First Name: ___________________________ Last Name: ___________________________
Student-No: ___________________________ Section: ___________________________

Grade: 

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**Riemann Sum and FTC**

1. **8 marks** Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2}{n^3(\frac{i}{n})^3 + 2}
\]

by first writing it as a definite integral and then evaluating it.

**Answer:** \(\ln(3) - \ln(2)\)

**Solution:** We identify \(a = 0, b = 1, \Delta(x) = \frac{1}{n}, x_i = \frac{i}{n},\) and

\[
f(x_i) = \frac{3x_i^2}{x_i^3 + 2}.
\]

This yields,

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3i^2}{n^3(\frac{i}{n})^3 + 2} = \int_{0}^{1} \frac{3x^2}{x^3 + 2} dx.
\]

To calculate the integral, let \(u = x^3 + 2\). Then \(du = 3x^2 dx\), \(u(0) = 2\), and \(u(1) = 3\). Then

\[
\int_{0}^{1} \frac{3x^2}{x^3 + 2} dx = \int_{2}^{3} \frac{1}{u} du = [\ln(u)]_{2}^{3} = \ln(3) - \ln(2).
\]

(b) Define \(F(x)\) and \(g(x)\) by \(F(x) = \int_{2\pi}^{x} t \sin t \, dt\) and \(g(x) = (x - \sqrt{\pi})F(x^2)\). Calculate \(g'(\sqrt{\pi})\).

**Answer:** \(3\pi\)

**Solution:** We first write:

\[
g'(x) = F(x^2) + (x - \sqrt{\pi})(F(x^2))'
\]

We do not need to evaluate \((F(x^2))'\) as when we take \(x = \sqrt{\pi}\), the second term on the rhs cancels out. So we only need to calculate \(F(x^2)\). By integration by parts with \(u = t\) and \(v' = \sin t\), we get:

\[
\int_{2\pi}^{x} t \sin t \, dt = \left[-t \cos t\right]_{2\pi}^{x} + \int_{2\pi}^{x} \cos t \, dt = \left[-t \cos t + \sin t\right]_{2\pi}^{x}
\]

\[
= -x^2 \cos x^2 + \sin x^2 + 2\pi
\]
Taking $x = \sqrt{\pi}$, we get:

$$g'(\sqrt{\pi}) = -\pi \cdot (-1) + 2\pi = 3\pi$$
Indefinite Integrals

2. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral \( \int 2(x + 3)^3 \sin ((x + 3)^2) \, dx \).

**Answer:** \(- (x + 3)^2 \cos ((x + 3)^2) + \sin ((x + 3)^2) + C\)

**Solution:** We start by substituting \( u = (x + 3)^2 \). This gives:

\[
\int u \sin(u) \, du.
\]

We now do integration by parts and obtain:

\[
\left( u(- \cos(u)) - \int - \cos(u) \, du \right)
\]

which simplifies to

\[
u(- \cos(u)) + \sin(u) + C.
\]

Now resubstitute \( u \) to get the final answer:

\[(x + 3)^2(- \cos ((x + 3)^2)) + \sin ((x + 3)^2) + C.\]

(b) Calculate the indefinite integral \( \int (5 + 2 \sin \theta) \frac{15}{2} \cos \theta \, d\theta \).

**Answer:** \( \frac{1}{17}(5 + 2 \sin \theta)^{17/2} + C \)

**Solution:** By substitution, with

\[
u(\theta) = 5 + 2 \sin \theta
\]

\[
u'(\theta) = 2 \cos \theta
\]

Then

\[
\int (5 + 2 \sin \theta)^{15/2} \cos \theta \, d\theta = \int \frac{1}{2} u^{15/2} \, du
\]

so that

\[
\frac{1}{2} \frac{2}{17}(5 + 2 \sin \theta)^{17/2} + C
\]
(c) (A Little Harder): Calculate the indefinite integral \( \int x^3 e^{x^2} \, dx \).

**Answer:** \( \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C \)

**Solution:** We use the substitution \( s = x^2 \), \( ds/dx = 2x \) so that \( x \, dx \) is replaced by \( 1/2 \, ds \). This gives

\[
I = \int x^3 e^{x^2} \, dx = \frac{1}{2} \int s e^s \, ds.
\]

Now do integration by parts. Set \( u = s \) and \( dv/ds = e^s \) so that \( du/ds = 1 \) and \( v = e^s \). This yields

\[
I = \frac{1}{2} \int s e^s \, ds = \frac{1}{2} \left[ s e^s - \int e^s \, ds \right]
\]

Performing the last integration and setting \( s = x^2 \) we get

\[
I = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C.
\]
Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{0}^{\pi/4} \sec^4(x) \tan(x) \, dx$.

$$\text{Answer: } 3/4$$

**Solution:** This is a trigonometric integral that is calculated as by holding one $\sec^2(x)$ and replacing the other $\sec^2(x)$ by $1 + \tan^2(x)$:

\[
I = \int_{0}^{\pi/4} \sec^4(x) \tan(x) \, dx = \int_{0}^{\pi/4} (1 + \tan^2(x)) \tan(x) \sec^2(x) \, dx \\
= \int_{0}^{\pi/4} (\tan(x) + \tan^3(x)) \sec^2(x) \, dx
\]

which gives, upon substituting $u = \tan(x)$ and $du = \sec^2(x) \, dx$:

\[
I = \int_{0}^{1} (u + u^3) \, du = \left[ \frac{u^2}{2} + \frac{u^4}{4} \right]_{0}^{1} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
\]

