First Name: ________________________ Last Name: ________________________

Student-No: ________________________ Section: ________________________

Grade:

The remainder of this page has been left blank for your workings.
Riemann Sum and FTC

1. **8 marks** Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2 e^{\frac{i^3}{n^3}+2}}{n^3}$$

by first writing it as a definite integral and then evaluating it.

**Answer:** $e^3 - e^2$

**Solution:** We identify $a = 0$, $b = 1$, $\Delta(x) = \frac{1}{n}$, $x_i = \frac{i}{n}$, and

$$f(x_i) = 3x_i^2 \exp(x_i^3 + 2).$$

This yields,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2 \exp\left(\frac{i^3}{n^3} + 2\right)}{n^3} = \int_{0}^{1} 3x^2 \exp(x^3 + 2) \, dx.$$

To calculate the integral, let $u = x^3 + 2$. Then $du = 3x^2 \, dx$, $u(0) = 2$, and $u(1) = 3$. Then

$$\int_{0}^{1} 3x^2 \exp(x^3 + 2) \, dx = \int_{2}^{3} \exp(u) \, du = [\exp(u)]_{2}^{3} = e^3 - e^2.$$

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_{0}^{x} (2t - 1) e^t \, dt$ and $g(x) = x^2 F(x)$. Calculate $g'(1/2)$.

**Answer:** $3 - 2e^{1/2}$

**Solution:** We use the product rule to get: $g'(x) = 2xF(x) + x^2 F'(x)$. From FTC I, we get:

$$F'(x) = (2x - 1)e^x$$

and we calculate $F(x)$ using integration by parts with $u = 2t - 1$, $v' = e^t$, such that:

$$F(x) = \int_{0}^{x} (2t - 1) e^t \, dt = [(2t - 1)e^t]_{0}^{x} - \int_{0}^{x} 2e^t \, dt = [(2t - 3)e^t]_{0}^{x} = (2x - 3)e^x + 3$$

Bringing pieces together, we can write:

$$g'(x) = 2x((2x - 3)e^x + 3) + x^2(2x - 1)e^x$$

Midterm F: Page 2 of 10
Taking $x = 1/2$, we finally get:

$$g'(1/2) = 2^{1/2} \left( \left( \frac{1}{2} - 3 \right) e^{1/2} + 3 \right) = 3 - 2e^{1/2}$$
Indefinite Integrals

2. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int (x + 2)(x - 7)^4 \, dx$.

Answer: $\frac{1}{6}(x - 7)^6 + \frac{9}{5}(x - 7)^5 + C$

Solution: Using the substitution $u = x - 7$, $u' = 1$ and writing $x = u + 7$, we get:

$$\int (x + 2)(x - 7)^4 \, dx = \int (u + 9)u^4 \, du = \int (u^5 + 9u^4) \, du = \frac{1}{6}u^6 + \frac{9}{5}u^5 + C$$

Substituting back $u = x - 2$, we get:

$$\int (x + 2)(x - 7)^4 \, dx = \frac{1}{6}(x - 7)^6 + \frac{9}{5}(x - 7)^5 + C$$

(b) Calculate the indefinite integral $\int (8 + 2 \sin \theta)^{\frac{3}{2}} \cos \theta \, d\theta$.

Answer: $\frac{1}{5}(8 + 2 \sin \theta)^{\frac{5}{2}} + C$

Solution: By substitution, with

$$u(\theta) = 8 + 2 \sin \theta$$

$$u'(\theta) = 2 \cos \theta$$

Then

$$\int (8 + 2 \sin \theta)^{\frac{3}{2}} \cos \theta \, d\theta = \int \frac{1}{2}u^{\frac{3}{2}} \, du$$

so that

$$\frac{1}{2^{\frac{5}{2}}} (8 + 2 \sin \theta)^{\frac{5}{2}} + C$$
(c) (A Little Harder): Calculate the indefinite integral $\int e^{-2x} \sin x \, dx$.

| Answer: $-\frac{1}{5} e^{-2x} \cos x - \frac{2}{5} e^{-2x} \sin x + C$ |

**Solution:** We integrate by parts once using $u = e^{-2x}$ and $dv/dx = \sin x$. Then, $du/dx = -2e^{-2x}$ and $v = -\cos x$, and we get

$$I = -e^{-2x} \cos x - 2 \int e^{-2x} \cos x \, dx$$

We then integrate by parts one more time using $u = e^{-2x}$ and $dv/dx = \cos x$. Then, $du/dx = -2e^{-2x}$ and $v = \sin x$. We get

$$I = -e^{-2x} \cos x - 2 \left[ e^{-2x} \sin x + 2 \int e^{-2x} \sin x \, dx \right] = -e^{-2x} \cos x - 2e^{-2x} \sin x - 4I$$

Solving for $I$, we get $5I = -e^{-2x} \cos x - 2e^{-2x} \sin x$, which upon dividing by five gives the final result.
Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate \( \int_0^{\pi/4} \sec^4(x) \tan^3(x) \, dx \).

