Math 401: Assignment 1 (Due Wed., Jan. 18)

1. (Fun with generalized functions)
   (a) Show \( f(x)\delta_y(x) = f(y)\delta_y(x) \) (where \( \delta_y(x) = \delta(x - y) \) denotes the delta function centred at the point \( y \)).
   (b) Show \( x\delta'(x) = -\delta(x) \).
   (c) Find the (generalized) first, second, and third derivatives of \( f(x) = \text{sgn}(x)x^2 \) (where \( \text{sgn}(x) \) denotes \(-1\) if \( x \leq 0 \) and \( 1 \) if \( x \geq 0 \)).
   (d) Express \( \delta(e^x - 2) \) as a (usual) delta function.

2. (Usefulness of distributions in PDE – slightly challenging)
   As you may know, for any smooth function \( f : \mathbb{R} \to \mathbb{R} \), \( u(x, t) = f(x - ct) \) (which is just a wave moving to the right with speed \( c \)) is a solution of the transport equation \( u_t + cu_x = 0 \) (you can easily check this using the chain rule). However, if \( f \) is not differentiable (for example a square, triangular, or sawtooth wave), \( u_x \) and \( u_t \) do not necessarily make any sense at all! Distributions (generalized functions) can get us out of this mess. Show that if \( f \) is merely a continuous (or even just piecewise continuous) function, then \( u(x, t) = f(x - ct) \) satisfies the PDE \( u_t + cu_x \) in the sense of distributions on \( \mathbb{R}^2 = \{(x, t) \mid x \in \mathbb{R}, \ t \in \mathbb{R}\} \).

3. (Review of separation of variables, and warm-up for Green’s functions)
   (a) Use the method of separation of variables and Fourier series to solve Poisson’s equation on the rectangle \([0,1] \times [0,1]\) (with zero boundary conditions) for \( u(x_1, x_2) \):
       \[
       \Delta u := u_{x_1x_1} + u_{x_2x_2} = f(x_1, x_2), \ 0 < x_1 < 1, \ 0 < x_2 < 1, \\
       u(x_1, x_2) = 0 \text{ for } x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1.
       \]
   (b) Now write your solution \( u(x_1, x_2) \) in the form
       \[
       u(x_1, x_2) = \int_0^1 \int_0^1 G(x_1, x_2; y_1, y_2)f(y_1, y_2)dy_1 dy_2,
       \]
       and in so doing, identify the Green’s function \( G(x_1, x_2; y_1, y_2) \).

(Jan. 13)

(Tips for 2):
1) Use \((f, t)\)=\(-(f, t')\) for a test function \( t \)
2) Use change of variables \( s=x-ct \), change \( t \) to \((x-s)/c\) and show that integral over \( x \) is 0