GREEN'S FUNCTIONS

Lecture (4): Green's functions by eigenfunction expansion

So far, we have shown that the Green's function can be determined in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) (free-space Green's function) or in a region that exhibits particular symmetry properties or a particular geometry (half-planes, quadrants, sphere, etc.). We also proved that Green's functions exist for any region \( D \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

When \( D \neq \mathbb{R}^2 \) or \( \mathbb{R}^3 \) and the method of images does not help, we can try to calculate the Green's function by eigenfunction expansion. We illustrate this here with two examples.

**Example 1**

Let us consider the following Poisson equation in a square \( C \subset \mathbb{R}^2 \) with Neumann BCs

\[
\Delta u + f(x_1, x_2) = 0 \quad (0 < x_1 < \pi, 0 < x_2 < \pi)
\]

\[
\frac{\partial u}{\partial x_1} = 0 \quad \text{at} \quad x_1 = 0 \text{ and } x_1 = \pi
\]

\[
\frac{\partial u}{\partial x_2} = 0 \quad \text{at} \quad x_2 = 0 \text{ and } x_2 = \pi
\]

The corresponding homogeneous problem \((f = 0)\) is likely to have a non-trivial solution \( u^* \).

Let's try to find \( u^* \) by eigenfunction expansion.

First, we determine the eigenvalues/eigenfunctions of the \( \Delta \) operator with Neumann BCs in \([0, \pi] \times [0, \pi]\), i.e. \( \Delta u = \lambda u \) with Neumann BCs.

By separation of variables, we write \( u(x_1, x_2) = F(x_1) G(x_2) \) and we solve two similar problems:
(a) \( \frac{\partial^2 F}{\partial x^2} = \lambda_1 F(x) \), \( F(0) = F(\pi) = 0 \)

(b) \( \frac{\partial^2 G}{\partial x^2} = \lambda_2 G(x) \), \( G(0) = G(\pi) = 0 \)

Let's quickly look at (a).

The roots of the characteristic equation are \( \lambda_{1,2} = \pm \sqrt{\lambda_1} \)
and the solution is of the form \( F(x) = A e^{-\sqrt{\lambda_1} x} + B e^{\sqrt{\lambda_1} x} \)

\( \lambda > 0 \): the solution can be written in the form

\[
F(x) = A \cosh(\sqrt{\lambda_1} x) + B \sinh(\sqrt{\lambda_1} x) \\
F'(x) = A \sqrt{\lambda_1} \sinh(\sqrt{\lambda_1} x) + B \sqrt{\lambda_1} \cosh(\sqrt{\lambda_1} x) \\
F(0) = 0 \implies B = 0 \\
F'(0) = 0 \implies A = 0
\]

\( \lambda = 0 \): \( F = \text{constant} \) is a solution.

\( \lambda < 0 \): the solution can be written in the form

\[
F(x) = A \cos(\sqrt{\lambda_2} x) + B \sin(\sqrt{\lambda_2} x) \\
F'(x) = -A \sqrt{\lambda_2} \sin(\sqrt{\lambda_2} x) + B \sqrt{\lambda_2} \cos(\sqrt{\lambda_2} x) \\
F(0) = 0 \implies B = 0 \\
F'(0) = 0 \implies A = 0 \\
\lambda_2 = -\lambda_{1,2} = \sqrt{\lambda_1}
\]

Remark: \( \lambda_2 = 0 \) recovers the case \( \lambda = 0 \) since \( \cos(\sqrt{\lambda_2} x) = \cos(0) = 1 \); normalized eigenfunction.

We can do the same for problem (b) and eventually derive that

the eigenvalues \( \lambda_{1,2} \) and eigenfunctions of the operator \( \Delta \sin \left[ 0, \pi \right] \times \left[ 0, \pi \right] \)
with Neumann BCs are \(- (\lambda_1^2 + \lambda_2^2)\) and \( \cos(\lambda_1 x) \cos(\lambda_2 x) \), \( \lambda_{1,2} = \pm \pi \).

As a consequence, the eigenvalues of \( \Delta + \delta \) are \( 8 - (\lambda_1^2 + \lambda_2^2) \) and
the eigenfunctions are the same.
For \( n_1 = n_2 = 2 \), the eigenvalue is 0 and the eigenfunction is \( \cos(2x_1) \cos(2x_2) \).

So \( u^*(x_1, x_2) = \cos(2x_1) \cos(2x_2) \) is a non-trivial solution of the homogeneous problem, and we need a solvability condition.

