GREEN'S FUNCTIONS:
Lecture (a) : General notions on solving ODEs and PDEs

• Introduction

Main goal of the course is to learn a new way to solve ODEs and PDEs in problems arising in science. Solve does not necessarily mean writing the solution explicitly, but instead finding an explicit formula for the solution and learn important qualitative properties of that solution.

Different techniques are available to solve ODEs and PDEs for which finding an explicit solution is difficult:
* Separation of variables and Fourier series
* General method of eigenfunction expansion
* Fourier and Laplace transforms (that turn ODEs into algebraic equations and PDEs into ODEs, but need to eventually compute the inverse transform)
* Perturbation methods

The first part of the course will be devoted to writing solution formulas using Green's function, while the second part will introduce variational methods.

In short, Green's functions express the effect of the data (BC, ICs) at one point on the solution at another point. We will mainly examine linear problems.

Variational methods imply that PDEs are solved by minimizing or maximizing some quantities. They are extremely powerful and even apply directly to some non-linear problems.
Let's start with Green's functions.

To introduce the notion of "the effect of the data on the solution" and "explicit formula" for the solution, we will start by writing solutions of ODE and PDE with known techniques.

**Example 1: Effect of the RHS on the solution of an 2nd-order ODE with non-constant coefficients (non-homogeneous equation)**

By the method of variation of parameters

Let assume that $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the homogeneous equation: $py'' + qy' + ry = 0 + IC$

We would like to determine a particular solution of

$$py'' + qy' + ry = g(t)$$

where $g(t)$ are continuous functions of $t$

We look for a solution like $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

so $y_p' = u_1'y_1 + u_2'y_2' + u_2'y_1 + u_1'y_2'$

and assume that $u_1'y_1 + u_2'y_2 = 0 \Rightarrow y_p' = u_2'y_1 + u_1'y_2'$

$y_p'' = u_2'y_1'' + u_1'y_2'' + u_2'y_1' + u_1'y_2'$

Plug this into the ODE:

$$p(t)(u_2'y_1'' + u_1'y_2'' + u_2'y_1' + u_1'y_2') + q(t)(u_2'y_1' + u_1'y_2') + r(t)(u_2'y_1 + u_1'y_2) = g(t)$$

$$\Rightarrow \quad u_2'y_1'' + u_1'y_2'' = \frac{g(t)}{p(t)} \quad (A)$$

Using $y_1'' = -u_1'\frac{y_2}{y_1}$ into $(B)$:

$$-u_2'\frac{y_2}{y_1}' + u_1'y_2' = \frac{g(t)}{p(t)}$$

We get:

$$u_2'\left(\frac{y_1y_2'}{y_2} - \frac{y_2y_1'}{y_1}\right) = \frac{g(t)}{p(t)} \quad y_1$$
Finally: \( u_1' = \frac{y_1}{W(y_1 y_2)} \cdot \frac{g(t)}{p} \) and \( u_2' = -\frac{y_2}{W(y_1 y_2)} \cdot \frac{g(t)}{p} \).

and \[ u_1 = -\int^t \frac{y_2}{W(y_1 y_2)} \cdot \frac{g(s)}{p} \, ds \quad \text{and} \quad u_2 = \int^t \frac{y_1}{W(y_1 y_2)} \cdot \frac{g(s)}{p} \, ds \]

Remark: do not worry about the constants, they are absorbed into the complementary solution constants.

So we get:
\[
\begin{aligned}
y_p(t) &= -y_1(t) \int^t \frac{y_2(s)}{W(y_1 y_2)(s)} \cdot \frac{g(s)}{p(s)} \, ds + y_2(t) \int^t \frac{y_1(s)}{W(y_1 y_2)(s)} \cdot \frac{g(s)}{p(s)} \, ds
\end{aligned}
\]

We can rewrite that solution as:
\[
y_p(t) = \int^t \frac{y_2(s) y_1(t) - y_2(t) y_1(s)}{p(s) W(y_1 y_2)(s)} \cdot g(s) \, ds
\]

and introduce the function \( G(t,s) = \frac{y_1(s) y_2(t) - y_1(t) y_2(s)}{p(s) W(y_1 y_2)(s)} \)

The solution formula becomes:
\[
y_p(t) = \int^t G(t,s) \cdot g(s) \, ds
\]

and is written as an integral of the data \( g(t) \), the r.h.s or source term. \( G(t,s) \) is the Green function.

