Indefinite Integrals

1. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.
   (a) Calculate the indefinite integral \( \int \frac{\ln x}{\sqrt{x}} \, dx \) for \( x > 0 \).

   Answer: \( 2\sqrt{x}(\ln x - 2) + C \)

   **Solution:** We do integration by parts with:

   \[ u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}, \]

   \[ v'(x) = x^{-1/2} \Rightarrow v(x) = 2x^{1/2}. \]

   \[ \int \frac{\ln x}{\sqrt{x}} \, dx = 2x^{1/2} \ln x - \int \frac{1}{x} \cdot 2x^{1/2} \, dx \]

   The second term on the rhs is simply \( 2 \int x^{-1/2} \, dx \), and we finally get:

   \[ \int \frac{\ln x}{\sqrt{x}} \, dx = 2\sqrt{x}(\ln x - 2) + C \]

   (b) Calculate the indefinite integral \( \int -2x\sqrt{3 + 2x} \, dx \) for \( x > -3/2 \).

   Answer: \( \frac{2}{5}(3 + 2x)^{3/2}(1 - x) + C \)

   **Solution:** We take \( u(x) = 3 + 2x \), then we have \( u'(x) = 2 \) and we replace \(-2x\) by \( 3 - u(x) \), such that we write

   \[ \int -2x\sqrt{3 + 2x} \, dx = \frac{1}{2} \int 2(-2x)\sqrt{3 + 2x} \, dx = \frac{1}{2} \int (3 - u)u^{1/2}u' \, dx \]

   and apply substitution rule as:

   \[ \int (3 - u)u^{1/2}u' \, dx = \left( \int 3u^{1/2} - u^{3/2} \, du \right)_{u=3+2x} \]

   Anti-differentiating the simple polynomial function \( 3u^{1/2} - u^{3/2} \) and eventually substituting \( u(x) = 3 + 2x \), we finally get:

   \[ \int -2x\sqrt{3 + 2x} \, dx = \left( (3 + 2x)^{3/2} - \frac{1}{5}(3 + 2x)^{5/2} \right) + C = \frac{2}{5}(3 + 2x)^{3/2}(1 - x) + C \]

   Note that this problem can also be solved by IBP (but more challenging) with:

   \[ u(x) = -2x \Rightarrow u'(x) = -2, \]

   \[ v'(x) = (3 + 2x)^{1/2} \Rightarrow v(x) = \frac{2}{3} \left( \frac{1}{2} \right) (3 + 2x)^{3/2} = \frac{1}{3}(3 + 2x)^{3/2}. \]
such that

\[ I = -2x \frac{1}{3} (3 + 2x)^{3/2} - \int (-2) \frac{1}{3} (3 + 2x)^{3/2} \, dx \]

\[ = \left( -\frac{2}{3}x \right) (3 + 2x)^{3/2} + \frac{2}{3} \int (3 + 2x)^{3/2} \, dx \]

Given that the anti-derivative of \( \int (3 + 2x)^{3/2} \, dx \) is \( \frac{2}{5} \left( \frac{1}{2} \right) (3 + 2x)^{5/2} + C = \frac{1}{5} (3 + 2x)^{5/2} + C \), we get:

\[ I = (3 + 2x)^{3/2} \left( -\frac{2}{3}x + \frac{2}{15} (3 + 2x) \right) + C = (3 + 2x)^{3/2} \left( \frac{6}{15} - \frac{2}{5} \right) + C \]

\[ = \frac{2}{5} (3 + 2x)^{3/2} (1 - x) + C \]
(c) (A Little Harder): Calculate the indefinite integral \( \int \frac{x^2 + x + 3}{x^3 + 4x - x^2 - 4} \, dx \).

Answer: \( \ln |x - 1| + \frac{1}{2} \arctan \left(\frac{x}{2}\right) + C \)

**Solution:** This is a partial fraction indefinite integral. We first recognize that \( x = 1 \) is a root of the denominator, such that we can write

\[
x^3 + 4x - x^2 - 4 = (x - 1)(ax^2 + bx + c) = ax^3 + (b - a)x^2 + (c - b)x - c
\]

which gives \( a = 1, \ b = 0 \) and \( c = 4 \), such that we can now write:

\[
\frac{x^2 + x + 3}{x^3 + 3x + x^2 + 3} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2 + 4) + (Bx + C)(x - 1)}{x^3 + 4x - x^2 - 4}
\]

which gives \( A = 1, \ B = 0 \) and \( C = 1 \). We can now calculate the indefinite integral as:

\[
\int \frac{x^2 + x + 3}{x^3 + 4x - x^2 - 4} \, dx = \int \frac{1}{x - 1} + \frac{1}{x^2 + 4} \, dx = \ln |x - 1| + \frac{1}{2} \arctan \left(\frac{x}{2}\right) + C
\]
Definite Integrals

2. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate \( \int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx \).

