First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Riemann Sum and FTC

1. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3(\frac{i^3}{n^3}+2)}$$

by first writing it as a definite integral and then evaluating it.

Answer: $\ln(3)-\ln(2)$

Solution: We identify $a=0$, $b=1$, $\Delta(x)=\frac{1}{n}$, $x_i=\frac{i}{n}$, and

$$f(x_i)=\frac{3x_i^2}{x_i^3+2}.$$

This yields,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3(\frac{i^3}{n^3}+2)}=\int_0^1\frac{3x^2}{x^3+2}dx.$$

To calculate the integral, let $u=x^3+2$. Then $du=3x^2dx$, $u(0)=2$, and $u(1)=3$. Then

$$\int_0^1\frac{3x^2}{x^3+2}dxdx=\int_2^3\frac{1}{u}du=[\ln(u)]_2^3=\ln(3)-\ln(2).$$

(b) Define $F(x)$ and $g(x)$ by $F(x)=\displaystyle\int_{2\pi}^{x}t\sin t\,dt$ and $g(x)=(x-\sqrt{\pi})F(x^2)$. Calculate $g'(\sqrt{\pi})$.

Answer: 3π

Solution: We first write:

$$g'(x)=F(x^2)+(x-\sqrt{\pi})(F(x^2))'$$

We do not need to evaluate $(F(x^2))'$ as when we take $x=\sqrt{\pi}$, the second term on the rhs cancels out. So we only need to calculate $F(x^2)$. By integration by parts with $u=t$ and $v'=\sin t$, we get:

$$\int_{2\pi}^{x^2}t\sin t\,dt=[-t\cos t]_{2\pi}^{x^2}+\int_{2\pi}^{x^2}\cos t\,dt=[-t\cos t+\sin t]_{2\pi}^{x^2}$$
$$=-x^2\cos x^2+\sin x^2+2\pi$$

Taking $x = \sqrt{\pi}$, we get:

$$g'(\sqrt{\pi}) = -\pi \cdot (-1) + 2\pi = 3\pi$$

Indefinite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int 2(x+3)^3 \sin\left((x+3)^2\right)\, \mathrm{d}x$.

Answer:
$-(x+3)^2\cos((x+3)^2)+\sin((x+3)^2)+C$

Solution: We start by substituting $u=(x+3)^2$. This gives:

$$\int u \sin(u)\,\mathrm{d}u.$$

We now do integration by parts and obtain:

$$\left(u(-\cos(u)) - \int -\cos(u)\,\mathrm{d}u\right)$$

which simplifies to

$$u(-\cos(u)) + \sin(u) + C.$$

Now resubstitute u to get the final answer:

$$(x+3)^2(-\cos\left((x+3)^2\right)) + \sin\left((x+3)^2\right) + C.$$

(b) Calculate the indefinite integral $\int (5+2\sin\theta)^{\frac{15}{2}} \cos\theta\, d\theta$.

Answer: $\frac{1}{17}(5+2\sin\theta)^{\frac{17}{2}} + C$

Solution: By substitution, with

$$u(\theta) = 5+2\sin\theta$$
$$u'(\theta) = 2\cos\theta$$

Then

$$\int (5+2\sin\theta)^{\frac{15}{2}}\cos\theta\, d\theta = \int \frac{1}{2} u^{\frac{15}{2}}\, du$$

so that

$$\frac{1}{2}\frac{2}{17}(5+2\sin\theta)^{\frac{17}{2}} + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int x^3 e^{x^2}\, dx$.

Answer: $\frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$

Solution: We use the substitution $s = x^2$, $ds/dx = 2x$ so that xdx is replaced by $1/2ds$. This gives

$$I = \int x^3 e^{x^2}\, dx = \frac{1}{2}\int s e^s\, ds\,.$$

Now do integration by parts. Set $u = s$ and $dv/ds = e^s$ so that $du/ds = 1$ and $v = e^s$. This yields

$$I = \frac{1}{2}\int s e^s\, ds = \frac{1}{2}\left[s e^s - \int e^s\, ds\right]$$

Performing the last integration and setting $s = x^2$ we get

$$I = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C\,.$$

Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/4} \sec^4(x)\tan(x)\,dx$.

Answer: 3/4

Solution: This is a trigonometric integral that is calculated as by holding one $\sec^2(x)$ and replacing the other $\sec^2(x)$ by $1+\tan^2(x)$:

$$\begin{aligned} I = \int_0^{\pi/4} \sec^4(x)\tan(x)\,dx &= \int_0^{\pi/4} (1+\tan^2(x))\tan(x)\sec^2(x)\,dx \\ &= \int_0^{\pi/4} (\tan(x)+\tan^3(x))\sec^2(x)\,dx \end{aligned}$$

which gives, upon substituting $u = \tan(x)$ and $du = \sec^2(x)dx$:

$$I = \int_0^1 (u+u^3)du = \left[\frac{u^2}{2} + \frac{u^4}{4}\right]_0^1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

(b) Calculate $\displaystyle\int_0^1 \frac{5x^2}{3x^2+3}\,dx$.

