1. Find the free-space Green’s function for the operator \(-\Delta + k^2\) (\(k > 0\) a constant) in \(\mathbb{R}^3\). That is, solve

\[ (-\Delta + k^2)G_x = \delta_x \]

in \(\mathbb{R}^3\), with \(G_x(y) \to 0\) as \(|y| \to \infty\). (Hint: look for \(G_x(y)\) in the form \(G_x(y) = g(r)/r\), \(r := |y - x|\).)

Writing \(G = g(r)/r\), we have, for \(r > 0\),

\[
0 = -\left(\frac{d}{dr} + \frac{2}{r}\right)(g/r)' + k^2 g/r = -\left(\frac{d}{dr} + \frac{2}{r}\right)(g'/r - g/r^2) + k^2 g/r \\
= -g''/r + k^2 g/r = (-g'' + k^2 g)/r,
\]

so \(g''(r) = k^2 g(r)\), and so \(g(r) = Ae^{kr} + Be^{-kr}\). We take \(A = 0\), to avoid \(G\) growing as \(r = |y - x| \to \infty\), hence

\[ G_x(y) = B \frac{e^{-k|y-x|}}{|y-x|}, \]

and it remains to find \(B\). For this, use, for some \(\delta > 0\),

\[
1 = \int_{|y-x| \leq \delta} \delta_x(y)dy = \int_{|y-x| \leq \delta} (-\Delta + k^2)G_x(y)dy \\
= - \int_{|y-x| \leq \delta} \nabla \cdot \nabla G_x(y)dy + k^2 \int_{|y-x| \leq \delta} G_x(y)dy.
\]

The absolute value of the second term on the right is (using \(|e^{-kr}| \leq 1\))

\[
|k^2 B(4\pi) \int_0^\delta \frac{e^{-kr}}{r} r^2 dr| \leq 2\pi k^2 |B| \delta^2 \to 0 \text{ as } \delta \to 0,
\]

while, using the divergence theorem, the first term on the right is

\[
- \int_{|y-x| = \delta} \frac{\partial}{\partial n} G_x(y)dS(y) = B \int_{|y-x| = \delta} e^{-k|y-x|} \left( \frac{1}{|y-x|^2} + \frac{1}{|y-x|} \right) dS(y) \\
= B \int_{|y-x| = \delta} e^{-k\delta} \left( \frac{1}{\delta^2} + \frac{1}{\delta} \right) dS(y) = 4\pi Be^{-k\delta}(1 + k\delta) \to 4\pi B \text{ as } \delta \to 0.
\]

Hence taking \(\delta \to 0\), we find \(B = 1/(4\pi)\), and so

\[ G_x(y) = \frac{e^{-k|y-x|}}{4\pi|y-x|}. \]

2. Let \(D_a\) be the disk of radius \(a\) in the plane, centred at the origin. Let \(C_a\) be its boundary (circle of radius \(a\)).
(a) Use the method of images to find the Dirichlet ($G = 0$ on the boundary) Green’s function for $\Delta$ on $D_a$.

Given $x \in D_a$ ($x \neq 0$), set $x^* := \frac{a^2}{|x|^2} x$ (so that $|x^*| = a^2/|x| > a$ and $x^* \notin D_a$),
and note that for $y \in \partial D_a$ (i.e. $|y| = a$),

$$|y - x^*|^2 = a^2 - 2 \frac{a^2}{|x|^2} x \cdot y + \frac{a^2}{|x|^2} |y - x|^2.$$  

Thus, since the 2D free-space Green’s function is $\frac{1}{|y - x|/(2\pi)}$, setting

$$G_x(y) := \frac{1}{2\pi} \ln |y - x| - \frac{1}{2\pi} \ln |y - x^*| + \frac{1}{2\pi} \ln \frac{a}{|x|} = \frac{1}{2\pi} \ln \left( \frac{a|y - x|}{|x||y - x^*|} \right)$$

(note that this makes sense even for $x \to 0$), we see that

- for $y \in D_a$, $\Delta_y G_x(y) = \delta_x(y) - \delta_{x^*}(y) = \delta_x(y)$,
- for $y \in \partial D_a$, $G_x(y) = 0$,

as required.

(b) For a given function $f(x)$ on $D_a$, solve $\Delta u = f(x)$ in $D_a$, $u = 0$ on $C_a$.

$$u(x) = (\delta_x, u) = (\Delta_y G_x(y), u) = (G_x, \Delta u) + \int_{\partial D_a} \left( \frac{\partial G_x}{\partial n} u - G_x \frac{\partial u}{\partial n} \right) dl(y)$$

$$= (G_x, f) = \frac{1}{2\pi} \int_{D_a} \ln \left( \frac{a|y - x|}{|x||y - x^*|} \right) f(y) \, dy.$$

(c) For a given function $g(\theta)$ on $C_a$ ($0 \leq \theta \leq 2\pi$ denoting the angle around $C_a$), solve $\Delta u = 0$ in $D_a$, $u = g(\theta)$ on $C_a$.

