GREEN'S FUNCTIONS

In Lecture (g), we recognized that explicitly solving the Green's function problem for a general domain \( D \subset \mathbb{R}^3 \) is almost impossible and instead we solved it for the whole \( \mathbb{R}^3 \) (without any BC). However, for very special domains (that are not \( \mathbb{R}^3 \)) with strong symmetry properties, we can solve the Green's function problem by the method of images. These special domains include:

- half-plane, quarter plane, half space, etc
- disks, balls
- some combinations of the above as half-disk, half-sphere.

The problem arises from the inability of the free-space Green's function to satisfy the BCs (Note: from now let's denote the free-space Green's function \( G_F \)).

The idea of the method of images is to balance \( G_F(x,y) \) with other copies of \( G_F \) with singularities at different points (generally outside the domain \( D \)) in order to satisfy the BCs.

The name of the method comes from the idea of placing an "image" charge, with the interpretation of free-space Green's functions as potentials generated by point electric charges. Let's consider two examples to illustrate the method.
Example 4: Solve the BVP for the Laplace equation on the half plane \( \{ x=(x_1,x_2) \mid x_2 > 0 \} \):

\[
\begin{align*}
\Delta u &= 0 \quad (x_2 > 0) \\
u(x) &= u(x_1,0) = g(x_1) \quad (x_2 = 0)
\end{align*}
\]

The corresponding Green's function problem is:

For \( x_2 > 0 \):

\[
\Delta G_{x}(y) = \delta(y-x), \quad y_2 > 0
\]

\[
G_{x}(y) = G_{x}(y_1,0) = 0, \quad y_2 = 0
\]

Here, the geometry suggests to place an image at \((x_1,-x_2)\) with opposite sign in order to satisfy \(G_{x}(y) = 0\) at \(y_2 = 0\).

So we define \( G_{x}(y) \) as:

\[
G_{x}(y) = G_{x}^{f}(y) - G_{x}^{f}(y), \quad \bar{x} = (x_1-x_2)
\]

i.e.

\[
G_{x}(y) = \frac{1}{2\pi} \left( \log |y-x| - \log |y-x| \right)
\]

\[
|y-x| = \sqrt{(y_1-x_1)^2 + (y_2+x_2)^2} \quad \text{and} \quad |y-x| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2}
\]

\[
|y-x| = |y-x| \quad \text{when} \quad y_2 = 0
\]

So we satisfy the BC \( G_{x}(y) = 0 \) at \( y_2 = 0 \).

What about the PDE?

We have \( \Delta G_{x}(y) = G_{x}^{f}(y) - G_{x}^{f}(y) = \delta(y-x) - \delta(y-x) \).

But \( \delta(y-x) = \delta(y_1-x_1) \cdot \delta(y_2+x_2) \).

For \( x_2 > 0 \), there is no \( y_2 > 0 \) such that \( \delta(y_2+x_2) \neq 0 \).

\( \Rightarrow \delta(y_2+x_2) = 0 \) and \( \delta(y-x) = 0 \).

So we indeed have \( \Delta G_{x}(y) = \delta(y-x) \).

Remark: other way to see that is simply to observe that...
\( \bar{x} \) lies outside the domain \( x_2 > 0 \) and \( x_2 \) does not contribute.

To derive the solution formula, we need to compute the normal derivative of \( G(x, y) \) at \( y_2 = 0 \).

From Lecture 4, we know that \( \nabla G(x, y) = \frac{4}{2\pi} \cdot \frac{4}{y-x} \)

So we get \( \nabla G(x, y) = \frac{4}{2\pi} \left( \frac{y-x}{|y-x|^2} - \frac{y-x}{|y-x|^2} \right) \)

Since \( y_2 = 0 \), \( |y-x| = |\bar{x} - y| \) \( \equiv \nabla G(x, y) = \frac{4}{2\pi} \left( \frac{\bar{x} - x}{|y-x|^2} \right) y - y_2 \)

\[ n = (0, -1) \quad \text{(outward, normal unit vector)} \]

