Math 401: Assignment 5 Solutions

1. Consider the heat-equation on the half-line with insulating boundary condition at the endpoint:

\[
\begin{align*}
  & \quad u_t = u_{xx} \quad x > 0, \quad t > 0 \\
  & \quad u_x(0,t) = 0 \\
  & \quad u(x,0) = u_0(x), \quad x > 0 \\
\end{align*}
\]

(a) Write down the problem that the corresponding (Neumann) Green’s function $G_x(y,\sigma) \ (\sigma = t-\tau)$ should solve.

For $x > 0$,

\[
\begin{align*}
  \frac{\partial}{\partial y} G_x - \frac{\partial^2}{\partial y^2} G_x &= \delta(y)\delta(\sigma), \quad y > 0, \quad \sigma > 0, \\
  \frac{\partial}{\partial y} G_x(0,\sigma) &= 0, \quad \sigma > 0, \\
  G_x &= 0, \quad \sigma < 0
\end{align*}
\]

(b) Use the method of images to find this Green’s function.

By the method of images, the Green’s function is

\[
G_x(y,\sigma) = (4\pi\sigma)^{-1/2} \left[ e^{-(y-x)^2/(4\sigma)} + e^{-(y+x)^2/(4\sigma)} \right]
\]

(c) Express the solution $u(x,t)$ of the above heat equation problem using this Green’s function.

For $0 < t < T$,

\[
\begin{align*}
  u(x,t) &= \int_0^T d\tau \int_0^\infty dy \ u(y,\tau)(-G_\tau - G_{yy}) \\
  &= \int_0^T d\tau \int_0^\infty dy \ (u_\tau - u_{yy}) G + \int_0^T d\tau (u_y G - u G_y)|_{y=\infty} - \int_0^\infty dy \ u G_{|\tau=T}^{\tau=0} \\
  &= \int_0^{\infty} G_x(y,t)u_0(y)dy = (4\pi t)^{-1/2} \int_0^{\infty} \left[ e^{-(y-x)^2/(4t)} + e^{-(y+x)^2/(4t)} \right] u_0(y)dy.
\end{align*}
\]

(d) Now suppose that heat is being pumped in at the endpoint at a fixed rate, so that the boundary condition becomes $u_x(0,t) = a \ (a \ a \ constant)$. Write the solution $u(x,t)$ in this case, and find an explicit formula for the temperature at the endpoint, $u(0,t)$.

As above,

\[
\begin{align*}
  u(x,t) &= \int_0^{\infty} G_x(y,t)u_0(y)dy - a \int_0^t G_x(0, t - \tau)d\tau,
\end{align*}
\]

so that

\[
\begin{align*}
  u(0,t) &= \int_0^{\infty} G_0(y,t)u_0(y)dy - a \int_0^t G_0(0, t - \tau)d\tau \\
  &= (\pi t)^{-1/2} \int_0^{\infty} e^{-y^2/(4t)}u_0(y)dy - a(\pi)^{-1/2} \int_0^t (t - \tau)^{-1/2}d\tau \\
  &= \frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-y^2/(4t)}u_0(y)dy - \frac{2a}{\sqrt{\pi t}} \sqrt{t}
\end{align*}
\]
2. Consider the following initial-boundary value problem for the heat equation in the 1/2-plane \( \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \} \):

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u \quad x \in V, \quad t > 0 \\
\quad u|_{x_2=0} = g(x_1), \\
\quad u(x, 0) = u_0(x) \quad x \in V
\end{cases}
\]

where \( g \) and \( u_0 \) are continuous functions which decay rapidly at infinity.

(a) Write the problem solved by the corresponding (Dirichlet) Green’s function \( G_x(y, \sigma) \) (\( \sigma = t - \tau \)).

For \( x \in V \),

\[
\begin{cases}
\frac{\partial G}{\partial \sigma} - \Delta_y G = \delta_x(y)\delta_0(\sigma), \quad y_2 > 0, \quad \sigma > 0, \\
\quad G = 0, \quad y_2 = 0, \\
\quad G = 0, \quad \sigma < 0
\end{cases}
\]

(b) Use the method of images to find that Green’s function.

\[
G_x(y, \sigma) = (4\pi \sigma)^{-1} \left[ e^{-|y-x|^2/(4\sigma)} - e^{-|y-\bar{x}|^2/(4\sigma)} \right]
\]

where \( \bar{x} = (x_1, -x_2) \).

(c) Express the solution \( u(x, t) \) in terms of the Green’s function you found.