Green's second identity gives:

\[
\int \left( (\Delta u)(x) u^* - u \frac{\partial u^*}{\partial n} \right) \, dx = \int (u \frac{\partial u^*}{\partial n} - u^* \frac{\partial u}{\partial n}) \, ds(x) \quad \text{as} \quad \frac{\partial u}{\partial n} - \frac{\partial u^*}{\partial n} = 0
\]

\[
\int u^* f \, dx = 0 \quad \Rightarrow \quad \int_0^\pi \int_0^\pi \cos(2x_1) \cos(2x_2) f(x_1, x_2) \, dx_1 \, dx_2 = 0
\]

(4)

So the problem has a modified Green's function \( \tilde{\Phi}(x, y) \).

Let's normalize the eigenfunctions such that \( (\delta_{ij}, \delta_{ij}) = 1 \).

For \( (n_1, n_2) = (0, 0) \), \( \Phi_{0,0}(x) = 1 \quad \Rightarrow \quad \int_0^\pi \int_0^\pi 1 \, dx_1 \, dx_2 = \pi^2 = (\Phi_{0,0}, \Phi_{0,0}) \)

\[ \Rightarrow \quad \tilde{\Phi}_{0,0}(x) = \frac{1}{\pi} \]

For \( (n_1, n_2) = (n_2, 0) \), \( \Phi_{n_2,0}(x) = \cos(n_2 x_2) \)

\[ \Rightarrow \quad (\Phi_{n_2,0}, \Phi_{n_2,0}) = \pi \int_0^\pi \cos^2(n_2 x_2) \, dx_2 = \pi \int_0^\pi \frac{1}{2} \left[ 1 + \cos(2n_2 x_2) \right] \, dx_2 \]

\[ = \pi \left[ \frac{x_2}{2} + \frac{1}{4n_2^2} \sin(2n_2 x_2) \right]_0^{\pi} = \frac{\pi^2}{2} \]

\[ \Rightarrow \quad \tilde{\Phi}_{n_2,0}(x) = \frac{\sqrt{2}}{\pi} \cos(n_2 x_2) \quad n_2 = 1, 2, \ldots \]

Similarly, we get:

For \( (n_1, n_2) = (0, n_2) \), \( \tilde{\Phi}_{0,n_2}(x) = \frac{\sqrt{2}}{\pi} \cos(n_2 x_2) \quad n_2 = 1, 2, \ldots \)

For \( (n_1, n_2), n_1 \neq 0, n_2 \neq 0 \), \( \tilde{\Phi}_{n_1,n_2}(x) = \frac{2}{\pi} \cos(n_1 x_1) \cos(n_2 x_2) \)
The modified Green's function $\tilde{G}_x(y)$ solves
\[(\Delta + \delta) \tilde{G}_x(y) = \delta_x(y) - \frac{u^+(x)}{u^+(\infty)} \cdot u^+(y) + \text{homogeneous Neumann BCs}\]

Here $u^+(\infty) = \Phi_{2,2}(x)$ and $(\Phi_{2,1}, \Phi_{2,2}) = 4$

\[= \left(\Delta + \delta\right) \tilde{G}_x(y) = \delta_x(y) - \Phi_{2,2}(x) \Phi_{2,2}(y) \quad (2)\]

Now let's expand $\tilde{G}_x(y)$ in terms of eigenfunctions:
\[\tilde{G}_x(y) = \sum_{n_1, n_2} c_{n_1 n_2}(x) \Phi_{n_1 n_2}(y)\]

and plug it into (2):
\[\sum_{n_1, n_2} c_{n_1 n_2}(x) \underbrace{(\Delta + \delta) \Phi_{n_1 n_2}(y)}_{(8 - (n_1^2 + n_2^2)) \Phi_{n_1 n_2}(y)} = \delta_x(y) - \Phi_{2,2}(x) \Phi_{2,2}(y)\]

and multiply both sides by $\Phi_{n_1 n_2}(y)$, we get
\[\underbrace{(8 - (n_1^2 + n_2^2)) c_{n_1 n_2}(x)}_{(8 - (n_1^2 + n_2^2))} = \delta_x(y) \Phi_{n_1 n_2}(y) - \Phi_{2,2}(x) \Phi_{2,2}(y) \Phi_{n_1 n_2}(y)\]

so that:
For $(n_1, n_2) \neq (2,2)$:
\[c_{n_1 n_2}(x) = \frac{1}{8 - (n_1^2 + n_2^2)} \Phi_{n_1 n_2}(x)\]

For $(n_1, n_2) = (2,2)$: we have $0 \cdot c_{2,2}(x) = \Phi_{2,2}(x) - \Phi_{2,2}(x) = 0$.