Finally:
\[
\frac{\partial G(x, y)}{\partial n} \bigg|_{y=y_2} = \frac{4}{2\pi} \cdot \frac{2x_2}{(y_2 - x_2)^2 + x_2^2} = \frac{x_2}{(y_2 - x_2)^2 + x_2^2} \cdot \frac{4}{\pi}
\]

And the solution formula is:
\[
\phi(x) = \int_{-\infty}^{\infty} \frac{x_2}{(y_2 - x_2)^2 + x_2^2} g(y_2) dy_2 = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{g(y_2)}{(y_2 - x_2)^2 + x_2^2} dy_2
\]

Is this expression really a solution of Problem 4?
\[
\Delta_2 \phi(x) = \frac{4}{\pi} \int_{-\infty}^{\infty} g(y_2) \cdot \Delta_2 \left( \frac{x_2}{(y_2 - x_2)^2 + x_2^2} \right) dy_2 \quad \text{(provided \( g \) is bounded and continuous)}
\]

It is easy to show that the function \( \frac{x_2}{x_4^2 + x_2^2} \) is harmonic for \( x_2 \)?

\[
\Delta \left( \frac{x_2}{x_4^2 + x_2^2} \right) = \frac{2}{x_4} \left( \frac{-2x_4 x_2}{x_4^2 + x_2^2} \right) + \frac{2}{x_2} \left( \frac{x_2^2 - x_2^2}{(x_4^2 + x_2^2)^2} \right)
\]
\[
= \frac{-2x_4 (x_4^2 + x_2^2) + 2x_2^2 (x_4^2 + x_2^2) - 4x_2 (x_4^2 - x_2^2)}{(x_4^2 + x_2^2)^3} = 0
\]

as well as to show that \( \frac{x_2}{(y_2 - x_2)^2 + x_2^2} \) is also harmonic for any \( y_2 \), so we have \( \Delta \phi(x) = 0 \).
For the BC \( u(x) = g(x) \) for \( x_2 = 0 \)

By a change of variable \( y_1 = x_1 + x_2 s, \quad dy_1 = x_2 ds \), we get

\[
u(x) = \frac{x_2}{\pi} \int_{-\infty}^{+\infty} \frac{g(x_1 + x_2 s)}{(x_2 s)^2 + x_2^2} \; x_2 ds = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(x_1 + x_2 s)}{1 + s^2} \; ds
\]

When \( x_2 = 0 \), we have \( u(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{g(x_1)}{1 + s^2} \; ds = g(x_1) \int_{-\infty}^{+\infty} \frac{1}{1 + s^2} \; ds = \pi \)

hence \( u(x) = g(x) \) for \( x_2 = 0 \)

Example 2: Solve the BVP for the Laplace equation in the 3D ball of radius \( a \)

\[
\begin{align*}
\Delta u &= 0, \quad |x| < a \\
u(x) &= g(x), \quad |x| = a
\end{align*}
\]

The corresponding Green's function problem is:

\[
\begin{align*}
\Delta G_{xy}(y) &= \delta(y-x), \quad |y| < a \\
G_{xy}(y) &= 0, \quad |y| = a
\end{align*}
\]

Due to the angular symmetry of the problem (3), in order to satisfy \( G_{xy}(y) = 0, \; |y| = a \), it makes sense to try to place an image charge in the radial direction i.e. the direction of \( x \)

\[
\text{Radial direction } \vec{r} = \frac{\vec{x}}{|x|}
\]

Let's try to put a charge of intensity \( \beta \) at \( x^* = 2x \), \( x > 1 \) (so that the image charge is outside the domain)

Hence we construct a Green's function \( G_{xy}(y) \) as

\[
G_{xy}(y) = G_k^e(y) + \beta G_{x^*}(y) = -\frac{1}{4\pi} \left[ \frac{1}{|y-x|} + \beta \frac{1}{|y-x^*|} \right], \quad x > 1, \beta \in \mathbb{R}, \beta \neq 0
\]
Remark: \( \beta G_\beta(x)(y) \) solves the following problem: \( \Delta G_\beta(x)(y) = \beta \delta(y-x) \) and characterizes the "response" to an impulse of "intensity" \( \beta \), as \( \int_{|y-x|<\varepsilon} \beta \delta(y-x)dy = \beta \) (instead of 1)