Remarks:
1) If we can compute \( G \), we can solve the problem (then compute \( y_1(t) \) and \( y_2(t) \)) provided we can integrate!
2) The Green's function is a function of two variables, \( t \) for the solution and \( s \) is integrated.
3) \( G(t,s) \) gives the effect of the data at one point \( s \) on the solution at another point \( t \).
4) Note that \( s \leq t \).
Example 2: Using Fourier transform to solve the 1D heat equation in free-space.

\[ \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty, \quad u(x,0) = u_0(x), \quad d > 0 \]

**Fourier Transform**

Given a function \( f: \mathbb{R} \to \mathbb{C} \), integrable, its Fourier transform \( F(\xi) \) is defined as:

\[
F(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \xi x} \, dx
\]

and the inverse Fourier transform by:

\[
f(x) = \int_{-\infty}^{+\infty} F(\xi) e^{2\pi i \xi x} \, d\xi
\]

Alternative definition using \( \omega = 2\pi \xi \) gives:

\[
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} \, dx
\]

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} \, d\omega
\]

An interesting property of Fourier transform is differentiation:

\[
F\prime(\omega) = i\omega F(\omega)
\]

Easy to prove by integration by parts:

\[
F\prime(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f\prime e^{-i\omega x} \, dx = \left[ fe^{-i\omega x} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(-i\omega) i\omega e^{-i\omega x} \, dx
\]

If \( f \) decays at \( \pm \infty \),

\[
= i\omega F(\omega)
\]

This property is useful as it will turn our PDE into an ODE.
Let's take the Fourier transform in space

\[ F_{xx}(\omega t) = -\omega^2 F_x(\omega t) \]

\[ F_t(\omega t) = \frac{\partial F_x(\omega t)}{\partial t} \] (easy to show)

Our PDE becomes:

\[ \frac{\partial F_x(\omega t)}{\partial t} + \omega^2 F_x(\omega t) = 0 \] (A)

and the initial condition \( x(\chi, 0) = u_0 \) implies

\[ F_x(\omega t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(\chi) e^{i\omega t} d\chi \] (A) is actually an ODE in \( t \) for fixed \( \omega \)

So we get:

\[ F_x(\omega t) = F_x(\omega, 0) e^{-\omega^2 t} \]

Now to get our solution \( u(x, t) \), we need to compute the inverse Fourier transform of \( F_x(\omega t) \)

\[ u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F_x(\omega, 0) e^{-i\omega x} e^{-\omega^2 t} d\omega \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(y) e^{-i\omega y} dy \right) e^{-\omega^2 t} e^{i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u_0(y) e^{i\omega(x-y)} e^{-\omega^2 t} d\omega dy \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u_0(y) e^{-\omega^2 t} \cos (\omega (x-y)) d\omega dy \]

\[ + \frac{i}{2\pi} \int_{-\infty}^{+\infty} u_0(y) e^{-\omega^2 t} \sin (\omega (x-y)) d\omega dy \]

\[ = 0 \]

\[ = \frac{1}{\pi} \int_{-\infty}^{+\infty} u_0(y) e^{-\omega^2 t} \cos (\omega (x-y)) d\omega dy \]

\[ = \frac{1}{\pi} \int_{-\infty}^{+\infty} u_0(y) \left( \int_{0}^{\infty} \cos (\omega (x-y)) e^{-\omega^2 t} d\omega \right) dy \]

Known result:

\[ \int_{0}^{\infty} e^{-s^2} \cos (2bs) ds = \frac{\sqrt{\pi}}{2} e^{-b^2} \]

Change of variable: \( s = w \sqrt{\frac{t}{2}} \) \( \omega = \frac{s}{\sqrt{2t}} \) \( dw = \frac{ds}{\sqrt{2t}} \)
Hence: \[ I = \int_0^\infty \cos(\omega(x-y)) e^{-\frac{s^2}{4\eta t}} \, ds \]
\[ = \int_0^\infty \cos(\frac{s}{\sqrt{4\eta t}}(x-y)) e^{-\frac{s^2}{4\eta t}} \, ds \]
\[ = \frac{x-y}{\sqrt{4\eta t}} \Rightarrow s^2 = \frac{(x-y)^2}{4\eta t} \]
\[ \text{and} \quad I = \frac{4}{\sqrt{4\eta t}} \int_0^\infty \cos(\frac{x-y}{\sqrt{4\eta t}}s) e^{-\frac{s^2}{2}} \, ds = \frac{\sqrt{\pi}}{2} e^{-\frac{(x-y)^2}{4\eta t}} \cdot \frac{4}{\sqrt{4\eta t}} \]

