GREEN'S FUNCTIONS

Lecture (j): The maximum principle

We will start by formulating the theorem.

Let $D$ be a region of $\mathbb{R}^n$ bounded by a smooth surface $\partial D$.

**Theorem**: Maximum principle for harmonic functions

Let $u$ be a harmonic function in $D$ ($\nabla^2 u = 0$) and continuous on $\overline{D} = D \cup \partial D$, then

1. $u$ attains its maximum and minimum values on the boundary $\partial D$
2. if $u$ also attains its maximum or minimum value at an interior point of $D$, then $u$ must be a constant function

Remarks:

1. $u$ must attain its maximum and minimum values somewhere on $\partial D$ since $u$ is a continuous function on a closed and bounded region.
2. the maximum part of the theorem extends to subharmonic functions ($\nabla^2 u \geq 0$) and the minimum part to superharmonic functions ($\nabla^2 u \leq 0$)

We will give two proofs of the theorem.

*Proof 1*

Let's suppose that $u$ has its maximum at an interior point $x_0 \in D$.

Let's consider a ball $B_{r_0}$ of radius $r_0$ centered at $x_0$.

From the mean-value property of harmonic functions, we have

$$u(x_0) = \frac{1}{|B_{r_0}|} \int_{B_{r_0}} u \, ds$$  \hspace{1cm} (1)
(i) \( u_0 \) cannot be \( > u(x_0) \) \( \Rightarrow \) it contradicts the fact that \( u(x) \) is the max of \( u \)

(ii) \( u_0 \) cannot be \( < u(x_0) \) \( \Rightarrow \) because of (i), (ii) would then not be satisfied

(iii) \( u_{xx} = u(x_0) \) is the only possibility, and \( u \) is constant on \( \partial \Omega \).

Repeating that argument, we can fill all of \( D \) and conclude that \( u \) is constant on \( D \), and by continuity constant on \( \overline{D} \).

* Proof 2

Once again, let's suppose \( u \) has its maximum at \( x_0 \in \Omega \)

Hence we have \( \Delta u(x_0) \leq 0 \), which is almost a contradiction to \( \Delta u = 0 \), but not quite \( (\leq \) rather \( <) \)

Let's introduce the function \( v(x) = u(x) + 3|x|^2, \ v \geq 0 \)

We have: \( \Delta v = \Delta u + 2 \Delta x, \ \Delta u = \Delta u + 2 \Delta p = 2 \Delta p > 0 \), where \( p \) is the space dimension.

Because \( \Delta u > 0 \), \( v \) cannot have a maximum at an interior point of \( D \). So for any \( x \in D \), we can write:

\[
u(x) \leq v(x) < \max_{y \in \Omega} v(y) = \max_{y \in \Omega} (u(y) + 3|y|^2) \leq \max_{y \in \Omega} u(y) + 3 \max_{y \in \Omega}|y|^2 \]

Taking \( \varepsilon \to 0 \), \( u(x) \leq \max_{y \in \Omega} u(y) \)

* Consequences of the maximum principle

1) Uniqueness of solutions of the Dirichlet BV Poisson's problem

Theorem: Let \( f \) be a continuous function of \( \Omega \) and \( g \) a continuous function on \( \partial \Omega \). There is at most one function \( u(x) \), twice continuously differentiable in \( \Omega \), continuous on \( \overline{\Omega} \), that solves
\[ \begin{cases} \Delta u = f, & x \in D \\ u = g, & x \in \partial D \end{cases} \] (2)

Proof: Let's suppose that \( u_1(x) \) and \( u_2(x) \) are both solutions of problem (2), we have
\[ \begin{cases} \Delta u_1 = f, & x \in D \\ u_1 = g, & x \in \partial D \end{cases} \quad \begin{cases} \Delta u_2 = f, & x \in D \\ u_2 = g, & x \in \partial D \end{cases} \]
so their difference \( v(x) = u_1(x) - u_2(x) \) solves
\[ \begin{cases} \Delta v = 0, & x \in D \\ v = 0, & x \in \partial D \end{cases} \]

From the maximum principle (\( v \) is harmonic), \( v \) attains its maximum and minimum values on \( \partial D \). Since \( v = 0 \) on \( \partial D \), \( v = 0 \) in \( D \) and \( u_1 = u_2 \) in \( D \).

2) Uniqueness of the Dirichlet Green's function

Theorem: There is at most one Dirichlet Green's function for a given region \( D \).

Proof: If \( G(x,y) \) is a Dirichlet Green's function for \( D \), we have
\[ G(x,y) = G^f(x,y) + H(x,y), \quad y \in D, \quad G(x,y) = 0, \quad y \in \partial D \] and \( H(x,y) \) solves:
\[ \begin{cases} \Delta H(x,y) = 0, & y \in D \\ H(x,y) = -\frac{G^f(x,y)}{\partial}, & y \in \partial D \]

From the previous theorem, \( H(x,y) \) is unique.
Since \( G^f(x,y) \) is also unique, so is \( G(x,y) \).