Now let's try to find \( \alpha \) and \( \beta \) to satisfy \( G_\beta(x)(y) = 0 \) for \( |y| = a \)
So we want \( |y-x| = -\frac{|y-2x|}{\beta} \) for \( |y| = a \)

\[ -\beta |y-x| = |y-2x| \]  
This tells that \( \beta < 0 \) so let's introduce \( \chi > 0, \chi = -\beta \), we get:

\[ \chi |y-x| = |y-2x| \]

Taking the square of both sides we obtain:

\[ \chi^2 (|y|^2 - 2y \cdot x + |x|^2) = |y|^2 - 2y \cdot x + 2^2 |x|^2 \]

Let's try \( \chi^2 = \chi \Rightarrow \chi = \sqrt{\alpha}, \chi > 0, \alpha > 1 \times d > 0 \)

and use \( |y| = a \), we get:

\[ d^2 |x|^2 - (a^2 + |x|^2) \alpha + \alpha^2 = 0 \]

\[ \Delta = (a^2 + |x|^2)^2 - 4a^2 |x|^2 = (a^2 - |x|^2)^2 \]

\[ d_1 = \frac{a^2 + |x|^2 + a^2 - |x|^2}{2|x|^2} = \frac{a^2}{|x|^2} \quad \text{and} \quad d_2 = \frac{a^2 + |x|^2 - a^2 + |x|^2}{2|x|^2} = 1 \]

\( \Delta \) does not satisfy \( d > 1 \) so \( d = d_1 = \frac{a^2}{|x|^2} \) and \( \beta = -\sqrt{\alpha} = -\frac{a}{|x|} \)

and our Green's function that satisfies \( G_\beta(x)(y) = 0 \) for \( |y| = a \) reads

\[ G_\beta(x)(y) = G_\beta(x)(y) - \frac{a}{|x|} G_\beta(x)(y) = -\frac{1}{4\pi} \left[ \frac{1}{|y-x|} - \frac{1}{|x|} \cdot \frac{1}{|y-x|^2} \right], x^2 = \frac{a^2}{|x|^2} \]

Remark: for \( |x| < a, |x|^2 > a \) and the change indeed lies outside the domain.

Finally, to write the solution formula, we need to calculate the outwards normal derivative of \( G_\beta(x)(y) \) at the boundary i.e. on the surface of the sphere at \( |y| = a \).
\[
\left( \frac{\partial G_n(y)}{\partial y} \right)_{y_l=a} = n(y)_{y_l=a} \cdot \nabla G_n(y)_{y_l=a}
\]

\[
\nabla G_n(y) = -\frac{4}{4\pi} \left[ -\frac{1}{1-y-x^2} \cdot \frac{y-x}{|y-x|^3} \cdot \frac{1}{1-y-x^2} \cdot \frac{y-x}{|y-x|^3} \right] = -\frac{4}{4\pi} \left[ \frac{y-x}{|y-x|^3} - \frac{a}{|x|} \cdot \frac{y-x}{|y-x|^3} \right] = \frac{4}{4\pi} \frac{y-x}{|y-x|^3} - \frac{a}{|x|} \frac{y-x}{|y-x|^3}
\]

At \( y_l=a, |y-x| = \frac{2d}{a} \cdot |y-x| \Rightarrow \frac{4}{(y-x)^3} = \frac{x}{a^2} \cdot \frac{4}{y-x^3} \Rightarrow \frac{a}{|x|} \frac{4}{(y-x)^3} = \frac{x}{a^2} \frac{4}{y-x^3} \]

Hence \( \nabla G_n(y)_{y_l=a} = \frac{4}{4\pi} \left[ \frac{y-x}{|y-x|^3} - \frac{x}{a^2} \cdot \frac{y-x}{|y-x|^3} \right] = \frac{4}{4\pi} \frac{y-x}{a^2} \left[ y-x - \frac{x^2}{a^2} (y-x) \right] \)

Since \( n(y)_{y_l=a} = \frac{y}{a} \) and \( x = \frac{a^2}{|x|^2} x \), we get

\[
\left( \frac{\partial G_n}{\partial y} \right)_{y_l=a} = \frac{4}{4\pi a (y-x)^3} \cdot y \left( y-x - \frac{x^2}{a^2} (y-x) \right)
\]

