Math 401: Assignment 6 (Due Wed., Mar. 23 at the start of class)

1. Wave equation on the half-line. Solve the wave equation on the half-line, with zero initial velocity, with Dirichlet BCs at the origin

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0, & 0 < x < \infty, \ t > 0 \\
  u(0,t) = 0 \\
  u(x,0) = g(x), \ u_t(x,0) = 0
\end{cases}
\]

and with Neumann BCs at the origin,

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0, & 0 < x < \infty, \ t > 0 \\
  u_t(0,t) = 0 \\
  u(x,0) = g(x), \ u_t(x,0) = 0
\end{cases}
\]

by using the method of images to find, for each case, the appropriate Green’s function, and then expressing the solution in terms of the Green’s function. Interpret your results in terms of (reflected) waves. Hints:

- Recall the Green’s function for the wave equation on the entire real line is
  \[ G^R_x(y,\sigma) = \frac{1}{2c} H(\sigma - |y - x|/c), \ \sigma = t - \tau \]

- You may use the relation \[ \frac{d}{dx} H(\sigma - |y - x|/c) = c[\delta(y - (x - c\sigma)) + \delta(y - (x + c\sigma))] \]
  (can you derive this?)

2. Energy conservation for the wave equation.

(a) Show that if \( u(x,t) \) solves the Cauchy problem for the wave equation on bounded domain \( D \) with Neumann BCs,

\[
\begin{cases}
  u_{tt} = c^2 \Delta u & x \in D, \ t > 0 \\
  \frac{\partial u}{\partial n} = 0 & x \in \partial D \\
  u(x,0) = u_0(x), \ u_t(x,0) = v_0(x)
\end{cases}
\]

then the energy

\[ E(t) := \frac{1}{2} \int_D \left( |\nabla u(x,t)|^2 + \frac{1}{c^2} (u_t(x,t))^2 \right) \, dx \]

is a constant function of time.

(b) Prove that solutions of problem (1) are unique. (Hint: consider the difference of two solutions, and look at its energy.)

3. Helmholtz equation.

(a) Suppose \( v(x,t) \) satisfies the wave equation \( v_{tt} = c^2 \Delta v \) and has the following periodic-in-time form: \( v(x,t) = e^{-\omega t} u(x) \), with \( \omega > 0 \). Show that \( u(x) \) satisfies the Helmholtz equation

\[
(\Delta + k^2)u = 0 \quad \quad k = \frac{\omega}{c}.
\]
(b) **3D free-space Green’s function for Helmholtz.** Find the constant $A$ so that $Ae^{ikr}/r$ and $Ae^{-ikr}/r$, where $r = |y - x|$, satisfy the requirements of being free-space Green’s functions in three dimensions for the Helmholtz operator $\Delta + k^2$. *Remark: how can we choose between the two? One way is to demand that the Green’s function describe an “outgoing” (“scattering”) wave, and note that $e^{-i\omega t}e^{ikr} = e^{i(kr - \omega t)}$ is an outgoing spherical wave for $t > 0$, while $e^{i\omega t}e^{-ikr}$ is not.*

4. For some bounded domain $D \subset \mathbb{R}^n$, let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ be the (Dirichlet) eigenvalues of the problem

\[
\begin{cases}
L\phi := -\nabla \cdot [p(x)\nabla \phi] + q(x)\phi = \lambda \phi & x \in D \\
\phi = 0 & x \in \partial D
\end{cases}
\]

and let $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ be corresponding ortho-normal eigenfunctions. Write the solutions $u(x, t)$, as eigenfunction expansions, of:

(a) the (generalized) heat equation:

\[
\begin{cases}
\partial_t u = -Lu, & x \in D, \ t > 0 \\
u = 0 & x \in \partial D \\
u(x, 0) = f(x)
\end{cases}
\]

(b) the (generalized) Schrödinger equation (here $u(x, t)$ is complex-valued):

\[
\begin{cases}
i\partial_t u = -Lu, & x \in D, \ t > 0 \\
u = 0 & x \in \partial D \\
u(x, 0) = f(x)
\end{cases}
\]

(c) the (generalized) wave equation:

\[
\begin{cases}
\partial_{tt} u = -Lu, & x \in D, \ t > 0 \\
u = 0 & x \in \partial D \\
u(x, 0) = f(x), \ u_t(x, 0) = g(x)
\end{cases}
\]