1. Consider the heat-equation on the half-line with insulating boundary condition at the endpoint:
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} & x > 0, \ t > 0 \\
\frac{\partial u}{\partial x}(0, t) &= 0 \\
u(x, 0) &= u_0(x), \ x > 0
\end{align*}
\]
(a) Write down the problem that the corresponding (Neumann) Green's function \(G_x(y, \sigma) \ (\sigma = t - \tau)\) should solve.
(b) Use the method of images to find this Green’s function.
(c) Express the solution \(u(x, t)\) of the above heat equation problem using this Green’s function.
(d) Now suppose that heat is being pumped in at the endpoint at a fixed rate, so that the boundary condition becomes \(u_x(0, t) = a\) (a a constant). Write the solution \(u(x, t)\) in this case, and find an explicit formula for the temperature at the endpoint, \(u(0, t)\).

2. Consider the following initial-boundary value problem for the heat equation in the 1/2-plane \(V := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}\):
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u & x \in V, \ t > 0 \\
u(x_2 = 0) &= g(x_1), \\
u(x, 0) &= u_0(x) & x \in V
\end{align*}
\]
where \(g\) and \(u_0\) are continuous functions which decay rapidly at infinity.
(a) Write the problem solved by the corresponding (Dirichlet) Green’s function \(G_x(y, \sigma) \ (\sigma = t - \tau)\).
(b) Use the method of images to find that Green’s function.
(c) Express the solution \(u(x, t)\) in terms of the Green’s function you found.
(d) What can you say about the steady-state \(\lim_{t \to \infty} u(x, t)\)?

3. The temperature \(u(x, t)\) of a (bounded) body \(V \subset \mathbb{R}^n\) subject to a distributed heat source \(f(x, t)\), with boundary temperatures \(g(x, t)\) given, and with initial temperature distribution \(u_0(x)\), solves
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + f(x, t) & x \in V, \ t > 0, \\
u(x, t) &= g(x), & x \in \partial V \\
u(x, 0) &= u_0(x), & x \in V
\end{align*}
\]
Use the maximum principle (for the heat equation) to show that if \(u_0(x) \geq 0\) (and not identically zero), and for \(0 \leq t \leq T\), \(f(x, t) \geq 0\) and \(g(x) \geq 0\), then \(u(x, t) > 0\) for \(0 < t \leq T\).
4. The temperature \( u(x, t) \) of a (bounded) body \( V \subset \mathbb{R}^n \) with insulated boundary, and with initial temperature distribution \( u_0(x) \), solves

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u & x \in V, \ t > 0, \\
\frac{\partial u}{\partial n} &= 0 & x \in \partial V \\
u(x, 0) &= u_0(x) & x \in V
\end{align*}
\]

(a) Prove uniqueness for this problem – that is, show that there is at most one solution \( u(x, t) \). (Hint: the maximum principle doesn’t help here, because of the Neumann BCs. Instead, compute \( \frac{d}{dt} \int_V u^2(x, t) dx \) and use the divergence theorem. Actually, do it for \( w = u_1 - u_2 \) assuming \( u_1 \) and \( u_2 \) are 2 solutions.)

(b) The steady-states for this problem – i.e. harmonic functions with Neumann BCs – are just constants. Assuming the solution \( u(x, t) \) converges to a constant as \( t \to \infty \), find that constant. (Hint: consider \( \frac{d}{dt} \int_V u(x, t) dx \).)

5. Consider the heat equation in the unit square \( V = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u & x \in V, \ t > 0, \\
u &= 0 & x \in \partial V \\
u(x, 0) &= u_0(x) & x \in V
\end{align*}
\]

(a) Find the Green’s function for this problem as an eigenfunction expansion

(b) Write the solution \( u(x, t) \) in terms of this Green’s function.

(c) In this infinite series solution, what is the leading (i.e. largest) term for long times (i.e. large \( t \))?