Math 401: Assignment 2 Solutions

1. Let \( L := a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x) \).

   (a) Show that \( L = L^* \) if and only if \( a'_0 = a_1 \)
   
   We have (carry out the integration by parts if it is unclear)
   
   \[
   L^* u = (a_0 u')' - (a_1 u)' + a_2 u = a_0 u'' + (2a_0 - a_1) u' + (a''_0 - a'_1 + a_2) u.
   \]
   
   So if \( L = L^* \), we must have \( 2a_0 - a_1 = a'_1 \) or \( a'_0 = a_1 \). Conversely, if \( a'_0 = a_1 \), then it follows from the above expression that \( L = L^* \).

   (b) Under what condition on the numbers \( \alpha \) and \( \beta \) is the problem
   
   \[
   (p(x)u')' + q(x)u = f(x), \quad 0 < x < 1, \quad u(0) = \alpha u(1), \quad u'(0) = \beta u'(1)
   \]
   
   self-adjoint?

   If \( u(x) \) and \( v(x) \) both satisfy the given BCs (which are homogeneous),
   
   \[
   (u, L v) = \int_0^1 u(x)[(p(x)v'(x))' + q(x)v(x)]dx
   \]
   
   \[
   = \int_0^1 [ -u'(x)p(x)v'(x) + q(x)u(x)v(x) ]dx + p(1)u(1)v'(1) - p(0)u(0)v'(0)
   \]
   
   \[
   = \int_0^1 [v(x)(p(x)u'(x))' + q(x)u(x)v(x)]dx
   \]
   
   \[
   - p(1)u'(1)v(1) + p(0)u'(0)v(0) + p(1)u(1)v'(1) - p(0)u(0)v'(0)
   \]
   
   \[
   = (Lu, v) + p(0)[\beta \alpha u'(1)v(1) - \alpha \beta u(1)v'(1)] + p(1)[u(1)v'(1) - u'(1)v(1)]
   \]
   
   \[
   = (Lu, v) + [u(1)v'(1) - u'(1)v(1)][p(1) - \alpha \beta p(0)]
   \]
   
   so the problem is self-adjoint if \( \alpha \beta = p(1)/p(0) \).

2. The steady-state temperature along the rod \( 0 \leq x \leq 1 \) is \( u(x) \). The thermal conductivity of the rod is \( e^x \). There is a heat source \( f(x) \). The left end of the rod is insulated, while the right end is held at temperature 1. Find \( u(x) \) – i.e., solve

   \[
   (e^x u')' = f(x), \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 1,
   \]

   by finding the Green’s function, \( G \), for the problem and expressing \( u \) in terms of \( G \) and \( f \).

   This problem is self-adjoint (it is of Sturm-Liouville type), so the Green’s function solves, for a given \( x \in (0, 1) \),

   \[
   (e^y G_x'(y))' = \delta_x(y), \quad 0 < y < 1,
   \]

   \[
   G_x'(0) = 0, \quad G_x(1) = 0
   \]
(the homogeneous version of the original BCs). So for $y < x$, $G'_x = Ae^{-y}$ hence $G_x = -Ae^{-y} + B$. The BC $G'_x(0) = 0$ means $A = 0$. Similarly, for $y > x$, $G_x = -Ce^{-y} + D$. The BC $G_x(1) = 0$ means $C = eD$. So

$$G_x(y) = \begin{cases} B & 0 \leq y < x \\ D(1 - e^{1-y}) & x < y \leq 1 \end{cases}.$$ 

Continuity at $y = x$ yields $B = D(1 - e^{1-x})$. Now the jump condition:

$$e^x G'_x(y)|^+_x = \int_x^1 \delta_x = 1.$$

Thus we have $eD = 1$. So

$$G_x(y) = \begin{cases} e^{-1} - e^{-x} & 0 \leq y < x \\ e^{-1} - e^{-y} & x < y \leq 1 \end{cases}.$$ 

OK, now we may express the solution by integrating by parts:

$$u(x) = \langle \delta_x, u \rangle = \langle (e^y G'_x)'(y), u \rangle = -(e^y G'_x(y), u') + e^y G'_x u|_0^1$$

$$(G_x, (e^y u')') + eG'_x(1) - eG'_x u|_0^1 = (G_x, f) + eG'_x(1)$$

$$(e^{-1} - e^{-x}) \int_0^1 f(y)dy + \int_x^1 (e^{-1} - e^{-y})f(y)dy + 1.$$

Remark: this problem is much simpler to integrate directly!

3. Consider the problem

$$u'' + u = f(x), \quad 0 < x < L,$$

$$u(0) = 0, \quad u(L) = 0.$$

(a) Find the Green’s function, and express the solution in terms of it.

