Math 401: Assignment 4 (Due Wed., March 2nd at the start of class)

4. Find the Dirichlet Green’s function for the square \([0, \pi] \times [0, \pi]\) (as an eigenfunction expansion), and use it to express the solution of the boundary value problem

\[
\begin{align*}
\Delta u &= 0 & 0 < x_1 < \pi, \ 0 < x_2 < \pi \\
u &= 0 & x_1 = 0, \ x_1 = \pi, \ x_2 = \pi \\
u &= g(x_1) & x_2 = 0
\end{align*}
\]

2. Find the Dirichlet Green’s function (as an eigenfunction expansion) for the semi-infinite strip \([0, 1] \times [0, \infty)\), and use it to express the solution of the problem

\[
\begin{align*}
\Delta u &= 0 & 0 < x_1 < 1, \ 0 < x_2 < \infty \\
u &= g(x_1) & x_2 = 0 \\
u &\to 0 & x_2 \to \infty \\
u &= 0 & x_1 = 0, \ x_1 = 1
\end{align*}
\]

3. A smooth function \(u\) is called subharmonic (respectively superharmonic) if \(\Delta u \geq 0\) (respectively \(\Delta u \leq 0\)).

   (a) Show that if \(u\) is subharmonic (say, in \(\mathbb{R}^3\), for simplicity – though the \(\mathbb{R}^n\) case is essentially the same) then

   \[u(x) \leq \text{the average of } u \text{ over any sphere centred at } x.\]

   Hint: let \(A(r)\) be the average of \(u\) over a sphere of radius \(r\) centred at \(x\). Compute \(dA/dr\) by first making a change of variables like \(y = x + rz\) in the integral – then apply the divergence theorem so that \(\Delta u\) appears.

   Remarks:
   - a similar proof can be used to establish the mean-value property of harmonic functions (without using Poisson’s formula)
   - of course an identical statement (with signs reversed) holds for superharmonic functions

   (b) Use part (a) to show that the maximum principle holds for subharmonic functions: if \(u\) is subharmonic on a (smooth, bounded, connected) region \(D\) (and continuous on \(\overline{D}\)), then if the maximum of \(u\) is attained at an interior point of \(D\) (i.e. not on \(\partial D\)), \(u\) must be constant.

   Remark: a superharmonic function obeys a corresponding “minimum principle”.
4. Let $D$ be a smooth, open, bounded, connected region in $\mathbb{R}^n$ ($n = 2$ or $3$ for concreteness). Use the maximum principle (for sub/superharmonic functions, as above) to:

(a) establish the following "comparison principle": if functions $u_1(x)$ and $u_2(x)$ satisfy

\[
\begin{align*}
\begin{cases}
\Delta u_1 = f_1(x) & \text{in } D \\
u_1 = g_1(x) & \text{on } \partial D
\end{cases},
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\Delta u_2 = f_2(x) & \text{in } D \\
u_2 = g_2(x) & \text{on } \partial D
\end{cases},
\end{align*}
\]

\[f_1(x) \leq f_2(x), \text{ and } g_1(x) \geq g_2(x),\]

then $u_1(x) \geq u_2(x)$.

(b) prove that the Dirichlet Green’s function for $\Delta$ on $D$ is non-positive: $G_x(y) \leq 0$ for $y \in D$. (Consider $u=0$ on $\partial D$. You can do the superharmonic case only if you want)