{\tau}(\varepsilon) - F_{\tau}(-\varepsilon) = 0 \text{ from causality condition}$$

(If $C = 0$ for $T < 0 \Rightarrow F_{\tau} = 0$, $T < 0$)

Finally, $\varepsilon \to 0$, we get $F_{\tau}(0) = C = \frac{4}{(2\pi)^{n}} e^{-iwx}$.

and we get $F_{\tau} = \frac{4}{(2\pi)^{n}} e^{-iwx} e^{-D|w|^2\tau}$.

Using property f) and g) (with $a = \frac{4}{2DT}$), we can invert back the Fourier transform $F_{\tau}$ to $G$ as:
\[ F^{-1}_e \omega \sigma(y) = \frac{a}{(2\sigma^2)^n} e^{-\frac{|y|^2}{4a^2}} \quad (\text{Property } g) \text{ with } a = \frac{1}{20\pi} \]  \( \tag{6} \)

* Multiplication by \( e^{-i\omega x} \) → shift by \( x \)

\[ G_{x,t}(y,T) = F^{-1}_e \varphi_T(w) = \frac{1}{(2\pi)^n} \cdot \frac{1}{(2\sigma^2)^n} e^{-\frac{|y-x|^2}{(2\sigma^2)(t-T)}} \quad \forall T \geq 0 \]

Changing back to \( T = t - \tau \), the free-space Green's function of the heat equation in \( \mathbb{R}^n \) is

\[ G_{x,t}(y,T) = \begin{cases} \frac{1}{(4\pi(t-T))^n} e^{-\frac{|y-x|^2}{4(t-T)}}, & t < T \\ 0, & t \geq T \end{cases} \]  \( \tag{4} \)

As a result, the solution formula for the heat equation in \( \mathbb{R}^n \)

\[ \frac{\partial u}{\partial t} - \Delta u = f(x,t), \quad x \in \mathbb{R}^n, \quad t > 0 \]

\[ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \]  \( \tag{5} \)

reads (plugging (4) into (2)):

\[ u(x,t) = \int \int \frac{1}{(4\pi(t-T))^n} e^{-\frac{|y-x|^2}{4(t-T)}} f(y,T) dydT 
+ \frac{1}{(4\pi(t-T))^n} \int \frac{1}{2(t-T)^n} \cdot u_0(y) dy \]  \( \tag{6} \)

and we have the following theorem:

Suppose that \( f(x,t) \) is continuously differentiable and bounded on \( \mathbb{R}^n \times [0,T] \) and \( u_0(x) \) is continuous on \( \mathbb{R}^n \). Then for \( 0 < t < T \), \( u(x,t) \) given by (6) is continuously differentiable in \( t \) and twice continuously differentiable in \( x \), and solves the heat equation in (5). Furthermore, for any \( x \in \mathbb{R}^n \), \( \lim_{t \to 0} u(x,t) = u_0(x) \)
The free-space solution formula (6) exhibits two important properties of diffusion. In fact, in the absence of sources ($f=0$), we have:

1) Instantaneous smoothing
   For $f=0$, $u(x,t)$ given by (6) becomes instantaneously smooth even if the initial data $(u_0(x))$ is not. This property derives from the fact that the fundamental solution (kernel) is infinitely differentiable in $x$ and $t$ for $t>0$, although it is a Dirac delta function in space at $t=0$.
   
   Using another definition of the Dirac delta function at the limit $S(x) = \lim_{c \to 0} \frac{1}{2\sqrt{\pi c}} e^{-\frac{x^2}{4c}}, x \in \mathbb{R}^n$ when $t=0$.

   Solution $\frac{4}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}} = S(x), x \in \mathbb{R}^n$ when $t=0$.

   In fact $\frac{4}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}} = \left( \frac{4}{2\sqrt{\pi Dt}} \right)^n e^{-\frac{|x|^2}{4Dt}} = \prod_{i=1}^{n} \frac{4}{2\sqrt{\pi Dt}} e^{-\frac{x_i^2}{4Dt}}$

   Now, let $x = 0t \ (x \to 0$ when $t \to 0)$, we get:
   
   $\lim_{t \to 0} \frac{1}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} = \frac{1}{2\sqrt{\pi D}} S(x_1) = S(x)$

   Similarly, we have $\frac{4}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}} \to S(0-x)$, which sheds some light on $\lim_{t \to 0} u(x,t) = u_0(x)$.

2) Infinite propagation speed
   For $f=0$, if $u_0(x) > 0$ in a ball of radius $t$ around the origin and vanishes outside that ball. Then for any $t > 0, x \in \mathbb{R}^n$, we have:
   
   $u(x,t) = \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4Dt}} u_0(y) dy > 0$
This means that the solution becomes positive everywhere in space (i.e. for any $x \in \mathbb{R}^n$) instantaneously for $t > 0$, i.e. the initial and localized disturbance is propagated with infinite speed (although with exponential (fast) decay in space).

Let's illustrate that in 1D:

$$u_0(x) = \begin{cases} 
0 & -1 < x < 1 \\
1 & \text{otherwise}
\end{cases}$$

Instantaneous smoothing

Positive everywhere with exponential decay