Math 401: Assignment 2 (Due Fri, Jan 26)

1. Let $L := a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x)$.
   
   (a) Show that $L = L^*$ if and only if $a'_0 = a_1$
   
   (b) Under what condition on the numbers $\alpha$ and $\beta$ is the problem
   
   $$(p(x)u')' + q(x)u = f(x), \quad 0 < x < 1, \quad u(0) = \alpha u(1), \quad u'(0) = \beta u'(1)$$
   
   self-adjoint?

2. The steady-state temperature along the rod $0 \leq x \leq 1$ is $u(x)$. The thermal conductivity of the rod is $c^2$. There is a heat source $f(x)$. The left end of the rod is insulated, while the right end is held at temperature 1. Find $u(x)$ — i.e., solve
   
   $$(e^x u')' = f(x), \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 1,$$
   
   by finding the Green’s function, $G$, for the problem and expressing $u$ in terms of $G$ and $f$.

3. Consider the problem
   
   $$u'' + u = f(x), \quad 0 < x < L,$$
   
   $$u(0) = 0, \quad u(L) = 0.$$
   
   (a) Find the Green’s function, and express the solution in terms of it.
   
   (b) Find the values of $L$ for which this solution breaks down, and for these values, determine the solvability condition on $f$, calculate the modified Green’s function, and (assuming the solvability condition is satisfied) find an integral representation for the solution.

4. Consider the equation
   
   $$-u'' + q^2 u = f(x)$$
   
   ($q > 0$ a constant). Determine the Green’s function for this problem on the line $-\infty < x < \infty$ associated with the “boundary conditions" $u(x) \to 0$ as $|x| \to \infty$.

5. Find a solvability condition for the problem
   
   $$u'' + q^2 u = f(x), \quad 0 < x < 1,$$
   
   $$u(0) = u(1), \quad u'(0) = -u'(1)$$
   
   ($q > 0$ a constant).

(Jan. 19)