First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Indefinite Integrals

1. 9 marks Each part is worth 3 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $\int x^2\sqrt{8-x^3}\,dx$ for $x<2$.

   Answer: $I = -2(8-x^3)^{3/2}/9 + C$.

   Solution: Let $u = 8 - x^3$, so that $x^2dx = -du/3$. Then,

   $$I = -\int \left(u^{1/2}/3\right)\,du = -2u^{3/2}/9 + C\,.$$

   Using $u = 8 - x^3$ we get $I = -2(8-x^3)^{3/2}/9 + C$.

   (b) Calculate the indefinite integral $\int x\sqrt{x-1}\,dx$ for $x>1$.

   Answer: $\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

   Solution: Let $u = x - 1$ so $du = dx$ and use $x = (1+u)$. Then,

   $$I = \int (u+1)u^{1/2}\,du = \int \left(u^{3/2} + u^{1/2}\right)\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C\,.$$

   Setting $u = x - 1$ this gives $I = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$.
   Method 2: Use integration by parts with $u = x$ and $dv/dx = \sqrt{x-1}$. Then, $du/dx = 1$ and $v = \frac{2}{3}(x-1)^{3/2}$. We get

   $$I = uv - \int v\frac{du}{dx}\,dx = \frac{2x}{3}(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\,dx = \frac{2x}{3}(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C\,.$$

   To show that these two methods give the same solution, we write the solution above as

   $$I = \frac{2(x-1)}{3}(x-1)^{3/2} + \frac{2}{3}(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C\,.$$

   Combining together we get

   $$I = \left(\frac{2}{3} - \frac{4}{15}\right)(x-1)^{3/2} + \frac{2}{3}(x-1)^{3/2} + C = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C\,.$$

(c) (A Little Harder): Calculate the indefinite integral $\int \ln\left(1+x^2\right)\, dx$.

Answer: $x\ln(1+x^2) - 2x + 2\arctan(x) + C$

Solution: Let $u = \ln(1+x^2)$ and $dv/dx = 1$. We calculate $du/dx = 2x/(1+x^2)$ and $v = x$, so that one step of integration by parts gives

$$I = uv - \int v\frac{du}{dx}\, dx = x\ln(1+x^2) - 2\int \frac{x^2}{(1+x^2)}\, dx\,.$$

This can be re-written in a form that is readily calculated as

$$I = x\ln(1+x^2) - 2\int \left[1 - \frac{1}{x^2+1}\right] dx = x\ln(1+x^2) - 2\left(x - \arctan(x)\right) + C\,.$$

Definite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^\pi \sin^3(x)\,dx$.

Answer: 4/3

Solution: Use $\sin^2(x) = 1 - \cos^2(x)$ to get $I = \int_0^\pi (1-\cos^2(x))\sin x\,dx$. Let $u = \cos x$, so that $du = -\sin x\,dx$. Then, since $x = 0$ maps to $u = 1$ while $x = \pi$ maps to $u = -1$, we get

$$I = -\int_1^{-1} (1-u^2)\,du = \int_{-1}^{1} (1-u^2)\,du = 2\int_0^1 (1-u^2)\,du = 2\,(1-1/3) = 4/3\,.$$

(b) Calculate $\int_{-1}^1 \left(x^2 e^{-x^3} + x^5\cos(x)\right)\,dx$.

Answer: $\frac{1}{3}\,(e^1 - e^{-1})$.

Solution: We write $I = \int_{-1}^1 x^2 e^{-x^3}\,dx + \int_{-1}^1 x^5\cos(x)dx$. The second integral vanishes since the integrand is odd and the integration is over a symmetric range of the origin. In the first integral put $u = x^3$ so that $(1/3)\,du = x^2 dx$. Since $x = \pm 1$ maps to $u = \pm 1$, we get $I = \int_{-1}^1 (e^{-u}/3)\,du$. We then integrate this expression to get $I = -e^{-u}/3|_{-1}^{1} = \frac{1}{3}\,(e^1 - e^{-1})$.

(c) (A Little Harder): Calculate $\int_1^e (\ln x)^2 \ dx$.

Answer: $e - 2$.

Solution: Let $u = (\ln x)^2$ and $dv/dx = 1$. Then, $du/dx = 2(\ln x)/x$ and $v = x$. By using one step of integration by parts (IBP) we get

$$I = uv|_1^e - \int_1^e v\frac{du}{dx}\, dx = x\,(\ln x)^2\,|_1^e - 2\int_1^e \ln x\, dx\,.$$

We then use IBP in the second integral. Let $u = \ln x$ and $dv/dx = 1$ so that $du/dx = 1/x$ and $v = x$. This gives

$$I = x\,(\ln x)^2\,|_1^e - 2\left(x \ln x|_1^e - \int_1^e dx\right)\,.$$

By putting in the limits, and by using $\ln(1) = 0$ and $\ln(e) = 1$, we get $I = e - 2\,[e - (e-1)] = e - 2$.

Riemann Sum, FTC, and Volumes

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i}{n^2\,(4+i^2/n^2)}$$

by first writing it as a definite integral. Then, **evaluate this integral**.

Answer: $\int_0^1 \frac{2x}{4+x^2}\,dx = \ln 5 - \ln 4$.

Solution: We identify $a=0$, $b=1$, $\Delta x = 1/n$, $x_i = i/n$, and $f(x_i) = 2x_i/(1+x_i^2)$. This yields

$$S \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i}{n^2\,(4+i^2/n^2)} = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta x) f(x_i) = \int_0^1 \frac{2x}{4+x^2}\,dx\,.$$

To calculate the integral we let $u = 4+x^2$, so that $S = \int 1/u\,du = \ln u$. This yields $S = \ln(4+x^2)|_0^1 = \ln 5 - \ln 4$.