Doing the "usual" integration by parts (as in question 1) we see that

\[
u(x, t) = \int_V G_x(y, t)u_0(y)dy - \int_0^t \int_{\partial V} g \left( \frac{\partial}{\partial n(y)} \right) G_x(y, t-\tau) \, dl(y) \, d\tau.
\]

The normal derivative of \( G \) on the boundary \( y_2 = 0 \) is

\[
G_n|_{\partial V} = -G_y|_{y_2=0} = -(4\pi \sigma)^{-1} e^{-((y_1-x_1)^2+x_2^2)/(4\sigma)} \frac{x_2}{\sigma},
\]

and so the solution of our problem is

\[
u(x, t) = (4\pi t)^{-1} \int_V \left[ e^{-|y-x|^2/(4t)} - e^{-|y-\bar{x}|^2/(4t)} \right] u_0(y)dy
\]

\[
+ \frac{x_2}{4\pi} \int_0^t (t-\tau)^{-2} \int_{-\infty}^{\infty} e^{-((y_1-x_1)^2+x_2^2)/(4(t-\tau))} g(y_1)dy_1 d\tau.
\]

If we like, in the second term we can integrate in \( \tau \) to get

\[
- \frac{x_2}{4\pi} \int_{-\infty}^{\infty} \frac{4}{(y_1-x_1)^2 + x_2^2} e^{-((y_1-x_1)^2+x_2^2)/(4(t-\tau))} \bigg|_{\tau=0}^{\tau=t} g(y_1)dy_1
\]

\[
= \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{e^{-((y_1-x_1)^2+x_2^2)/(4t)}}{(y_1-x_1)^2 + x_2^2} g(y_1)dy_1,
\]

so our solution is

\[
u(x, t) = (4\pi t)^{-1} \int_V \left[ e^{-|y-x|^2/(4t)} - e^{-|y-\bar{x}|^2/(4t)} \right] u_0(y)dy + \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{e^{-((y_1-x_1)^2+x_2^2)/4t}}{(y_1-x_1)^2 + x_2^2} g(y_1)dy_1.
\]
(d) What can you say about the steady-state \( \lim_{t \to \infty} u(x, t) \)?

As \( t \to \infty \), clearly the first term above tends to zero (the effect of the initial data disappears). The second term tends to

\[
\lim_{t \to \infty} u(x, t) = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{g(y_1)}{(y_1 - x_1)^2 + x_2^2} dy_1
\]

which is (recall) the harmonic function in \( V \) with boundary values \( g(x_1) \). In other words, as \( t \to \infty \), the solution of our heat equation problem tends toward the solution of the corresponding time-independent equation (\( \Delta u = 0 \)) with the appropriate boundary values.

3. The temperature \( u(x, t) \) of a (bounded) body \( V \subset \mathbb{R}^n \) subject to a distributed heat source \( f(x, t) \), with boundary temperatures \( g(x, t) \) given, and with initial temperature distribution \( u_0(x) \), solves

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + f(x, t) & x \in V, \ t > 0, \\
u(x, t) = g(x), & x \in \partial V \\
u(x, 0) = u_0(x), & x \in V
\end{cases}
\]

Use the maximum principle (for the heat equation) to show that if \( u_0(x) \geq 0 \) (and not identically zero), and for \( 0 \leq t \leq T \), \( f(x, t) \geq 0 \) and \( g(x) \geq 0 \), then \( u(x, t) > 0 \) for \( 0 < t \leq T \).

Since \( u_t - \Delta u = f \geq 0 \), \( u \) is a supersolution of the heat equation, and so obeys the (parabolic) minimum principle: the minimum value of \( u \) for \( 0 \leq t \leq T \) is attained either at the initial time \( t = 0 \) (where \( u = u_0 \geq 0 \)) or else on the spatial boundary \( x \in \partial V \) (where \( u = g \geq 0 \)). Hence \( u(x, t) \geq 0 \) for \( (x, t) \in V \times [0, T] \). Moreover, if \( u(x_0, t_0) = 0 \) for some \( x_0 \in V \), \( t_0 \in (0, T] \), we must have \( u \equiv C \geq 0 \) (constant) in \( V \times (0, T] \). \( C = 0 \) would imply \( u_0(x) \equiv 0 \), contradicting the assumption. Thus we conclude \( u(x, t) > 0 \) in \( V \times (0, T] \).

4. The temperature \( u(x, t) \) of a (bounded) body \( V \subset \mathbb{R}^n \) with insulated boundary, and with initial temperature distribution \( u_0(x) \), solves

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u & x \in V, \ t > 0, \\
\frac{\partial u}{\partial n} = 0 & x \in \partial V \\
u(x, 0) = u_0(x), & x \in V
\end{cases}
\]

(a) Prove uniqueness for this problem – that is, show that there is at most one solution \( u(x, t) \). (Hint: the maximum principle doesn’t help here, because of the Neumann BCs. Instead, compute \( \frac{d}{dt} \int_V u^2(x, t)dx \) and use the divergence theorem.)

If \( u_1(x, t) \) and \( u_2(x, t) \) are 2 solutions, their difference \( w(x, t) = u_1 - u_2 \) solve the same problem, but with zero initial condition: \( w(x, 0) \equiv 0 \). Then, using the divergence theorem (in a by now quite familiar way)

\[
\frac{d}{dt} \int_V w^2(x, t)dx = 2 \int_V w \dot{w} dx = 2 \int_V w \Delta w \ dt = 2 \int_V (\nabla \cdot (w \nabla w) - |\nabla w|^2) \ dt = 2 \int_{\partial V} w \frac{\partial w}{\partial n} dS - 2 \int_V |\nabla w|^2 dx = -2 \int_V |\nabla w|^2 dx \leq 0.
\]
Therefore for any \( t > 0 \),
\[
\int_V w^2(x,t) \, dx \leq \int_V w^2(x,0) \, dx = 0,
\]
so \( w(x,t) \equiv 0 \). That is, \( u_1(x,t) \equiv u_2(x,t) \).

