First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Grade:

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Riemann Sum and FTC

1. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2\cos(\frac{i^3}{n^3}+2)}{n^3}$$

by first writing it as a definite integral and then evaluating it.

Answer: $\sin(3)-\sin(2)$

Solution: We identify $a=0$, $b=1$, $\Delta(x)=\frac{1}{n}$, $x_i=\frac{i}{n}$, and

$$f(x_i)=3x_i^2\cos(x_i^3+2).$$

This yields,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2\cos(\frac{i^3}{n^3}+2)}{n^3}=\int_0^1 3x^2\cos(x^3+2)dx.$$

To calculate the integral, let $u=x^3+2$. Then $du=3x^2dx$, $u(0)=2$, and $u(1)=3$. Then

$$\int_0^1 3x^2\cos(x^3+2)dx=\int_2^3\cos(u)du=[\sin(u)]_2^3=\sin(3)-\sin(2).$$

(b) Define $F(x)$ and $g(x)$ by $F(x)=\displaystyle\int_0^x\frac{1}{2t^2+2}\,dt$ and $g(x)=x^2F(x)$. Calculate $g'(1)$.

Answer: $\frac{1}{4}(\pi+1)$

Solution: We first write:

$$g'(x)=2xF(x)+x^2F'(x)=2x\int_0^x\frac{1}{2t^2+2}\,dt+\frac{x^2}{2x^2+2}$$

Then we calculate the first term on the rhs to get:

$$g'(x)=2x\frac{1}{2}\int_0^x\frac{1}{t^2+1}\,dt+\frac{x^2}{2(x^2+1)}$$

and using the fact that $\arctan(0)=0$, finally:

$$g'(x)=x\arctan x+\frac{x^2}{2(x^2+1)}$$

Taking $x = 1$, we get:

$$g'(1) = 1 \cdot \arctan 1 + \frac{1^2}{2(1^2+1)} = \frac{1}{4}(\pi + 1)$$

Indefinite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int \frac{4x}{\sqrt{2x-1}}\,dx$.

Answer: $\frac{4}{3}(x+1)(2x-1)^{1/2}+C$

Solution: Using the substitution $u = 2x-1$, $u' = 2$ and writing $2x = u+1$, we get:

$$\int \frac{4x}{\sqrt{2x-1}}\,dx = \int 2\frac{2x}{\sqrt{2x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$
$$= u^{1/2}\left(\frac{2}{3}u + 2\right) + C$$

Substituting back $u = 2x-1$, we get:

$$\int \frac{4x}{\sqrt{2x-1}}\,dx = (2x-1)^{1/2}\left(\frac{2}{3}(2x-1)+2\right) + C = \frac{4}{3}(x+1)(2x-1)^{1/2} + C$$

(b) Calculate the indefinite integral $\int (6+8\sin\theta)^{\frac{5}{2}}\cos\theta\,d\theta$.

Answer: $\frac{1}{28}(6+8\sin\theta)^{\frac{7}{2}}+C$

Solution: By substitution, with

$$u(\theta) = 6+8\sin\theta$$
$$u'(\theta) = 8\cos\theta$$

Then

$$\int (6+8\sin\theta)^{\frac{5}{2}}\cos\theta\,d\theta = \int \frac{1}{8}u^{\frac{5}{2}}\,du$$

so that

$$\frac{1}{8}\frac{2}{7}(6+8\sin\theta)^{\frac{7}{2}} + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int x^3 \sin(x^2)\,dx$.

Answer: $-\frac{1}{2}x^2\cos(x^2) + \frac{1}{2}\sin(x^2) + C$

Solution: First set $s = x^2$ so that $ds/dx = 2x$. Therefore, x^3dx is replaced by $\frac{1}{2}sds$. This yields

$$I = \int x^3 \sin(x^2)\,dx = \frac{1}{2}\int s\sin s\,ds\,.$$

Now do one step of integration by parts. Let $u = s$ and $dv/ds = \sin s$ so that $du/ds = 1$ and $v = -\cos s$. We get

$$I = \frac{1}{2}\left[-s\cos s + \int \cos s\,ds\right]\,.$$

Performing the final integration, adding the constant, and replacing $s = x^2$ gives the result

$$I = -\frac{1}{2}x^2\cos(x^2) + \frac{1}{2}\sin(x^2) + C\,.$$

Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{-\pi}^{\pi} (\sin x + x^2)\sin(x)\,dx$.

Answer: π

Solution: Upon splitting the integral, the second integral vanishes because the function is odd and domain is symmetric, and we only need to compute

$$I = \int_{-\pi}^{\pi} (\sin x + x^2)\sin(x)\,dx = \int_{-\pi}^{\pi} \sin^2(x)\,dx$$

First we recognize that the integrand is even and we use the trig identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we get

$$I = 2\int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx = \int_0^{\pi} (1-\cos(2x))\,dx = \left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi} = \pi$$

(b) Calculate $\displaystyle\int_0^1 \arctan(3x)\,dx$.