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_1^x \ln t\,dt$ and $g(x) = x\,F(x^2)$ for $x > 1$. Calculate $g'(e)$.

Answer: $g'(e) = 5e^2 + 1$.

Solution: We use the product rule to get $g'x) = F(x^2) + 2x^2F'(x^2)$. Now by FTC I, we get $F'(x^2) = \ln(x^2) = 2\ln x$. This yields,

$$g'(x) = F(x^2) + 4x^2 \ln x\,. \qquad (1)$$

Now let $x = e$ and calculate using integration by parts that

$$F(e^2) = \int_1^{e^2} \ln t\,dt = t\ln t|_1^{e^2} - \int_1^{e^2} (1)\,dt = 2e^2 - (e^2-1) = e^2+1\,.$$

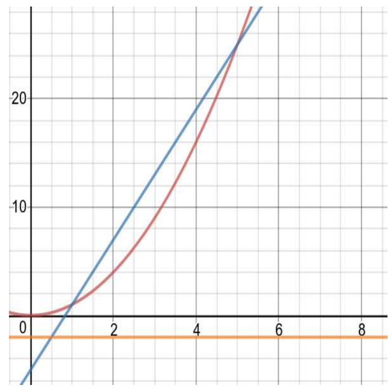
Therefore, from (1) and using $\ln(e) = 1$, we get

$$g'(e) = F(e^2) + 4e^2 = 5e^2 + 1\,.$$

(c) Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = x^2$ and $y = 6x - 5$ about the horizontal line $y = -2$. **Do not evaluate the integral**.

Answer: $V = \pi \int_1^5 \left[(6x - 5 + 2)^2 - (x^2 + 2)^2\right]\, dx.$

Solution:

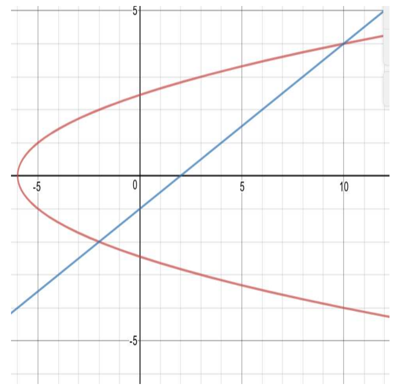


The two curves intersect when $x^2 = 6x - 5$, which yields $(x - 5)(x - 1) = 0$. This gives $x = 1$ and $x = 5$ for which $y = 1$ and $y = 25$, respectively. The intersection points are in the first quadrant. Define $y_T = 6x - 5$ (top blue curve) and $y_B = x^2$ (bottom red curve). Then, at each x in $[1, 5]$, we have that $(y_T + 2)$ and $(y_B + 2)$ are the distances of the two curves from the axis of rotation $y = -2$ shown by the orange curve. This yields $V = \pi \int_1^5 \left[(y_T + 2)^2 - (y_B + 2)^2\right]\, dx$.

4. (a) [2 marks] Plot the finite area enclosed by $y^2 = 6 + x$ and $2y = x - 2$.

Solution:

The area is the region enclosed between the blue and red curves:



(b) [4 marks] Write a definite integral with specific limits of integration that determines this area. **Do not evaluate the integral**.

Answer: $\int_{-2}^{4}(2y - y^2 + 8)\,dy$.

Solution: To find the intersection points we set $x = y^2 - 6 = 2 + 2y$. This yields, $y^2 - 2y - 8 = (y-4)(y+2) = 0$, which gives $y = -2$ and $y = 4$. We label $x_T = 2y + 2$ (blue curve) and $x_B = y^2 - 6$ (red curve), and observe that $x_T > x_B$ on $-2 < y < 4$. The area is best calculated as an integral in y, so that $A = \int_{-2}^{4} (x_T - x_B)\, dy = \int_{-2}^{4} (2y + 2 - (y^2 - 6))\, dy$.

5. A solid has as its base the region in the xy-plane between $y = 1 - x^2/16$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are isosceles right triangles (i.e. $45 - 45 - 90$ triangles) with the longest side (i.e. the hypoteneuse) in the base.

(a) 4 marks Write a definite integral that determines the volume of the solid.

Answer: $V = \frac{1}{4}\int_{-4}^{4}\left[1 - \frac{x^2}{16}\right]^2 dx\,.$

Solution: The intersection points with the x-axis are $x = \pm 4$. This gives, $V = \int_{-4}^{4} A(x)\,dx$ as the volume, where $A(x)$ is the cross-sectional area of the solid at position x. This cross-section is a $45 - 45 - 90$ triangle that has area $A(x) = [y(x)]([y(x)]/2)/2 = [y(x)]^2/4$. Here we have used the fact that the area of a $45-45-90$ triangle with baselength b is $bh/2$ where $h = b/2$ is the altitude of the triangle. This gives,

$$V = \frac{1}{4}\int_{-4}^{4}[y(x)]^2\,dx = \frac{1}{4}\int_{-4}^{4}\left[1 - \frac{x^2}{16}\right]^2 dx\,.$$

(b) 2 marks **Evaluate the integral** to find the volume of the solid.

Answer: 16/15

Solution: Since the integrand is even, we write $V = \frac{1}{2}\int_{0}^{4}\left[1 - \frac{x^2}{16}\right]^2 dx$. Now put $x = 4u$, so that $dx = 4du$, and so

$$V = 2\int_0^1 (1-u^2)^2\,du = 2\int_0^1 \left(1 - 2u^2 + u^4\right)\,du = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}\,.$$