**Answer:** 4

**Solution:** This is a trigonometric integral that is calculated as:

\[
I = \int_{-\pi/2}^{\pi/2} 3 \cos^3 x \, dx = 3 \int_{-\pi/2}^{\pi/2} \cos x \cos^2 x \, dx = 3 \int_{-\pi/2}^{\pi/2} \cos x (1 - \sin^2 x) \, dx
\]

which gives:

\[
I = 3 \left[ \sin x - \frac{\sin^3 x}{3} \right]_{-\pi/2}^{\pi/2} = 4 \left( \frac{2}{3} + \frac{2}{3} \right) = 4
\]

(b) Calculate \( \int_{-2}^{1} \frac{x + 2}{\sqrt{-4x - 2 - x^2}} \, dx \).

**Answer:** \( \sqrt{2} - 1 \)

**Solution:** We can rewrite \(-4x - 2 - x^2\) as \(2 - (x + 2)^2\) and use a trigonometric substitution as

\[
x + 2 = \sqrt{2} \sin \theta, \quad x' = \frac{dx}{d\theta} = \sqrt{2} \cos \theta,
\]

\[
x = -2 \Rightarrow \theta = 0, \quad x = -1 \Rightarrow \theta = \pi/4
\]

to get:

\[
I = \int_{-2}^{1} \frac{x + 2}{\sqrt{-4x - 2 - x^2}} \, dx = \int_{0}^{\pi/4} \frac{\sqrt{2} \sin \theta}{\sqrt{2 - 2 \sin^2 \theta}} \sqrt{2} \cos \theta \, d\theta
\]

Now we replace \(\sqrt{1 - \sin^2 \theta}\) by \(\sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta\) as \(\cos \theta\) is positive on \([0, \pi/4]\) and finally calculate:

\[
I = \sqrt{2} \int_{0}^{\pi/4} \sin \theta \, d\theta = \sqrt{2} \left[ -\cos \theta \right]_{0}^{\pi/4} = \sqrt{2} - 1
\]

Note that this problem can also be solved by standard substitution: \(u(x) = -4x - 2 - x^2, u'(x) = -4 - 2x = -2(x + 2), u(-2) = 2, u(-1) = 1\) as

\[
\int_{-2}^{1} \frac{x + 2}{\sqrt{-4x - 2 - x^2}} \, dx = -\frac{1}{2} \int_{-2}^{1} \frac{-2(x + 2)}{\sqrt{-4x - 2 - x^2}} \, dx = -\frac{1}{2} \int_{-2}^{1} u'u^{-1/2} \, dx
\]
and then

\[-\frac{1}{2} \int_{-2}^{-1} u'u^{-1/2} \, dx = -\frac{1}{2} \int_{2}^{1} u^{-1/2} \, du = \left[-u^{1/2}\right]_{-2}^{1} = -1 + 2^{1/2}\]
Riemann Sum and FTC

3. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\sqrt{x_i^2 + 9n^2}}{x_i^2} \)?

(A) \( \int_{0}^{3} \frac{\sqrt{\pi^2 + 1}}{\pi} \) 
(B) \( 3 \int_{0}^{1} \frac{\sqrt{\pi^2 + 1}}{\pi^2} \) 
(C) \( \frac{1}{3} \int_{0}^{1} \sqrt{\pi^2 + 1} \) 
(D) \( \int_{0}^{1} \frac{\sqrt{\pi^2 + 9}}{\pi^2} \) 
(E) \( \int_{0}^{3} \frac{\sqrt{\pi^2 + 9}}{\pi^2} \)

Answer: D

Solution: Pick \( x_i = \frac{i}{n} \), so \( x_0 = 0, x_n = 1 \) and \( \Delta x = \frac{1}{n} \). Then we can rewrite the summation as:

\[
\sum_{i=1}^{n} \frac{\sqrt{n^2 x_i^2 + 9n^2}}{n^2 x_i^2} = \sum_{i=1}^{n} \frac{\sqrt{x_i^2 + 9}}{n x_i^2} = \sum_{i=1}^{n} \frac{\sqrt{x_i^2 + 9}}{x_i^2} \Delta x
\]

which corresponds to the Right Riemann Sum for option (D).