Finally \( \left( \frac{\partial G_n}{\partial y} \right)_{y_l=a} = \frac{a^2 - |x|^2}{4\pi a |y-x|^3} \)

and the solution formula is:

\[
\tag{4}
\mu(x) = \frac{a^2 - |x|^2}{4\pi a} \int_{y_l=a} \frac{G(y)}{|y-x|^3} \, ds(y)
\]

Remarks:

1. (4) is the Poisson's formula for harmonic functions \( (\Delta u = 0) \) in a 3D ball (sphere)

2. It is possible once again to show that \( u(x) \) given by (4)
is really a solution of problem (2), provided \( g(y) \) is a continuous function on the sphere \( |y| = a \).

3. Furthermore, any function harmonic on the sphere of radius \( a \) and with continuous boundary value \( g \) is given by (4).

4. If we take \( x = 0 \) in (4), we get:

\[
\begin{align*}
\n(5) \quad u(0) = \frac{a}{4\pi} \int_{|y| = a} g(y) \frac{1}{|y|} \, ds(y) = \frac{4}{4\pi a^2} \int_{|y| = a} g(y) \, ds(y)
\end{align*}
\]

(5) is the mean value formula for harmonic functions on a ball of radius \( a \) that shows that the value at the centre of the ball is equal to its average over the boundary of the ball.

5. In Example 4, we could have played the same game as in Example 2 to determine \( \alpha \) and \( \beta \).

In fact, we could have started with an image charge of intensity \( \beta \) placed at \( \bar{x} = (x_4 - dx_2) \), \( \beta > 0 \).

Hence \( G_x(y) = \frac{4}{2\pi} \left[ \log |y - \bar{x}| + \beta \log |y - \bar{\infty}| \right], \beta > 0 \)

At \( y_2 = 0 \), \( G_x(y) = 0 \) \( \Rightarrow \) \( |y - \bar{x}| = |y - \bar{\infty}|^\beta \) or \( |y - x_2|^2 = (y - \bar{x})^2 \beta \)

i.e. \( (y_2 - x_2)^2 + x_2^2 = (y_2 - x_2)^2 + x_2^2 \beta \)

\( \Rightarrow \) \( \alpha = 4, \beta = -1 \) and hence \( G_x(y) = G_x^0(y) - G_x^\beta(y) \)

\( \bar{x} = (x_4 - x_2) \)

as derived previously.

6. Let's change the Dirichlet BC at \( x_2 = 0 \) by a Neumann BC a the normal derivative of the form \( \frac{\partial u}{\partial n} = g(x_4) \)

The corresponding BC for the Green's function is \( \frac{\partial G_x(y)}{\partial n(y)} = 0 \).
Where should we place the image charge and how should we choose its intensity to satisfy \((\frac{\partial G}{\partial n})_{y_2=0} = 0\)?

\[ x_2 \]
\[ + (x_2, x_2) \]
\[ \rightarrow x_1 \]
\[ + x = (x_1, x_2) \]
\[ + x = (x_2, x_2) \]
\[ + x = (x_2, x_2) \]
\[ x_0 = (x_2, x_2) \]
\[ I = +1 \]
\[ x_0 = (x_2, x_2) \]
\[ I = +1 \]

It is easy to show that an image charge placed at \(x\) (same location as for Dirichlet BC) but with an intensity \(+1\) (\(\beta = 1\)) will do the job.

So we construct:

\[ G_{2i}(y) = G_{2i}^P(y) + G_{2i}^P(y) \]

In fact, we have:

\[ \frac{\partial G}{\partial n} = n \cdot \nabla G = n \cdot \left( \frac{4}{2\pi} \left( \frac{y - x_2^2}{|y - x|^2} + \frac{y - x_2}{|y - x|^2} \right) \right) \]

At \(y_2 = 0\), \(n = (0, -1) \Rightarrow \frac{\partial G}{\partial n} \bigg|_{y_2=0} = \frac{4}{2\pi} \left( \frac{x_2 - x_2}{|y - x|^2} + \frac{x_2}{|y - x|^2} \right)_{y_2=0} = \frac{4}{2\pi} \left( \frac{x_2}{|y - x|^2} - \frac{x_2}{|y - x|^2} \right) \) as \(x_2 = -x_2\).