(b) Calculate $\int_{0}^{1} \frac{5x^2}{3x^2 + 3} \, dx$.

$$\text{Answer: } \frac{5}{3} \left(1 - \frac{\pi}{4}\right)$$

**Solution:** We first rewrite the definite integral as

\[
I = \int_{0}^{1} \frac{5x^2}{3x^2 + 3} \, dx = \frac{5}{3} \int_{0}^{1} \frac{x^2}{x^2 + 1} \, dx = \frac{5}{3} \int_{0}^{1} \frac{x^2 + 1 - 1}{x^2 + 1} \, dx \\
= \frac{5}{3} \int_{0}^{1} \left(1 - \frac{1}{x^2 + 1}\right) \, dx
\]

In this form, the integrand is very easy to anti-differentiate and we finally get:

\[
I = \frac{5}{3} \left[x - \arctan(x)\right]_{0}^{1} = \frac{5}{3} \left(1 - \frac{\pi}{4}\right)
\]
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out:** (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.

4. (a) [2 marks] Sketch by hand the finite area enclosed between the curves defined by the functions \( y = x^2 + 2 \) and \( y + x = 2 \)

**Answer:**

**Solution:** The area is the region enclosed between the red and blue curves:

(b) [4 marks] Write the definite integral with specific limits of integration that determines this finite area.

**Answer:** \(-\int_{-1}^{0} (x + x^2) \, dx\)

**Solution:** We first find the intersection points of the two curves, given by the solution of:

\[ x^2 + 2 = 2 - x \iff x(x + 1) = 0. \]

The intersection points are therefore \((0, 2)\) and \((-1, 3)\). We then label the curve \(y_R = x^2 + 2\) and \(y_B = 2 - x\) and notice that \(y_B \geq y_R\) for \(-1 \leq x \leq 0\). The area is therefore given by the following definite integral:

\[ A = \int_{-1}^{0} (2 - x - x^2 - 2) \, dx = \int_{-1}^{0} (-x - x^2) \, dx = -\int_{-1}^{0} (x + x^2) \, dx \]
(c) \(2\) marks Evaluate the integral.

Answer: \(\frac{1}{6}\)

Solution:

\[
A = - \int_{-1}^{0} (x + x^2) \, dx = - \left[ \frac{x^2}{2} + \frac{x^3}{3} \right]_{-1}^{0} = - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
\]
5. **4 marks** Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between \( y = 3\sqrt{x} + 1 \) and \( y = x + 3 \) about the vertical line \( x = -2 \). **Do not evaluate the integral.**

\[
\text{Answer: } \pi \int_{4}^{7} (y - 1)^2 - \left(\frac{(y-1)^2}{9} + 2\right)^2 dy
\]

**Solution:** Intersection points are given by \( 3\sqrt{x} + 1 = x + 3 \).
Solving for \( x \), we determine the 2 intersection points

\[ I_1 = (1, 4) \quad , \quad I_2 = (4, 7). \]

We integrate in \( y \), hence we write \( x \) as a function of \( y \) for the 2 curves and apply a shift of +2, we finally establish:

\[
\pi \int_{4}^{7} (y - 1)^2 - \left(\frac{(y-1)^2}{9} + 2\right)^2 dy.
\]
6. A tank of height $H$ and of square cross section of edge length $L$ is half full with water of density $\rho = 1000 \text{kg/m}^3$. The top of the tank features a spout of height $h$. We take the vertical axis $y$ upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10 \text{m/s}^2$.

We take $H = 4 \text{m}$, $L = 10 \text{m}$ and $h = 2 \text{m}$.

(a) [2 marks] Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $10^6 \int_0^2 (6 - y) \, dy$

Solution: The cross section of the tank as a function of $y$ is constant and equal to $L^2$. So the elementary volume, mass and force of a slice of height $\Delta y$ read:

$\Delta V = L^2 \Delta y$
$\Delta M = \rho L^2 \Delta y$
$\Delta F = g \rho L^2 \Delta y$

The displacement of a slice of height $\Delta y$ at position $y$ is $H + h - y$, and the elementary work of that slice is:

$\Delta W = g \rho L^2 (H + h - y) \Delta y = g \rho L^2 (6 - y) \Delta y$
Now we integrate from bottom $y = 0$ to half height $H/2 = 4/2 = 2$ as

$$W = \int_0^2 g \rho L^2 (6 - y) \, dy = 10^6 \int_0^2 (6 - y) \, dy$$

(b) 2 marks Evaluate the definite integral.

**Answer: $10^7 J$**

**Solution:**

$$W = 10^6 \int_0^2 (6 - y) \, dy = 10^6 \left[ 6y - \frac{y^2}{2} \right]_0^2 = 10^6 \left( 6 \cdot 2 - \frac{2^2}{2} \right)$$

$$= 10^6 \cdot 10 = 10^7 J$$