Math 401: Assignment 3  (Due Fri., Feb. 5th)

1. Find the free-space Green’s function for the operator \(-\Delta + k^2\) \((k > 0\) a constant\) in \(\mathbb{R}^3\). That is, solve
   \[(\Delta + k^2)G_x = \delta_x\]
   in \(\mathbb{R}^3\), with \(G_x(y) \to 0\) as \(|y| \to \infty\). (Hint: look for \(G_x(y)\) in the form \(G_x(y) = g(r)/r\), \(r := |y - x|\).)

2. Let \(D_a\) be the disk of radius \(a\) in the plane, centred at the origin. Let \(C_a\) be its boundary (circle of radius \(a\)).
   
   (a) Use the method of images to find the Dirichlet (\(G = 0\) on the boundary) Green’s function for \(\Delta\) on \(D_a\).
   
   (b) For a given function \(f(x)\) on \(D_a\), solve \(\Delta u = f(x)\) in \(D_a\), \(u = 0\) on \(C_a\).

   (c) For a given function \(g(\theta)\) on \(C_a\) \(0 \leq \theta \leq 2\pi\) denoting the angle around \(C_a\), solve \(\Delta u = 0\) in \(D_a\), \(u = g(\theta)\) on \(C_a\).

3. Let \(D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}\) be the first quadrant in the plane. Use the method of images to solve the problem \(\Delta u = 0\) in \(D\), \(\frac{\partial}{\partial x_2} u(x_1, 0) = f(x_1), x_1 > 0\), and \(\frac{\partial}{\partial x_1} u(0, x_2) = g(x_2), x_2 > 0\) by finding the Neumann Green’s function \((\frac{\partial}{\partial n} G = 0\) on the boundary).