Using the computation from the previous part, we see

$$u(x) = \int_{\partial D_a} \frac{\partial G_x}{\partial n} u \, dl.$$

Compute

$$\nabla_y G_x(y)|_{y \in \partial D_a} = \frac{1}{2\pi} \left( \frac{y - x}{|y - x|^2} - \frac{y - x^*}{|y - x^*|^2} \right)$$

$$= \frac{1}{2\pi |y - x|^2} \left( y - x - \frac{|x|^2}{a^2} \left( y - \frac{a^2}{|x|^2} x \right) \right) = \frac{1 - |x|^2/a^2}{2\pi |y - x|^2} y,$$

and since $\hat{n} = y/a$, we have

$$\frac{\partial G_x}{\partial n}|_{y \in \partial D_a} = \frac{a^2 - |x|^2}{2\pi a |y - x|^2},$$

and so

$$u(x) = \frac{a^2 - |x|^2}{2\pi a} \int_0^{2\pi} g(\theta) \left( \frac{\cos \theta - x_1}{(a \cos \theta - x_1)^2 + (a \sin \theta - x_2)^2} \right)^2 d\theta,$$

which is Poisson’s formula giving the harmonic function on the disk $D_a$ with boundary values $g$.  

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3. Let $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}$ be the first quadrant in the plane. Use the method of images to solve the problem $\Delta u = 0$ in $D$, $\frac{\partial}{\partial x_2} u(x_1, 0) = f(x_1)$, $x_1 > 0$, and $\frac{\partial}{\partial x_1} u(0, x_2) = g(x_2)$, $x_2 > 0$ by finding the Neumann Green’s function ($\frac{\partial}{\partial n} G = 0$ on the boundary).

For $x \in D$, set $\bar{x} = (x_1, -x_2)$, and take

$$G_x(y) = \frac{1}{2\pi} \left( \ln |y - x| + \ln |y - \bar{x}| + \ln |y + x| + \ln |y + \bar{x}| \right) = \frac{1}{2\pi} \ln (|y - x||y - \bar{x}||y + x||y + \bar{x}|) .$$

Then since $\pm \bar{x}$ and $-x$ lie outside $D$, we have $\Delta_y G_x(y) = \delta_x(y)$ for $y \in D$. Moreover, if $y_2 = 0$, then

$$\frac{\partial}{\partial y_2} G_x(y) \bigg|_{y_2=0} = \frac{1}{2\pi} \left( \frac{1}{|y - x|^2} (-x_2 + x_2 + 1/|y + x|^2 (x_2 - x_2)) \right) = 0 .$$

Similarly, $\frac{\partial}{\partial y_2} G_x(y) \bigg|_{y_2=0} = 0$. So $\frac{\partial}{\partial n} G|_{y \in \partial D} = 0$, as required. Thus $G_x$ is our Neumann Green’s function. Now by the usual ’integration by parts” (as in 2(b) above),

$$u(x) = -\int_{\partial D} \frac{\partial u(y)}{\partial n(y)} G_x(y) \, dl(y)$$

$$= \frac{1}{2\pi} \int_0^\infty f(y_1) \ln \left( \left( (y_1 - x_1)^2 + x_2^2 \right) (y_1 + x_1)^2 + x_2^2 \right) \, dy_1$$

$$+ \frac{1}{2\pi} \int_0^\infty g(y_2) \ln \left( \left( x_1^2 + (y_2 - x_2)^2 \right) (x_1^2 + (y_2 + x_2)^2) \right) \, dy_2 .$$

Remark: note that we did not need a solvability condition to get this formula, even though any (non-zero) constant is a non-zero solution of the homogeneous problem.

However, consider the first term and notice that for $x_1 \gg y_1$,

$$\ln \left( \left( (y_1 - x_1)^2 + x_2^2 \right) (y_1 + x_1)^2 + x_2^2 \right) \approx 4 \ln |x|$$

and so the leading term for large $|x|$ is

$$\frac{2}{\pi} \int_0^\infty f(y_1) \, dy_1 \ln |x| ,$$

and so in order to have a solutions which decays (rather than grows) as $|x| \to \infty$, we require the ”solvability condition” $\int_0^\infty f(y_1) \, dy_1 = 0$ (and similarly $\int_0^\infty g(y_2) \, dy_2 = 0$ for the second term).