Answer: \( \frac{5}{12} \)

**Solution:** This is a trigonometric integral that is calculated as:

\[
I = \int_0^{\pi/4} \sec^4(x) \tan^3(x) \, dx = \int_0^{\pi/4} \sec^3(x) \tan(x) \sec^2(x) \tan^2(x) \sec(x) \tan(x) \, dx
\]

\[
= \int_0^{\pi/4} \sec^3(x)(\sec^2(x) - 1) \sec(x) \tan(x) \, dx
\]

\[
= \int_0^{\pi/4} (\sec^5(x) - \sec^3(x)) \sec(x) \tan(x) \, dx
\]

which gives, upon substituting \( u = \sec(x) \) and \( du = \sec(x) \tan(x) \, dx \):

\[
I = \left[ \frac{1}{6} \sec^6(x) - \frac{1}{4} \sec^4(x) \right]_0^{\pi/4} = \frac{1}{6} \left( \left( \frac{2}{\sqrt{2}} \right)^6 - 1 \right) + \frac{1}{4} \left( \left( \frac{2}{\sqrt{2}} \right)^4 - 1 \right)
\]

\[
= \frac{7}{6} - \frac{3}{4} = \frac{5}{12}
\]

(b) Calculate \( \int_0^1 \frac{7x^2}{5x^2 + 5} \, dx \).

Answer: \( \frac{7}{5} \left( 1 - \frac{\pi}{4} \right) \)

**Solution:** We first rewrite the definite integral as

\[
I = \int_0^1 \frac{7x^2}{5x^2 + 5} \, dx = \frac{7}{5} \int_0^1 \frac{x^2}{x^2 + 1} \, dx = \frac{7}{5} \int_0^1 \frac{x^2 + 1 - 1}{x^2 + 1} \, dx
\]

\[
= \frac{7}{5} \int_0^1 \left( 1 - \frac{1}{x^2 + 1} \right) \, dx
\]

In this form, the integrand is very easy to anti-differentiate and we finally get:

\[
I = \frac{7}{5} \left[ x - \arctan(x) \right]_0^1 = \frac{7}{5} \left( 1 - \frac{\pi}{4} \right)
\]
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions**, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.

4. (a) **2 marks** Sketch by hand the finite area enclosed between the curves defined by the functions $y^2 + x = 1$ and $x = y - 1$

**Answer:**

**Solution:** The area is the region enclosed between the red and blue curves:

(b) **4 marks** Write the definite integral with specific limits of integration that determines this finite area.

**Answer:** $\int_{-2}^{1} (2 - y - y^2) \, dy$

**Solution:** We first find the intersection points of the two curves, given by the solution of:

$$1 - y^2 = y - 1 \Leftrightarrow (y - 1)(y + 2) = 0.$$ 

The intersection points are therefore $(-3, -2)$ and $(0, 1)$. We then label the curve $x_B = 1 - y^2$ and $x_R = y - 1$ and notice that $x_B \geq x_R$ for $-2 \leq y \leq 1$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{1} (1 - y^2 - y + 1) \, dy = \int_{-2}^{1} (2 - y - y^2) \, dy$$
(c) 2 marks Evaluate the integral.

Answer: \( \frac{9}{2} = 4.5 \)

Solution:

\[
A = \int_{-2}^{1} (2 - y - y^2) \, dy = \left[ 2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1} = 6 + \frac{3}{2} - 3 = \frac{9}{2}
\]
5. **4 marks** Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between \( x = \frac{(y+1)^2}{25} \) and \( x = y - 3 \) about the horizontal line \( y = -2 \). **Do not evaluate the integral.**

\[
\text{Answer: } \pi \int_1^{16} (5\sqrt{x} + 1)^2 - (x + 5)^2 \, dx
\]

**Solution:** Intersection points are given by \( \frac{(y+1)^2}{25} = y - 3 \).

Solving for \( y \), we determine the 2 intersection points \( I_1 = (1, 4) \), \( I_2 = (16, 19) \).

We integrate in \( x \), hence we write \( y \) as a function of \( x \) for the 2 curves and apply a shift of +2, we finally establish:

\[
\pi \int_1^{16} (5\sqrt{x} + 1)^2 - (x + 5)^2 \, dx.
\]
6. A tank of height $H$ and of square cross section of edge length $L$ is half full with water of density $\rho = 1000 \text{kg/m}^3$. The top of the tank features a spout of height $h$. We take the vertical axis $y$ upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10 \text{m/s}^2$.

We take $H = 8 \text{m}$, $L = 2 \text{m}$ and $h = 3 \text{m}$.

(a) **2 marks** Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $4 \cdot 10^4 \int_0^4 (11 - y) \, dy$

**Solution:** The cross section of the tank as a function of $y$ is constant and equal to $L^2$. So the elementary volume, mass and force of a slice of height $\Delta y$ read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g \rho L^2 \Delta y$$

The displacement of a slice of height $\Delta y$ at position $y$ is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g \rho L^2 (H + h - y) \Delta y = g \rho L^2 (11 - y) \Delta y$$
Now we integrate from bottom $y = 0$ to half height $H/2 = 8/2 = 4$ as

$$W = \int_0^4 g\rho L^2 (11 - y) \, dy = 4 \cdot 10^4 \int_0^4 (11 - y) \, dy$$

(b) \text{2 marks} Evaluate the definite integral.

\begin{align*}
\text{Answer:} & \quad 1.44 \cdot 10^6 J \\
\text{Solution:} & \\
W & = 4 \cdot 10^4 \int_0^4 (11 - y) \, dy = 4 \cdot 10^4 \left[11y - \frac{y^2}{2}\right]_0^4 \\
& = 4 \cdot 10^4 \left(11 \cdot 4 - \frac{4^2}{2}\right) \\
& = 4 \cdot 10^4 \cdot 36 = 1.44 \cdot 10^6 J
\end{align*}