Answer: $\frac{5}{3}\left(1-\frac{\pi}{4}\right)$

Solution: We first rewrite the definite integral as

$$\begin{aligned} I = \int_0^1 \frac{5x^2}{3x^2+3}\,dx = \frac{5}{3}\int_0^1 \frac{x^2}{x^2+1}\,dx &= \frac{5}{3}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx \\ &= \frac{5}{3}\int_0^1 \left(1-\frac{1}{x^2+1}\right)\,dx \end{aligned}$$

In this form, the integrand is very easy to anti-differentiate and we finally get:

$$I = \frac{5}{3}\left[x-\arctan(x)\right]_0^1 = \frac{5}{3}\left(1-\frac{\pi}{4}\right)$$

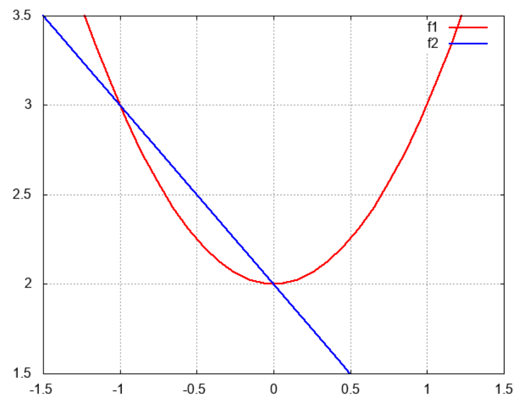
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) [2 marks] Sketch by hand the finite area enclosed between the curves defined by the functions $y = x^2 + 2$ and $y + x = 2$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) [4 marks] Write the definite integral with specific limits of integration that determines this finite area.

Answer: $-\int_{-1}^{0} (x + x^2)\, dx$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$x^2 + 2 = 2 - x \Leftrightarrow x\,(x+1) = 0.$$

The intersection points are therefore $(0, 2)$ and $(-1, 3)$. We then label the curve $y_R = x^2 + 2$ and $y_B = 2 - x$ and notice that $y_B \geq y_R$ for $-1 \leq x \leq 0$. The area is therefore given by the following definite integral:

$$A = \int_{-1}^{0} \left(2 - x - x^2 - 2\right)\, dx = \int_{-1}^{0} \left(-x - x^2\right)\, dx = -\int_{-1}^{0} \left(x + x^2\right)\, dx$$

(c) 2 marks Evaluate the integral.

Answer: $\frac{1}{6}$

Solution:

$$A = -\int_{-1}^{0} \left(x + x^2\right)\, dx = -\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^{0} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

5. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = 3\sqrt{x} + 1$ and $y = x + 3$ about the vertical line $x = -2$. **Do not evaluate the integral.**

Answer: $\pi \int_4^7 (y-1)^2 - \left(\frac{(y-1)^2}{9} + 2\right)^2 \, dy$

Solution: Intersection points are given by $3\sqrt{x} + 1 = x + 3$.

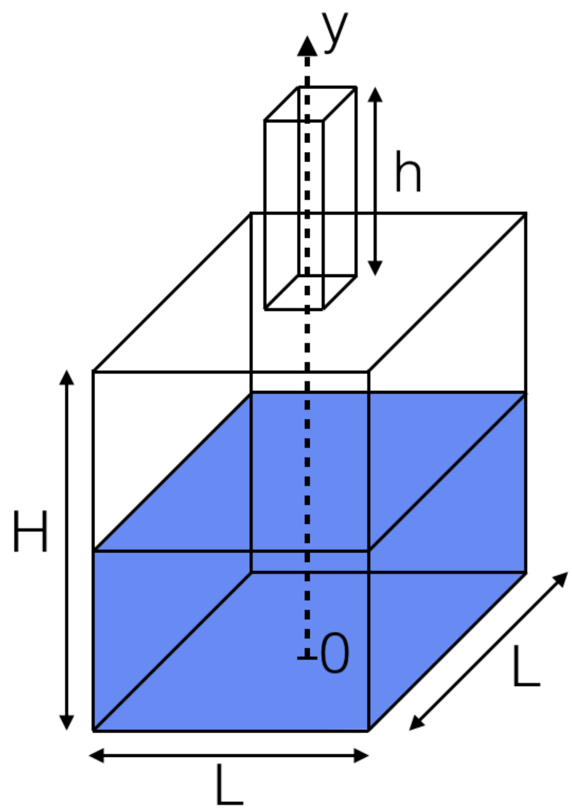
Solving for x, we determine the 2 intersection points

$$I_1 = (1, 4) \quad , \quad I_2 = (4, 7).$$

We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of $+2$, we finally establish:

$$\pi \int_4^7 (y-1)^2 - \left(\frac{(y-1)^2}{9} + 2\right)^2 \, dy.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000 kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10 m/s^2$.
We take $H = 4m$, $L = 10m$ and $h = 2m$.



(a) [2 marks] Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $10^6 \int_0^2 (6 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (6 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 4/2 = 2$ as

$$W = \int_0^2 g\rho L^2(6-y)\,dy = 10^6 \int_0^2 (6-y)\,dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $10^7 J$

Solution:

$$W = 10^6 \int_0^2 (6-y)\,dy = 10^6 \left[6y - \frac{y^2}{2}\right]_0^2 = 10^6 \left(6\cdot 2 - \frac{2^2}{2}\right)$$
$$= 10^6 \cdot 10 = 10^7 J$$