Math 401: Assignment 2 (Due Wed, Jan27)

1. Let \( L := a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x) \).
   
   (a) Show that \( L = L^* \) if and only if \( a_0' = a_1 \).
   
   (b) Under what condition on the numbers \( \alpha \) and \( \beta \) is the problem
       \[(p(x)u')' + q(x)u = f(x), \quad 0 < x < 1, \quad u(0) = \alpha u(1), \quad u'(0) = \beta u'(1)\]
       self-adjoint?

2. The steady-state temperature along the rod \( 0 \leq x \leq 1 \) is \( u(x) \). The thermal conductivity of the rod is \( e^x \). There is a heat source \( f(x) \). The left end of the rod is insulated, while the right end is held at temperature 1. Find \( u(x) \) - i.e., solve
       \[(e^x u')' = f(x), \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 1,\]
       by finding the Green’s function, \( G \), for the problem and expressing \( u \) in terms of \( G \) and \( f \).

3. Consider the problem
   \[ u'' + u = f(x), \quad 0 < x < L, \]
   \[ u(0) = 0, \quad u(L) = 0. \]
   
   (a) Find the Green’s function, and express the solution in terms of it.
   
   (b) Find the values of \( L \) for which this solution breaks down, and for these values, determine the solvability condition on \( f \), calculate the modified Green’s function, and (assuming the solvability condition is satisfied) find an integral representation for the solution.

4. Consider the equation
   \[ -u'' + q^2 u = f(x) \]
   \((q > 0 \text{ a constant})\). Determine the Green’s function for this problem on the line \(-\infty < x < \infty \) associated with the “boundary conditions” \( u(x) \to 0 \) as \( |x| \to \infty \).

5. Find a solvability condition for the problem
   \[ u'' + q^2 u = f(x), \quad 0 < x < 1, \]
   \[ u(0) = u(1), \quad u'(0) = -u'(1) \]
   \((q > 0 \text{ a constant})\).

(Jan. 22)