(b) The steady-states for this problem – i.e. harmonic functions with Neumann BCs – are just constants. Assuming the solution \( u(x,t) \) converges to a constant as \( t \to \infty \), find that constant. (Hint: consider \( \frac{d}{dt} \int_V u(x,t) \, dx \).

Suppose \( \lim_{t \to 0} u(x,t) = C \). Notice
\[
\frac{d}{dt} \int_V u(x,t) \, dx = \int_V u_t \, dx = \int_V \Delta u \, dx = \int_{\partial V} \frac{\partial u}{\partial n} \, dS = 0,
\]
so for any \( t > 0 \), \( \int_V u(x,t) \, dx = \int_V u(x,0) \, dx = \int_V u_0(x) \, dx \), and hence
\[
C|V| = \int_V C \, dx = \lim_{t \to \infty} \int_V u(x,t) \, dx = \int_V u_0(x) \, dx,
\]
so
\[
C = \frac{1}{|V|} \int_V u_0(x) \, dx = \text{average of } u_0.
\]

5. Consider the heat equation in the unit square \( V = [0,1] \times [0,1] \subset \mathbb{R}^2 \):
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u & x \in V, \quad t > 0, \\
\quad u = 0 & x \in \partial V \\
\quad u(x,0) = u_0(x) & x \in V
\end{cases}
\]

(a) Find the Green’s function for this problem as an eigenfunction expansion

The (unnormalized) eigenfunctions of \( \Delta \) in \( V = [0,1]^2 \) with Dirichlet BCs are
\[
\phi_{j,k}(x) = \sin(j \pi x_1) \sin(k \pi x_2), \quad j, k = 1, 2, 3, \ldots,
\]
so expand the Greens function \( G_x(y, \sigma) \) (\( \sigma = t - \tau \)) for the heat equation in \( V \) (with Dirichlet BCs) as
\[
G_x(y, \sigma) = \sum_{j,k=1}^{\infty} c_{j,k}(\sigma ; x) \phi_{j,k}(y).
\]

We need
\[
\delta_x(y) \delta_0(\sigma) = \left( \frac{\partial}{\partial \sigma} - \Delta_y \right) G = \sum_{j,k=1}^{\infty} ((c_{j,k})_{\sigma} + \pi^2 (j^2 + k^2) c_{j,k}) \phi_{j,k}(y).
\]
Multiplying both sides by \( \phi_{l,m}(y) \), integrating in \( y \), and using \( (\phi_{j,k}, \phi_{l,m}) = \frac{1}{4} \delta_{j,l} \delta{k,m} \), we find
\[
4 \phi_{l,m}(x) \delta_0(\sigma) = (c_{l,m})_{\sigma} + \pi^2 (l^2 + m^2) c_{l,m}.
\]
For $\sigma > 0$, we have $(c_{l,m})_\sigma = -\pi^2(l^2 + m^2) c_{l,m}$, so

$$c_{l,m}(\sigma;x) = A_{l,m}(x)e^{-\pi^2(l^2 + m^2)\sigma}.$$ 

Integrating the full ODE from 0− to 0+ in $\sigma$, and using $c_{l,m} = 0$ for $\sigma < 0$ (causality) gives

$$4\phi_{l,m}(x) = c_{l,m}|_{\sigma=0^+} = A_{l,m}(x).$$

So

$$G_x(y, \sigma) = 4 \sum_{j,k=1}^{\infty} e^{-\pi^2(j^2+k^2)\sigma} \sin(j\pi x_1) \sin(k\pi x_2) \sin(j \pi y_1) \sin(k \pi y_2).$$

(b) Write the solution $u(x, t)$ in terms of this Green’s function.

The representation formula (derived in the notes, or you can easily do it yourself) is (interchanging order of summation and integration just for fun)

$$u(x, t) = \int_V G_x(y, t) u_0(y) dy$$

$$= 4 \sum_{j,k=1}^{\infty} e^{-\pi^2(j^2+k^2)t} \sin(j\pi x_1) \sin(k\pi x_2) \int_0^1 \int_0^1 \sin(j \pi y_1) \sin(k \pi y_2) u_0(y_1, y_2) dy_1 dy_2.$$ 

(c) In this infinite series solution, what is the leading (i.e. largest) term for long times (i.e. large $t$)?

The term with the slowest exponential decay is $j = k = 1$, and so for $t \gg 1$,

$$u(x, t) \approx 4e^{-2\pi^2 t} \sin(\pi x_1) \sin(\pi x_2) \int_0^1 \int_0^1 \sin(\pi y_1) \sin(\pi y_2) u_0(y_1, y_2) dy_1 dy_2.$$