GREEN'S FUNCTIONS

Lecture (f): Green's functions and eigenfunction expansion

Let's consider again the Sturm-Liouville problem with homogeneous BCs

\[ Lu = (p(x)u')' + q(x)u = f(x), \quad x_0 < x < x_4, \quad p(x) > 0 \]

BCs: \( L_0 u(x_0) + \beta_0 u'(x_0) = 0 \), \( L_4 u(x_4) + \beta_4 u'(x_4) = 0 \)

(1)

The Sturm-Liouville problem has eigenvalues \( \lambda_i, i = 0, 1, \ldots \) and corresponding eigenfunctions \( \phi_j(x) \) such that

\[ L \phi_j = \lambda_j \phi_j, \quad \phi_j \text{ satisfying BCs} \]

\[ \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \]

The eigenfunctions can be taken orthonormal, i.e.

\[ (\phi_j, \phi_k) = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \]

(2)

and are complete, meaning that any function \( g(x) \) on \([x_0, x_4]\) can be expanded as

\[ g(x) = \sum_{j=0}^{\infty} c_j \phi_j(x) \]

\[ c_j = (\phi_j, g) = \text{"Fourier coefficient"} \]

Remarks: 1. This series does not (in general) converge at every point \( x \) in \([x_0, x_4]\). For example, it cannot converge at \( x = x_0 \) and \( x = x_4 \) unless \( g \) satisfies the BCs. However, it converges in the \( L^2 \) sense, i.e., provided \( g(x) \) is square-integrable on \([x_0, x_4]\) \((\int_{x_0}^{x_4} g^2(x) dx < \infty)\), we have

\[ \lim_{N \to \infty} \int_{x_0}^{x_4} (g(x) - \sum_{j=0}^{N} c_j \phi_j(x)) dx = 0 \]
2. The type of convergence (L^2 uniform) depends on the properties of g. Apart from BCs, if g has a jump discontinuity the eigenfunction expansion might not converge either at that point even if N → ∞.

3. If g is continuous and piecewise differentiable, and satisfies BCs, then uniform convergence is obtained. This is the case of our Green's functions.

OK, let's express \( G_x(z) \) for problem (1) as such an eigenfunction expansion:

\[
G_x(z) = G(xz) = \sum_{j=0}^{\infty} c_j(x) \phi_j(z) \tag{3}
\]

Now let's determine the coefficients \( c_j(x) \).

Applying the operator \( L \) to (3), we get:

\[
L G_x(z) = \sum_{j=0}^{\infty} c_j(x) (L \phi_j)(z) = \sum_{j=0}^{\infty} c_j(x) \lambda_j \phi_j(z) = \delta(z-x)
\]

\[
(\delta_j \phi_j(z))
\]

Multiplying by \( \phi_j(z) \) and using the orthonormality property, we get:

\[
c_j(x) \lambda_j (\phi_j(z), \phi_j(z)) = (\delta(z-x), \phi_j(z)) \quad \text{as all the other (} \phi_j, \phi_k \text{ but } j \text{)} = 0
\]

\[
\Rightarrow c_j(x) = \frac{1}{\lambda_j} \phi_j(x)
\]

And finally, the expression of the Green's function as an eigenfunction expansion is:

\[
G_x(z) = \sum_{j=0}^{\infty} \frac{1}{\lambda_j} \phi_j(x) \phi_j(z) \tag{4}
\]

and the corresponding solution formula for \( u(x) \) is:

\[
u(x) = (G_x \phi) = \int_{x_1}^{x_2} \sum_{j=0}^{\infty} \frac{1}{\lambda_j} \phi_j(x) \phi_j(z) dz = \sum_{j=0}^{\infty} \frac{1}{\lambda_j} \phi_j(x) \phi_j(z) \tag{5}
\]
Remarks: 4. The reciprocity (symmetry of $a$ in $x$ and $z$) is even more visible in (4).

2. If, for some $j$, $d_j = 0$, expression (4) is ill-defined. In fact, in this case, $\phi_j$ is a non-trivial solution to the homogeneous problem, in other words, it is a $u^+$, and we already know that there is no (standard) Green's function.

3. Once again, if, for some $j$, $d_j = 0$, the only way for (5) to make sense is if we also have $(\phi_j, f) = 0$, which is exactly the solvability condition on $f$ on Lecture (e).

4. We could have commented earlier already on $u^+$ being an eigenfunction of problem (4) corresponding to the eigenvalue $\lambda = 0$. In fact, $u^+$ is a non-trivial solution to (4) (Sturm-Liouville problem with homogeneous BCs) and it actually solves $Lu = \lambda u$ with $\lambda = 0$. 