Back to \( u(x,t) \):
\[ u(x,t) = \frac{4}{\pi} \int_{-\infty}^{+\infty} \Phi_0(y) \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{(x-y)^2}{4\eta t}} \cdot \frac{4}{\sqrt{4\eta t}} \, dy \]

Finally:
\[ u(x,t) = \frac{1}{2\sqrt{\pi \eta t}} \int_{-\infty}^{+\infty} \Phi_0(y) \cdot e^{-\frac{(x-y)^2}{4\eta t}} \, dy \]

Let's rewrite the solution as:
\[ u(x,t) = \int_{-\infty}^{+\infty} G(x,t,y,0) \cdot \Phi_0(y) \, dy \]

with \( G(x,t,y,0) = \frac{1}{2\sqrt{\pi \eta t}} \cdot e^{-\frac{(x-y)^2}{4\eta t}} \)

Once again, \( G \) is our Green function with two sets of variables \((x,t)\) and \((y,s)\) (with \( s = 0 \) here).

We will rederive this expression for free-space Green's function applied to unsteady problems, and generalize to problems with a rhs (forcing) term and more than one dimension.
Example 3: Solve the 1D unsteady BVP for the non-homogeneous heat equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \), \( 0 < x < L, \ t > 0 \)

\[ u(0,t) = u(L,t) = 0 \quad (BC) \]
\[ u(x,0) = u_0(x) \quad (IC) \]

Let's first have a look at the homogeneous equation \( f \equiv 0 \).

We look for solutions of the form: \( u(x,t) = F(x)G(t) \)

So we get: \( F(x) \frac{dG}{dt} = G(t) \frac{d^2F}{dx^2} \)

i.e. \( \frac{4}{F} \frac{d^2F}{dx^2} = \frac{4}{G} \frac{dG}{dt} = -\lambda \)

Equation in time: \( \frac{dG}{dt} + \lambda G = 0 \Rightarrow G(t) = ce^{-\lambda t} \)

Equation in space: \( \frac{d^2F}{dx^2} = \lambda F \)

\[ F = e^{\lambda x} \Rightarrow \lambda^2 = -\lambda \Rightarrow \lambda_{1,2} = \pm \sqrt{-\lambda} \]

3 cases:

1) \( \lambda > 0 \) : \( \lambda_{1,2} = \pm \sqrt{\lambda} \)

\[ F = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x} = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x) \]

\[ F(0) = c_1 = 0 \]
\[ F(L) = c_2 \sinh(\sqrt{\lambda} L) = 0 \Rightarrow c_2 = 0 \] \( \lambda > 0 \) No non-trivial solution

2) \( \lambda = 0 \) : \( \frac{d^2F}{dx^2} = 0 \Rightarrow F = Ax + B \)

\[ F(0) = 0 = B \]
\[ F(L) = AL = 0 \Rightarrow A = 0 \]

Once again, only the trivial solution \( F = 0 \) is solution

3) \( \lambda < 0 \) : \( \lambda_{1,2} = \pm \sqrt{-\lambda} \)

\[ F = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x} = c_1 \cosh(\sqrt{-\lambda} x) + c_2 \sinh(\sqrt{-\lambda} x) \]

\[ F(0) = c_1 = 0 \]
\[ F(L) = c_2 \sinh(\sqrt{-\lambda} L) = 0 \Rightarrow c_2 = 0 \] \( \lambda < 0 \) No non-trivial solution

\[ F = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x} = c_1 \cosh(\sqrt{-\lambda} x) + c_2 \sinh(\sqrt{-\lambda} x) \]

\[ F(0) = c_1 = 0 \]
\[ F(L) = c_2 \sinh(\sqrt{-\lambda} L) = 0 \Rightarrow c_2 = 0 \] \( \lambda < 0 \) No non-trivial solution

