Math 401: Assignment 4 (Due Wed., Feb 13th at the start of class)

4. Find the Dirichlet Green's function for the square \([0, \pi] \times [0, \pi]\) (as an eigenfunction expansion), and use it to express the solution of the boundary value problem

\[
\begin{cases}
\Delta u = 0 & 0 < x_1 < \pi, \ 0 < x_2 < \pi \\
u = 0 & x_1 = 0, \ x_1 = \pi, \ x_2 = \pi \\
u = g(x_1) & x_2 = 0
\end{cases}
\]

Tip: the final solution looks as if it does not satisfy the BC \(u = g(x_1)\) for \(x_2 = 0\) but it does. It is actually satisfied at the limit \(x_2 \to 0\) and has to do with the convergence of the sum of \(\sin(nx)/n\) for \(n=1\) to infinity which is the Fourier series of \((\pi-x)/2\) over \([0:2\pi]\) and jumps from 0 to \(\pi/2\) at \(x = 0\). Quite tough to prove that the solution you obtain indeed satisfies BCs. Trust your calculations!

2. Find the Dirichlet Green's function (as an eigenfunction expansion) for the semi-infinite strip \([0, 1] \times [0, \infty)\), and use it to express the solution of the problem

\[
\begin{cases}
\Delta u = 0 & 0 < x_1 < 1, \ 0 < x_2 < \infty \\
u = g(x_1) & x_2 = 0 \\
u \to 0 & x_2 \to \infty \\
u = 0 & x_1 = 0, \ x_1 = 1
\end{cases}
\]

Tip: write an eigenfunction expansion in the \(x_1\) direction only and solve for the coefficients that are function of the other direction \(x_2\).