Answer: $\arctan(3) - \frac{1}{6}\ln(10)$

Solution: We use integration by parts with $u = \arctan(3x)$ and $v' = 1$. We get $u = \frac{3}{1+9x^2}$ and $v = x$. This gives

$$I = \int_0^1 \arctan(3x)\,dx = [x\arctan(3x)]_0^1 - \int_0^1 \frac{3x}{1+9x^2}\,dx$$
$$= \arctan(3) - \frac{1}{6}\int_0^1 \frac{18x}{1+9x^2}\,dx$$

Using the substitution $u = 1 + 9x^2$, $u' = 18x$, we get:

$$I = \arctan(3) - \frac{1}{6}\left[\ln(u)\right]_1^{10} = \arctan(3) - \frac{1}{6}\ln(10)$$

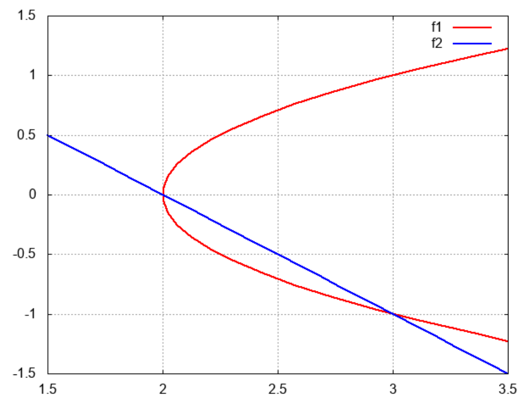
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) 2 marks Sketch by hand the finite area enclosed between the curves defined by the functions $y^2 + 2 = x$ and $y + x = 2$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) 4 marks Write the definite integral with specific limits of integration that determines this finite area.

Answer: $-\int_{-1}^{0} (y + y^2)\ dy$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$y^2 + 2 = 2 - y \Leftrightarrow y\,(y+1) = 0.$$

The intersection points are therefore (2, 0) and (3, −1). We then label the curve $x_R = y^2 + 2$ and $x_B = 2 - y$ and notice that $x_B \geq x_R$ for $-1 \leq y \leq 0$. The area is therefore given by the following definite integral:

$$A = \int_{-1}^{0} \left(2 - y - y^2 - 2\right)\ dy = \int_{-1}^{0} \left(-y - y^2\right)\ dy = -\int_{-1}^{0} \left(y + y^2\right)\ dy$$

(c) 2 marks Evaluate the integral.

Answer: $\frac{1}{6}$

Solution:

$$A = -\int_{-1}^{0} \left(y + y^2\right)\, dy = -\left[\frac{y^2}{2} + \frac{y^3}{3}\right]_{-1}^{0} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

5. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = \frac{(y+1)^2}{16}$ and $x = y - 2$ about the horizontal line $y = 1$. **Do not evaluate the integral.**

Answer: $\pi \int_1^9 (4\sqrt{x} - 2)^2 - (x+1)^2 \, dx$

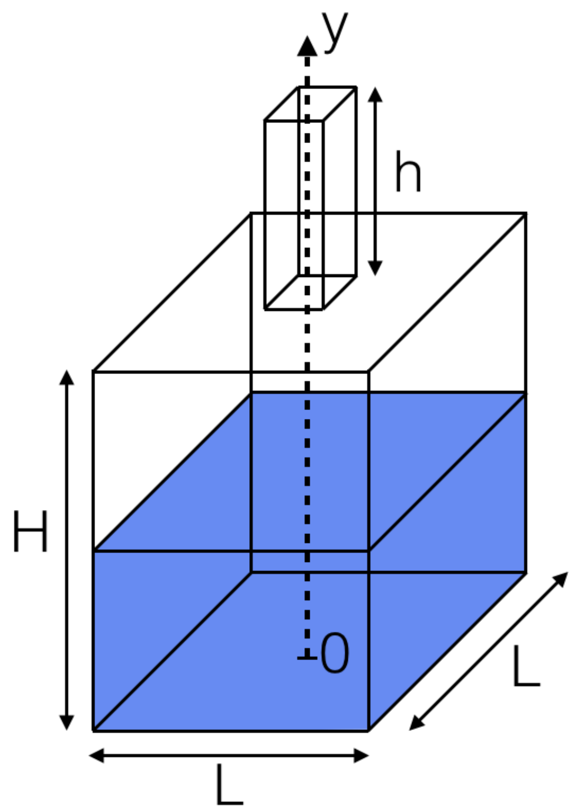
Solution: Intersection points are given by $\frac{(y+1)^2}{16} = y - 2$.
Solving for y, we determine the 2 intersection points

$$I_1 = (1, 3) \quad , \quad I_2 = (9, 11).$$

We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of -1, we finally establish:

$$\pi \int_1^9 (4\sqrt{x} - 2)^2 - (x+1)^2 \, dx.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10m/s^2$.
We take $H = 8m$, $L = 3m$ and $h = 4m$.



(a) 2 marks Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $9 \cdot 10^4 \int_0^4 (12 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (12 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 8/2 = 4$ as

$$W = \int_0^4 g\rho L^2(12-y)\,dy = 9 \cdot 10^4 \int_0^4 (12-y)\,dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $3.6 \cdot 10^6 J$

Solution:

$$W = 9 \cdot 10^4 \int_0^4 (12-y)\,dy = 9 \cdot 10^4 \left[12y - \frac{y^2}{2}\right]_0^4 = 9 \cdot 10^4 \left(12 \cdot 4 - \frac{4^2}{2}\right)$$
$$= 9 \cdot 10^4 \cdot 40 = 3.6 \cdot 10^6 J$$