So $c_{2,2}(x)$ is free, we can take it to 0 for convenience.

Finally, $\tilde{G}_x(y)$ writes
\[\tilde{G}_x(y) = \sum_{n_1, n_2 \geq 0} \frac{1}{8 - (n_1^2 + n_2^2)} \Phi_{n_1 n_2}(x) \Phi_{n_1 n_2}(y)\]

\[\Rightarrow \tilde{G}_x(y) = \sum_{n_1, n_2 \geq 0} \frac{1}{8 - (n_1^2 + n_2^2)} \cdot \text{an} \cdot \cos(n_1 x_1) \cos(n_2 x_2) \cos(n_1 y_1) \cos(n_2 y_2)\]
with
\[ a_{n_1 n_2} = \begin{cases} \frac{4}{\pi^2} & n_1 = n_2 = 0 \\ \frac{2}{\pi^2} & n_1 = 0, n_2 = 1, 2, 3, \ldots \text{ or } n_1 = 1, 2, 3, \ldots, n_2 = 0 \\ \frac{4}{\pi^2} & n_1, n_2 \neq 0 \text{ and } n_1 \neq n_2 \end{cases} \]
\[
\Delta u = f(x, \theta), \quad \Delta > 0, \quad 0 < \theta < \pi \\
\mu = 0, \quad \theta = 0 \text{ and } \theta = \pi
\]

Example 2

Let us consider the following Poisson equation in an infinite wedge of angle \( \alpha \). It is convenient to write the problem in cylindrical coordinates:

\[ \Delta u = f(x, \theta), \quad \Delta > 0, \quad 0 < \theta < \pi \]
\[ \Delta > 0 \quad \mu = 0, \quad \theta = 0 \text{ and } \theta = \pi \]

The Dirichlet Green's function solves:

\[ \Delta G_{\mu \theta}(s, \eta) = \delta(s-r) \delta(\eta-\theta), \quad \mu > 0 \]
\[ G_{\mu \theta}(s, 0) = G_{\mu \theta}(s, \pi) = 0 \]

Remark:
1. The delta Dirac function in cylindrical coordinates writes
   \[ S_{\mu \theta}(s, \eta) = \frac{1}{\sqrt{\lambda}} S(s-r) S(\eta-\theta) \]
   as \( ds \, dr \, d\theta \)
   \[ \Rightarrow \Delta G_{\mu \theta}(s, \eta) = \frac{1}{\sqrt{\lambda}} S(s-r) S(\eta-\theta) \]

2. We will drop the subscripts \( \mu \) \( \theta \) for clarity when writing \( G \)

So we look for \( G(s, \eta) \) solving:

\[ \frac{\partial^2 G}{\partial s^2} + \frac{4}{s} \frac{\partial G}{\partial s} + \frac{4}{s^2} \frac{\partial^2 G}{\partial \eta^2} = \frac{1}{\lambda} S(s-r) S(\eta-\theta), \quad \mu > 0, \quad 0 < \eta < \pi \]
\[ G(s, 0) = G(s, \pi) = 0 \]

We know that the eigenvalues and eigenfunctions for the angular part of \( \Delta \), i.e. \( \frac{\partial^2}{\partial \theta^2} \), with zero BCs at \( \eta = 0 \) and \( \eta = \pi \).
are \(-\frac{(n\pi}{a})^2\) and \(\sin\left(\frac{n\pi}{a}\eta\right)\) so we try a \(G\) in the form of the following eigenfunction expansion:

\[
G(s, \eta) = \sum_{n=1}^{\infty} c_n(s) \sin\left(\frac{n\pi}{a}\eta\right)
\]

(4)

Now we need to determine the coefficients \(c_n(s)\).