4) \( \lambda = 0 \) : \( \frac{d^2F}{dx^2} = 0 \Rightarrow F = Ax + B \)

\[ F(0) = 0 = B \]
\[ F(L) = AL = 0 \Rightarrow A = 0 \]

Once again, only the trivial solution \( F = 0 \) is solution
3) \( \lambda > 0 \): 
\[ f(x) = c_1 e^{i \sqrt{\lambda} x} + c_2 e^{-i \sqrt{\lambda} x} = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \]

\[ f(0) = 0 \Rightarrow c_1 = 0 \]

\[ f(L) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda} L) = 0 \]

To get a non-trivial solution, we need \( \sin(\sqrt{\lambda} L) = 0 \)

\[ \Rightarrow \sqrt{\lambda} L = n\pi, \quad n = 1, 2, \ldots \]

\[ \lambda = \left( \frac{n\pi}{L} \right)^2, \quad n = 1, 2, \ldots \]

The form of the homogeneous solution suggests that the solution of the non-homogeneous equation is of the form:

\[ u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{L} \right) \]  

(sine Fourier series)

Let's write \( f \) and \( u_0 \) as Fourier series:

\[ f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{L} \right), \quad f_n(t) = \frac{2}{L} \int_0^L f(y,t) \sin \left( \frac{n\pi y}{L} \right) \, dy \]

\[ u_0(x) = \sum_{n=1}^{\infty} g_n \sin \left( \frac{n\pi x}{L} \right), \quad g_n = \frac{2}{L} \int_0^L u_0(y) \sin \left( \frac{n\pi y}{L} \right) \, dy \]

Plugging these expressions into the PDE, we get:

\[ \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{L} \right) = \sum_{n=1}^{\infty} \frac{(n\pi)^2}{L^2} a_n(t) \sin \left( \frac{n\pi x}{L} \right) \]

\[ + \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{L} \right) \]

\[ \Rightarrow \forall n \in \mathbb{N}, \quad a_n(t) + \frac{(n\pi)^2}{L^2} a_n(t) = f_n(t), \quad a_n(0) = g_n \]

This is an ODE that is easy to solve; in fact we have

\[ a_n(t) = e^{-\frac{(n\pi)^2}{L^2} t} \left( \int_0^t e^{\frac{(n\pi)^2}{L^2} s} f_n(s) \, ds + g_n \right) \]

\[ g_n \] from IC

So the solution becomes:

\[ u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{(n\pi)^2}{L^2} t} \left( \int_0^t e^{\frac{(n\pi)^2}{L^2} s} f_n(s) \, ds + g_n \right) \sin \left( \frac{n\pi x}{L} \right) \]
Now replacing \( f_n(s) \) and \( g_n \) by their expression:

\[
\begin{align*}
\mu(x,t) &= \sum_{n=1}^{\infty} \frac{a_n}{L} e^{-\left(\frac{n\pi x}{L}\right)^2 t} \left( \int_0^L \left. \left( \frac{L}{\pi} \right) s \cdot \left( \frac{a_n}{L} \int_0^L f(y,s) \sin\left(\frac{n\pi y}{L}\right) dy \right) ds \right. \\
&\quad + \frac{2}{L} \int_0^L w_0(y) \sin\left(\frac{n\pi y}{L}\right) dy \sin\left(\frac{n\pi x}{L}\right) \right)
\end{align*}
\]

which can be arranged as:

\[
\begin{align*}
\mu(x,t) &= \frac{2}{L} \sum_{n=1}^{\infty} \frac{a_n}{L} e^{-\left(\frac{n\pi x}{L}\right)^2 t} \int_0^L \left. \left( \frac{L}{\pi} \right) s \cdot \left( \frac{a_n}{L} \int_0^L f(y,s) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dy \right) ds \right. \\
&\quad + \int_0^L w_0(y) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dy
\end{align*}
\]

\[
\mu(x,t) = \int_0^L \int_0^L G(x,t,y,s) f(y,s) \, dy \, ds + \int_0^L G(x,t,y,0) w_0(y) \, dy
\]

with \( G(x,t,y,s) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi x}{L}\right)^2 (t-s)} \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \)

Once again, \( G \) is an Green's function that expresses the effect of the data (IC and source term) at one point \((y,s)\) on the solution at another point \((x,t)\).