Again, this is a self-adjoint problem, so the Green’s function $G_x(y)$ solves the problem

$$G''_x(y) + G_x(y) = \delta_x(y), \quad 0 < y < L,$$

$$G_x(0) = G_x(L) = 0.$$ 

For $0 < y < x$, $G''_x = -G_x$ implies $G_x(y) = A \sin(y) + B \cos(y)$. The boundary condition $G_x(0) = 0$ yields $B = 0$. For $x < y \leq L$, we have $G_x(y) = C \sin(y) + D \cos(y)$. The BC $G_x(L) = 0$ yields

$$C \sin(L) + D \cos(L) = 0,$$

and so $C = -\cot(L)D$ (if $\sin(L) \neq 0$, and $D = 0$ if $\sin(L) = 0$). Continuity of $G_x$ at $x$ then requires

$$A \sin(x) = [-\cot(L) \sin(x) + \cos(x)]D$$

(or $A = C$ if $\sin(L) = 0$). Finally, we have a jump condition:

$$G''_x|^{x_+}_{x_-} = \int_{x_-}^{x_+} G''_x(y)dy = \int_{x_-}^{x_+} (\delta_x(y) - G_x(y))dy = 1.$$
This means

\[-\cot(L) \cos(x) - \sin(x)]D = A \cos(x) + 1 = 1 + \cot(x)[\cos(x) - \cot(L) \sin(x)]D

(or \(A \cos(x) + 1 = A \cos(x)\) if \(\sin(L) = 0\), which is impossible!). Solving for \(D\) gives

\[
\frac{1}{D} = -\cot(x) \cos(x) - \sin(x) = -\csc(x)
\]

or \(D = -\sin(x)\). Thus \(A = \cot(L) \sin(x) - \cos(x)\) and \(C = \cot(L) \sin(x)\). Hence

the Green’s function is

\[
G_x(y) = \frac{1}{\sin(L)} \left\{ \begin{array}{ll}
[\cos(L) \sin(x) - \sin(L) \cos(x)] \sin(y) & 0 \leq y \leq x \\
\sin(x) \cos(L) \sin(y) - \sin(L) \cos(y) & x \leq y \leq L
\end{array} \right.
\]

and the solution is

\[
u(x) = \frac{1}{\sin(L)} \left[ \cos(L) \sin(x) - \sin(L) \cos(x) \right] \int_0^x \sin(y) f(y) dy
\]

\[
+ \sin(x) \int_x^L \left( \cos(L) \sin(y) - \sin(L) \cos(y) \right) f(y) dy.
\]

(b) Find the values of \(L\) for which this solution breaks down, and for these values, determine the solvability condition on \(f\), calculate the modified Green’s function, and (assuming the solvability condition is satisfied) find an integral representation for the solution.

This solution formula breaks down if \(\sin(L) = 0\), that is if \(L = n\pi\) for some integer \(n\). In this case,

\[u^*(x) = \sin(x)\]

is a non-zero solution of the homogeneous problem (both ODE \((u^*)'' + u^* = 0\), and \(BCs\) \(u^*(0) = 0 = u^*(n\pi)\)), and our original problem has a solvability condition for \(f\):

\[0 = (u^*, f) = \int_0^{n\pi} \sin(x) f(x) dx.\]

A modified Greens function \(\tilde{G}_x(y)\) should solve

\[
\tilde{G}_x''(y) + \tilde{G}_x(y) = \delta_x(y) + c(x) \sin(y), \quad \tilde{G}_x(0) = 0 = \tilde{G}_x(n\pi),
\]

where

\[
c(x) = -\frac{\sin(x)}{\int_0^{n\pi} \sin^2(x) dx} = -\frac{2}{n\pi} \sin(x).
\]

For \(y \neq x\), we must solve

\[
\tilde{G}_x''(y) + \tilde{G}_x(y) = -\frac{2}{n\pi} \sin(x) \sin(y).
\]

Since \(\sin(y)\) is a solution of the homogeneous problem, we try for a particular solution of the form \(\tilde{G} = Ay \sin(y) + By \cos(y)\) (method of undetermined coefficients). Thus \(\tilde{G}' = A \sin(y) + B \cos(y) + A y \cos(y) - B y \sin(y), \tilde{G}'' = 2A \cos(y) - 2B \sin(y) - A y \sin(y) - B y \cos(y),\) and \(\tilde{G}'' + \tilde{G} = 2A \cos(y) - 2B \sin(y).\)
So take $A = 0$ and $B = \sin(x)/(n\pi)$. The general solution is our particular solution plus the general solution of the homogeneous problem:

$$
\tilde{G}_x(y) = \begin{cases}
\frac{1}{n\pi} \sin(x)y\cos(y) + a(x) + b(x) & 0 \leq y < x \\
\frac{1}{n\pi} \sin(x)y\cos(y) + c(x) - d(x) & x < y \leq n\pi
\end{cases}.
$$

Imposing the zero BCs implies $b = 0$ and $0 = (\sin(x) + d) \cos(n\pi) \implies d = -\sin(x)$, so

$$
\tilde{G}_x(y) = \begin{cases}
\frac{1}{n\pi} \sin(x)y\cos(y) + a(x) & 0 \leq y < x \\
\frac{1}{n\pi} \sin(x)y\cos(y) + c(x) - \sin(x)\cos(y) & x < y \leq n\pi
\end{cases}.
$$