Plugging (4) into (3), we get:

\[
\sum_{n=1}^{\infty} \left( c_n'' + \frac{4}{s} c_n' - \left(\frac{n\pi}{a}\right)^2 c_n \right) \sin\left(\frac{n\pi}{a}\eta\right) = \frac{4}{a} \sin\left(\frac{k\pi}{a}\eta\right) \eta = \frac{4}{a} s(s-\eta) \sin\left(\frac{k\pi}{a}\eta\right)
\]

Multiplying both sides by \(\sin\left(\frac{k\pi}{a}\eta\right)\), and integrating between 0 and \(a\), we get:

\[
\sum_{n=1}^{\infty} \left( c_n'' + \frac{4}{s} c_n' - \left(\frac{n\pi}{a}\right)^2 c_n \right) \int_0^a \sin\left(\frac{n\pi}{a}\eta\right) \sin\left(\frac{k\pi}{a}\eta\right) d\eta = \frac{4}{a} \sin\left(\frac{k\pi}{a}\eta\right) \eta
\]

\[
= \begin{cases} 
\frac{a}{2} & \text{if } k=n \\
0 & \text{if } k \neq n
\end{cases}
\]

\(\Rightarrow \quad c_n'' + \frac{4}{s} c_n' - \left(\frac{k\pi}{a}\right)^2 c_n = 0 \quad k \geq 1
\]

(5)

For \(s \neq a\), we have:

\(c_n'' + \frac{4}{s} c_n' - \left(\frac{k\pi}{a}\right)^2 c_n = 0 \quad k \geq 1
\)

This is an Euler ODE whose solutions are of the form \(s^b\)

\(\Rightarrow \quad 0 = b(b-1) + b - \left(\frac{k\pi}{a}\right)^2 = (b - \frac{k\pi}{a})(b + \frac{k\pi}{a})
\)

and so \(b = \pm \frac{k\pi}{a}\)

and \(c_n(s) = \begin{cases} 
A s^{\frac{k\pi}{a}} + B s^{-\frac{k\pi}{a}} & \text{for } 0 < s < a \\
C s^{\frac{k\pi}{a}} + D s^{-\frac{k\pi}{a}} & \text{for } a < s < \infty
\end{cases}
\)

\(G_0(0) = 0 \Rightarrow c_k(0) = 0 \Rightarrow B = 0
\)

To avoid growth as \(s \to \infty\), we take \(C = 0\)
Hence, we have
\[ c_k(s) = \begin{cases} 
A s^{\frac{k-1}{k}} & 0 < s < \lambda \\
0 s^{\frac{k-1}{k}} & \lambda < s < \infty
\end{cases} \]

Matching conditions at \( s = \lambda \)

+ Continuity: \( A\lambda = D\lambda^{\frac{k}{k+1}} \)

\[ \Rightarrow D = A \lambda^{\frac{2}{k+1}} \] (6)

+ Jump condition:
\[ \int_{\lambda^{-\varepsilon}}^{\lambda^{+\varepsilon}} \left[ c_k''(s) + \frac{4}{5} c_k'(s) - \left( \frac{k+n}{ds} \right)^2 c_k(s) \right] ds = \int_{\lambda^{-\varepsilon}}^{\lambda^{+\varepsilon}} \frac{2}{dn} (s-\lambda)^{n-1} \min \left( \frac{k+n}{\lambda} \theta \right) ds \]

\[ \Rightarrow \left[ c_k'(\lambda^{-\varepsilon}) + \int_{\lambda^{-\varepsilon}}^{\lambda^{+\varepsilon}} \left( \frac{4}{5} c_k'(s) - \left( \frac{k+n}{ds} \right)^2 c_k(s) \right) ds = \frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \right] \]

\[ \xrightarrow{\varepsilon \to 0} \]

\[ \Rightarrow c_k'(\lambda^{+}) - c_k'(\lambda^{-}) = \frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \]

Using the expression of \( c_k(s) \), we get
\[ \frac{2}{dn} \left( \frac{k}{\lambda^{k-1}} \right) = \frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \]

\[ \Rightarrow -\frac{k+n}{\theta} \left( 1 - \frac{k}{\lambda^{k-1}} \right) A \lambda^{\frac{k}{k+1}} = -\frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \] (7)

Plugging (6) into (7) we get:
\[ A \lambda^{\frac{2}{k+1}} - k \lambda^{k-1} + A \lambda^{\frac{k}{k+1}} = -\frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \]

\[ \Rightarrow A = -\frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \]

and
\[ D = -\frac{2}{dn} \min \left( \frac{k+n}{\lambda} \theta \right) \]
And finally:

\[
C_k(s) = -\frac{4}{\pi^k} \sin \left( \frac{k\pi s}{a} \right)
\]

\[
\begin{cases}
\left( \frac{s}{\sqrt{a}} \right)^{k-1/2}, & 0 < s \leq a \\
\left( \frac{a}{s} \right)^{k-1/2}, & a \leq s < \infty
\end{cases}
\]

which can be rewritten as:

\[
C_k(s) = -\frac{4}{\pi^k} \sin \left( \frac{k\pi s}{a} \right) \cdot \rho
\]

\[
\rho = \frac{\min(s, a)}{\max(s, a)}
\]

And the Green's function reads:

\[
G(s_1, \eta) = -\frac{4}{\pi^k} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi s}{a} \right) \sin \left( \frac{n\pi \eta}{a} \right) \cdot \rho
\]

But this is not the end! We can rewrite & using trigonometric identity and complex numbers.

