GREEN'S FUNCTIONS

Lecture (e): Modified Green's functions and solvability condition

In the previous lecture, we ran into trouble to construct a Green's function for the Sturm-Liouville problem:

\[ Lu = (p(x)u')' + q(x)u = f(x), \quad 0 < x < 1, \quad p(x) > 0 \tag{1} \]

BCs: \( \lambda_0 u(0) + \beta_0 u'(0) = 0, \quad \lambda_4 u(4) + \beta_4 u'(4) = 0 \]

if the corresponding homogeneous problem (i.e. \( f = 0 \)) has a non-trivial solution \( u^* \):

\[ Lu^* = 0, \quad \lambda_0 u^*(0) + \beta_0 u^*(0) = 0, \quad \lambda_4 u^*(4) + \beta_4 u^*(4) = 0 \tag{2} \]

It is easy to show that we should have \( (f, u^*) = 0 \) to have any hope of finding a solution \( u \).

In fact:

\[ 0 = (u, Lu^*) = (Lu, u^*) = (f, u^*) \]

So the source term \( f \) should satisfy the following solvability condition:

\[ (f, u^*) = \int_0^4 f(x) u^*(x) dx = 0 \tag{3} \]

**Theorem:** Fredholm alternative (for the Sturm-Liouville problem)

1. Problem (1) has exactly one solution, or
2. There is a non-zero solution \( u^* \) of the corresponding homogeneous problem (2). In this case, (1) has a solution if and only if the solvability condition (3) is satisfied. Also, the solution is not unique (you can add any multiple of \( u^* \) to get another one).

**Proof:** 1. If there is no \( u^* \), we have a solution constructed with a standard Green's function, so existence is OK.
For uniqueness, if there are two solutions $u_1$ and $u_2$, their difference $u_1 - u_2$ satisfies the homogeneous problem (2), i.e., it is either a $u^+ + c(z)$ or it can only be $0 \implies u_1 - u_2 = 0$, $u_1 = u_2$ and the solution is unique.

2. If there is a non-zero $u^+$, then (3) should hold. If (3) is satisfied, let's show how to construct a modified Green's function that leads to a solution formula for $u$.

If $u^+$ exists and (3) holds, let us introduce the following function $\tilde{G}_x(z) = \tilde{G}(x, z)$ solution of:

$$L \tilde{G}_x(z) = S(x, z) + C(z) \mu^+(z)$$

BCs: $\alpha_0 \tilde{G}_x(0) + \beta_0 \tilde{G}_x'(0) = 0$, $\alpha_4 \tilde{G}_x(4) + \beta_4 \tilde{G}_x'(4) = 0$

where $C(z)$ is a constant w.r.t. $z$.

Given such a $\tilde{G}_x(z)$, the solution formula is:

$$u(x) = (S(x, z)i(x, z)) = (\tilde{G}_x f) - C(z)(\mu^+ i)$$

(4)

Now we determine $C(z)$ such that (4) is indeed a solution of (2) and the solvability condition (3) is satisfied.

$L_x u(x) = (L_x \tilde{G}_x(z) f(z)) - (\mu^+ i) L_x C(z)$

Using reciprocity of $\tilde{G}$: $\tilde{G}_x(z) = \tilde{G}_z(x)$, we get:

$L_x u(x) = (L_x \tilde{G}_z(x) f(z)) - (\mu^+ i) L_x C(z)$

$$= (S(x, z) + C(z) \mu^+(x)) f(z) - (\mu^+ i) L_x C(z)$$

$$= f(x) + \mu^+(x) (C(x) f(z)) - (\mu^+ i) L_x C(z)$$

Also we have:

$$O = (L_x \tilde{G}_x(z), \mu^+(z)) = (L_x \tilde{G}_z(x), \mu^+(z))$$

$$= (S(x, z) + C(z) \mu^+(z), \mu^+(z)) = \mu^+(x) + C(z) (\mu^+, \mu^+)$$
Which gives
\[ c(x) = -\frac{u^*(x)}{(u^*_x)'u^*} \]

and we have:
\[ \log c(x) = -\frac{1}{(u^*_x)'u^*} \int c(x) u^*(x) = 0 \]

and
\[ (c(x), f(x)) = -\frac{1}{(u^*_x)'u^*} (u^*(x), f(x)) = 0 \]

Solvability condition

and we have
\[ \log u(x) = f(x) \text{ and the solution formula becomes:} \]
\[ u(x) = (\nabla_x f) + \frac{(u^*_x)'u^*}{(u^*_x)'u^*} u^* \]

Note that \( \frac{(u^*_x)'u^*}{(u^*_x)'u^*} \) is an arbitrary constant, as we can always add any multiple of \( u^* \) and still have a solution. So we can write
\[ u(x) = (\nabla_x f) + \lambda u^*, \lambda \in \mathbb{R} \]

Let’s take an example to clarify this.