(b) Define \( F(x) \) and \( g(x) \) by \( F(x) = \int_{\pi}^{2\pi} \cos^2 t \) and \( g(x) = x F(x) \). Calculate \( g'(\pi) \).

Answer: \( 3\frac{\pi}{2} \)

Solution: We use the product rule to get: \( g'(x) = F(x) + x F'(x) \), and FTC I and the chain rule to calculate \( F'(x) = 2 \cos^2(2x) - \cos^2 x \). So we have:

\[
g'(x) = F(x) + x(2 \cos^2(2x) - \cos^2 x)
\]

and \( g'(\pi) = F(\pi) + \pi(2 \cos^2(2\pi) - \cos^2 \pi) = F(\pi) + \pi \). Now we calculate \( F(\pi) \) as

\[
F(\pi) = \int_{\pi}^{2\pi} \cos^2 t \) dt = \int_{\pi}^{2\pi} \frac{1}{2} dt + \int_{\pi}^{2\pi} \frac{\cos(2t)}{2} dt = \frac{\pi}{2} + \left[ \frac{\sin(2t)}{4} \right]_{\pi}^{2\pi} = \frac{\pi}{2}
\]

and get \( g'(\pi) = \frac{\pi}{2} + \pi = 3\frac{\pi}{2} \).
(c) Let $F(x) = \int_{x^2}^{x^3} 7e^{t^2} \, dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

**Answer:** $y = 7e(x - 1)$

**Solution:** We first write $F(x)$ for any real number $c$ as:

$$F(x) = -\int_{x^2}^{c} 7e^{t^2} \, dt + \int_{c}^{x^3} 7e^{t^2} \, dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -7e^{x^4}2x + 7e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_{1}^{1} 7e^{t^2} \, dt = 0$ and $F'(1) = 7e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 7e(x - 1)$$
Areas and volumes

Please write your answers in the boxes. Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.

4. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between \( x = -(y - 4)^2 \) and \( x = -2 - y \) about the vertical line \( x = 1 \). Do not evaluate the integral.

\[
\text{Answer: } \pi \int_2^7 (-3 - y)^2 - (-1 - (y - 4)^2)^2 \, dy
\]

Solution: Intersection points are given by \(-2 - y = -(y - 4)^2\).

Solving for \( y \), we determine the 2 intersection points

\[ I_1 = (-4, 2), \quad I_2 = (-9, 7). \]

We integrate in \( y \), hence we write \( x \) as a function of \( y \) for the 2 curves and apply a shift of \(-1\), we finally establish:

\[
\pi \int_2^7 (-3 - y)^2 - (-1 - (y - 4)^2)^2 \, dy.
\]
5. (a) **2 marks** Sketch by hand the finite area enclosed by \( y^2 = 3 - x \) and \( 3y = x + 1 \)

**Answer:**

**Solution:** The area is the region enclosed between the red and blue curves:

(b) **4 marks** Write a definite integral with specific limits of integration that determines this finite area.

**Answer:** \( \int_{-4}^{1} (-y^2 - 3y + 4) \, dy \)

**Solution:** We first find the intersection between the two curves, given by the solution of:

\[
3 - y^2 = 3y - 1 \iff (y + 4)(y - 1) = 0.
\]

We then label the curve \( x_R = 3 - y^2 \) and \( x_B = 3y - 1 \) and notice that \( x_B \leq x_R \) for \( -4 \leq y \leq 1 \). The area is therefore given by the following definite integral:

\[
A = \int_{-4}^{1} (3 - y^2 - 3y + 1) \, dy = \int_{-4}^{1} (-y^2 - 3y + 4) \, dy
\]
(c) 2 marks Evaluate the integral to compute the area enclosed.

Answer: $\frac{125}{6}$

Solution:

$$ A = \left[ -\frac{y^3}{3} - \frac{3y^2}{2} + 4y \right]^{1}_{-4} = \frac{125}{6} $$