Continuity at $y = x$ requires $a(x) = (c - \cos(x))\sin(x)$, so $c = a + \cos(x)$ and hence

$$
\tilde{G}_x(y) = a(x) + \begin{cases}
\frac{1}{n\pi} \sin(x)y\cos(y) & 0 \leq y < x \\
\frac{1}{n\pi} \sin(x)y\cos(y) + \cos(x)\sin(x) - \sin(x)\cos(y) & x < y \leq n\pi
\end{cases}
$$

and we may as well take $a = 0$ (recall we can always add any multiple of the homogeneous solution $\sin(y)$) and recognize $\cos(x)\sin(y) - \sin(x)\cos(y) = \sin(y - x)$ to arrive at

$$
\tilde{G}_x(y) = \begin{cases}
\frac{1}{n\pi} \sin(x)y\cos(y) & 0 \leq y < x \\
\frac{1}{n\pi} \sin(x)y\cos(y) + \sin(y - x) & x < y \leq n\pi
\end{cases}.
$$

Notice now that the jump condition

$$
1 = \tilde{G}'_x|_{x^+} = \cos(0) = 1
$$

indeed holds. Finally, assuming the solvability condition above holds, the family of solutions to our original problem is, for any constant $C$,

$$
u(x) = (G_x, f) + C \sin(x) = \int_0^{n\pi} \tilde{G}_x(y)f(y)dy + C \sin(x)
$$

$$
= \frac{1}{n\pi} \sin(x) \int_0^{n\pi} y\cos(y)f(y)dy + \int_x^{n\pi} \sin(y - x)f(y)dy + C \sin(x).
$$

4. Consider the equation

$$
-u'' + q^2u = f(x)
$$

($q > 0$ a constant). Determine the Green’s function for this problem on the line $-\infty < x < \infty$ associated with the “boundary conditions” $u(x) \to 0$ as $|x| \to \infty$.

The Green’s function $G_x(y)$ should solve $-G''_x + q^2G_x = \delta_x$. For $y \neq x$, $G''_x = q^2G_x$ so $G_x = Ae^{qy} + Be^{-qy}$. For $y < x$, we take $B = 0$ to ensure $\lim_{y \to -\infty} G_x(y) = 0$, while for $y > x$, we take $A = 0$ to ensure $\lim_{y \to +\infty} G_x(y) = 0$, leaving

$$
G_x(y) = \begin{cases}
Ae^{qy} & -\infty < y < x \\
Be^{-qy} & x < y < \infty
\end{cases}.
$$
Continuity at \( y = x \) requires \( Ae^{qx} = Be^{-qx} \), so writing \( A = ae^{-qx} \) we have

\[
G_x(y) = a \begin{cases} 
  e^{q(y-x)} & -\infty < y < x \\
  e^{q(x-y)} & x < y < \infty
\end{cases} = ae^{-q|y-x|}.
\]

The jump condition

\[
1 = -G_x(x^+) = 2aq
\]

implies \( a = 1/(2q) \) and so

\[
G_x(y) = \frac{1}{2q}e^{-q|y-x|}.
\]

5. Find a solvability condition for the problem

\[
\begin{align*}
  u'' + q^2 u &= f(x), \quad 0 < x < 1, \\
  u(0) &= u(1), \quad u'(0) = -u'(1)
\end{align*}
\]

\((q > 0 \text{ a constant})\).

The operator \( L = \frac{d^2}{dx^2} + q^2 \) is self-adjoint, but the BCs are not. Let’s find the (homogeneous) adjoint BCs. Suppose \( u(0) = u(1) \) and \( u'(0) = -u'(1) \), and compute by integrating twice by parts,

\[
(Lu, v) = \int_0^1 (u''(x) + q^2 u(x))v(x)dx = (u'v - v'u)'|_0^1 + (u, Lv)
\]

\[
= u'(1)v(1) - v'(1)u(1) - u'(0)v(0) + v'(0)u(0) + (u, Lv)
\]

\[
= u'(1)[v(1) + v(0)] + u(1)[v'(0) - v'(1)] + (u, Lv).
\]

The adjoint BCs on \( v \) are those which make the boundary terms here vanish: \( v(0) = -v(1), \) and \( v'(0) = v'(1) \). So we have a solvability condition if we have a non-zero solution \( u^*(x) \) of

\[
(u^*)'' + q^2 u^* = 0, \quad u^*(0) = -u^*(1), \quad (u^*)'(0) = (u^*)'(1).
\]

The general solution is \( u^* = A \sin(qx) + B \cos(qx) \). The first BC yields \( B = -A \sin(q) - B \cos(q) \) while the second yields \( qA = qA \cos(q) - qB \sin(q) \). A non-zero solution is \( A = 1 + \cos(q), \quad B = -\sin(q) \), giving

\[
u^*(x) = \sin(q(x - 1/2)) \sin(qx) - \sin(q) \cos(qx),
\]

and so the solvability condition

\[
(u^*, f) = (1 + \cos(q)) \int_0^1 \sin(qx) f(x)dx - \sin(q) \int_0^1 \cos(qx) f(x)dx = 0
\]

(if you are good with trig identities, you might recognize \( u^* \) as a multiple of \( \sin(q(x - 1/2)) \), which if we were clever, we might have guessed directly from the BCs).