Example: \( u^*(x) = f(x), 0 < x < 1 \)

BCs: \( u'(0) = u'(1) = 0 \)

Obviously \( u^*(x) = \lambda, \lambda \in \mathbb{R}, \lambda \neq 0 \) is a non-trivial solution of the corresponding homogeneous problem (Note that the original problem is anyway already homogeneous). Let’s take \( u^* = 1 \).

Hence the solvability condition is:
\[ 0 = (f, u^*) = \int_0^1 f(x) dx \]

Remark: physically, if this equation represents the spatial \( T \) in a 1D rod with insulated ends, there is no steady-state solution unless the heat source term is 0 on average.

With \( u^*(x) = 1 \), we get
\[ c(x) = -\frac{4}{(4, x)} = -1 \]
And the modified Green's function \( \tilde{G}_x(\zeta) \) should solve
\[
\tilde{G}_x''(\zeta) = 8(\zeta-x) - 1, \quad \tilde{G}_x(0) = \tilde{G}_x'(1) = 0
\]
For \( z \neq x \), we have \( \tilde{G}_x''(\zeta) = -1 \), i.e., \( \tilde{G}_x(\zeta) \) is of the form
\[
-\frac{\zeta^2}{2} + \mu \zeta + \eta \quad \text{and} \quad \tilde{G}_x'(\zeta) = -\zeta + \mu, \quad \mu \in \mathbb{R}, \eta \in \mathbb{R}
\]
Using BC's, we have:
\[
\tilde{G}_x(\zeta) = \begin{cases} 
-\frac{\zeta^2}{2} + A & 0 \leq \zeta < x \\
-\frac{\zeta^2}{2} + z + B & x < \zeta \leq 1
\end{cases}
\]
Continuity of \( \tilde{G}_x \) at \( x \) gives: \[ A = x + B \]
Jump condition:
\[
\int_{x-\epsilon}^{x+\epsilon} \tilde{G}_x''(\zeta) \, d\zeta = \int_{x-\epsilon}^{x+\epsilon} 8(\zeta-x) \, d\zeta + \int_{x-\epsilon}^{x+\epsilon} 1 \, d\zeta
\]
\[
= \tilde{G}_x(x+\epsilon) - \tilde{G}_x'(x-\epsilon) = 4 + \left[ 2 \right]_{x-\epsilon}^{x+\epsilon} = 2 \epsilon
\]
Letting \( \epsilon \to 0 \):
\[
\tilde{G}_x'(x^+) - \tilde{G}_x'(x^-) = 4
\]
\[
1 - x - (-x) = 4 \quad \Rightarrow \quad 1 - x + x = 4 \quad \Rightarrow \quad 1 = 4
\]
The jump condition already holds and this leaves one free parameter, characteristic of modified Green's function problems.
This was expected since any multiple of \( u^+ \) can be added.
Finally, the solution formula is:
\[
u(x) = (\tilde{G}_x(\zeta), f(\zeta)) + x \cdot t + \lambda \in \mathbb{R}
\]
\[
= \int_0^x (-\frac{\zeta^2}{2} + A) f(\zeta) \, d\zeta + \int_x^1 (-\frac{\zeta^2}{2} + z + A - x) f(\zeta) \, d\zeta + \lambda
\]
\[
= \int_0^x (-\frac{\zeta^2}{2} + A) f(\zeta) \, d\zeta + \int_x^1 (x-x) f(\zeta) \, d\zeta + \lambda
\]
\[
= \text{constant} \Theta + \lambda \in \mathbb{R}
\]
\[
= (x \Theta) - x \int_x^1 f(\zeta) \, d\zeta + \int_x^1 2 f(\zeta) \, d\zeta
\]
Since \[ \int_0^1 f(z)dz = 0 = \int_0^x f(z)dz - \int_0^x f(z)dz \]

and finally:

\[ u(x) = x\int_0^x f(z)dz + \int_0^x z f(z)dz + D \]

with \( D = x + 0, \quad D \in \mathbb{R} \)

It is easy to verify that \( u(x) \) is a solution of our problem:

\[ u'(x) = xf(x) + \int_0^x f(z)dz - x f(x) = \int_0^x f(z)dz \]

\[ u''(x) = f(x) \]

\[ u'(0) = \int_0^0 f(z)dz = 0 \]

\[ u'(1) = \int_0^1 f(z)dz = 0 \quad (\text{solvability condition}) \]

Final remark:
So far we discussed the issues of solvability condition and modified Green's function for self-adjoint problems only.
For a general problem (not self-adjoint because \( L \neq L^* \) or from the BCs), the question of solvability comes from looking at the following homogeneous adjoint problem:

\[ L^*u^* = 0 \quad \text{with homogeneous adjoint BCs on } u \]

If this problem has a non-trivial solution \( u^* \), then the solvability condition for \( Lu = f \) with homogeneous BCs is

\[ 0 = (u, L^*u^*) = (Lu, u^*) = (f, u^*) \]

So the solvability condition is unchanged but the problem that \( u^* \) solves (problem (5)) is different.