First, notice that

\[
\sin \left( \frac{n\pi s}{a} \right) \sin \left( \frac{n\pi \eta}{a} \right) = \frac{1}{2} \left[ \cos \left( \frac{n\pi (s-\eta)}{a} \right) - \cos \left( \frac{n\pi (s+\eta)}{a} \right) \right]
\]

\[
= \frac{1}{2} \Re \left[ e^{i(n\pi(s-\eta)/a)} - e^{i(n\pi(s+\eta)/a)} \right]
\]

so that

\[
G(s_1, \eta) = -\frac{4}{2\pi^k} \Re \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( (pe^{i(n\pi(s-\eta)/a)})^{\frac{1}{n}} - (pe^{i(n\pi(s+\eta)/a)})^{\frac{1}{n}} \right) \right]
\]

Let us set \( z_1 = (pe^{i(n\pi(s-\eta)/a)})^{\frac{1}{n}} \) and \( z_2 = (pe^{i(n\pi(s+\eta)/a)})^{\frac{1}{n}} \), \(|z_1| = |z_2| = p^{\frac{1}{n}} < 1 \) for \( s \neq 1 \), and \( -\sum_{n=1}^{\infty} \frac{1}{n} = \log(1-z) \) (this series converges for \(|z|<1\)).

So we get:

\[
G(s_1, \eta) = \frac{4}{2\pi^k} \Re \left[ \log(1-z_1) - \log(1-z_2) \right]
\]

Using the fact that \( \Re[\log z] = \log |z| \), we get

\[
G(s_1, \eta) = \frac{4}{2\pi^k} \log \left| \frac{1-z_1}{1-z_2} \right| = \frac{4}{2\pi^k} \log \left| \frac{1-\frac{z_1}{z_2}}{1-1} \right|^2
\]
Now we have:

\[ |1-z_2|^2 = |1 - e^{\frac{i \pi}{\alpha} \cos((\eta - \theta) \frac{\pi}{\alpha})} - e^{\frac{i \pi}{\alpha} \sin((\eta - \theta) \frac{\pi}{\alpha})}|^2 \]

\[ = (1 - e^{\frac{i \pi}{\alpha} \cos((\eta - \theta) \frac{\pi}{\alpha})})^2 + (e^{\frac{i \pi}{\alpha} \sin((\eta - \theta) \frac{\pi}{\alpha})})^2 \]

\[ = 1 + e^{\frac{2i \pi}{\alpha}} - 2 e^{\frac{i \pi}{\alpha} \cos((\eta - \theta) \frac{\pi}{\alpha})} \]

\[ = e^{\frac{i \pi}{\alpha}} (e^{\frac{-i \pi}{\alpha}} + e^{\frac{i \pi}{\alpha}} - 2 \cos((\eta - \theta) \frac{\pi}{\alpha})) \]

and \( e^{\frac{-i \pi}{\alpha}} + e^{\frac{i \pi}{\alpha}} = e^{\frac{\pi}{\alpha} \log e} + e^{\frac{-\pi}{\alpha} \log e} = 2 \cosh\left(\frac{\pi}{\alpha} \log e\right) \)

Doing the same for \( |1-z_2|^2 \), we finally obtain:

\[
G(s, \eta) = \frac{1}{4} \left( \frac{\cosh\left(\frac{\pi}{\alpha} \log e\right)}{\sinh\left(\frac{\pi}{\alpha} \log e\right)} - \cos\left((\eta - \theta) \frac{\pi}{\alpha}\right) \right)
\]

This is an explicit formula for the Dirichlet Green's function in an infinite wedge of angle \( \alpha \). It is pretty rare to get such an explicit formula. We should be happy.

If \( \alpha = \pi \) or \( \alpha = \pi / 2 \), we recover the formulas that we derived by the method of images.