Also, we note that at \(y_2 = 0\), \(|y - x| = |y - x_2|\)

and we get \(\frac{\partial G}{\partial n} \bigg|_{y_2=0} = 0\).

7. In a quarter-plane (quadrant), we can combine more \(G_{2i}^P(y)\) to satisfy the appropriate BCs.

Ex: Dirichlet BCs in the positive quadrant:

\[ \Delta G_x(y) = \delta(y - x) \quad ; y_1 \geq 0, y_2 \geq 0 \]

\[ G_x(y) = 0 \quad ; y_2 = 0 \]

\[ G_x(y) = 0 \quad ; y_2 = 0 \]

It is easy to show that \(G_x(y)\) defined as:

\[ G_x(y) = G_x^P(y) - G_x^P(y) - G_x^P(y) + G_x^P(y) \]

satisfies the BCs.
Replacing the various \(G_k\) by their expression, we get:

\[
G_x(y) = \frac{4}{\pi n} \left( \log |y-x| - \log |y-x_b| - \log |y-x_1| + \log |y-x_0| \right)
\]

8. "Thumbrule" for the sign of the volume change
- Dirichlet \(\Rightarrow\) opposite sign (negative)
- Neumann \(\Rightarrow\) same sign (positive)

This principle works in simple geometries (a single symmetry surface). For combinations in more complicated geometries, look at the collective "compensations".

- Solvability condition and modified Green's functions

We will just do an example

Let's consider the problem:

\[
\begin{align*}
\Delta u &= f, \quad x \in D \subset \mathbb{R}^{2n+3} \\
\frac{\partial u}{\partial n} &= g, \quad x \in S = \partial D
\end{align*}
\]

The corresponding homogeneous problem \((g=0)\) has a non-trivial solution \(u^+(x) = 1\) (on a, \(x \in \mathbb{R}, d=0\)). From Green's second identity, we have

\[
0 = \int_D \Delta u^+ v dx = \int_D u^+ \Delta v dx + \int_{\partial D} \frac{\partial u^+}{\partial n} v ds - \int_{\partial D} u^+ \frac{\partial v}{\partial n} ds
\]

\[
\Rightarrow \int_D f(x) dx = \int_{\partial D} g(x) ds(x)
\]

This is the solvability condition for problem (6). Problem (6) does hence not admit a classical Green's function, but a modified Green's function solution of

\[
\begin{align*}
\Delta \tilde{G}_x(y) &= \delta(y-x) + C, \quad y \in D \\
\frac{\partial \tilde{G}_x(y)}{\partial n} &= 0, \quad y \in \partial D
\end{align*}
\]
where $C \in \mathbb{R}$ can be determined using the divergence theorem:

$$1 = \int_D s(y-x) \, dy = \int_D (\Delta \tilde{u}(y) - C) \, dy = \int_{\partial D} \tilde{u}(y) \, dS(y) - C \, |D|$$

so we have $C = -\frac{1}{|D|}$ where $|D|$ = volume/surface of $D$ in $\mathbb{R}^3/\mathbb{R}^2$ respectively.

From Green's second identity, we have:

$$\int_D \left( \frac{\partial \tilde{u}(y)}{\partial y} - \tilde{u}(y) \frac{\partial u}{\partial y} \right) \, dy = \int_{\partial D} \left( \frac{\partial \tilde{u}(y)}{\partial y} + \tilde{u}(y) \frac{\partial u}{\partial y} \right) \, dS(y)$$

$$= S(y-x) + C$$

$$\Rightarrow u(x) + C \int \tilde{u}(y) \, dy = \int \tilde{u}(y) f(y) \, dy - \int_{\partial D} \tilde{u}(y) g(y) \, dS(y)$$

= constant, that can be chosen arbitrarily (see lecture(e) in 3-D problems)

Finally, the solution formula for $u(x)$ is:

$$u(x) = \int_D \tilde{u}(y) f(y) \, dy - \int_{\partial D} \tilde{u}(y) g(y) \, dS(y) + A$$, $A \in \mathbb{R}$