Math 221, Midterm 1

Instructor: V. Vatsal (Sec 103)

1300–1345, Monday, October 3, 2016

There are 4 questions, with marks as indicated. Answer all of them. No notes, calculators, textbooks, or other aids are permitted.

Name (print):

Student ID number:

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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1. Consider the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 3 & -2 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 2 \\ -2 \\ 10 \end{pmatrix}.$$

a) (1 mark) Write the system of linear equations corresponding to the matrix equation $Ax = b$ above.

$$\begin{align*}
2x_1 + x_2 - x_3 + x_4 &= 2 \\
-2x_1 + x_2 + x_3 + x_4 &= -2 \\
4x_1 + 3x_2 - 2x_3 + x_4 &= 10
\end{align*}$$

b) (1 mark) Write the vector equation corresponding to the matrix equation $Ax = b$ as above.

$$x_1 \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 10 \end{pmatrix}$$

c) (2 marks) Reduce the matrix $A$ from the start of this question to echelon form and identify the pivots, basic variables, and free variables.

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 3 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 \end{pmatrix}$$

Pivots circled, basic $x_2, x_1, x_4$, free $x_3$. 

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d) (5 marks) Find all solutions of the system from part a) and write the answer in vector parametric form.

\[
\begin{pmatrix}
2 & 1 & -1 & 1 & 2 \\
-2 & 0 & 1 & 1 & 2 \\
4 & 3 & -2 & 1 & 2 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & -3 & 6
\end{pmatrix}
\]

\[x_4 = 6 \implies x_4 = -2\]

\[x_2 + 2x_4 = 0 \implies x_2 = 4\]

\[2x_1 + x_2 - x_3 + x_4 = 2\]

\[2x_1 + 4 - x_3 = 2 \implies 2x_1 - x_3 = 0\]

\[\Rightarrow x_1 = x_3/2\]

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
= \begin{pmatrix}
x_{3/2} \\
4 \\
x_3 \\
-2 \\
0
\end{pmatrix}
+ x_3 \begin{pmatrix}
1/2 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
\]
2. a) (2 marks) Find the values of \( s, t \) such that the system of two equations

\[
2x_1 + x_2 = 4 \\
sx_2 = t
\]

is inconsistent.

\[
\begin{pmatrix}
2 & 1 \\
0 & s
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
4 \\
t
\end{pmatrix}
\]

Inconsistent \( \iff \) \( s \neq 0 \) \( t \neq 0 \)

b) (3 marks) Consider the four vectors \( v_1 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \) and \( v_4 = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \). Are these vectors linearly independent?

\[
\begin{pmatrix}
4 & 4 & 4 & 4 \\
0 & 3 & 3 & 3 \\
0 & 0 & 2 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Pivots in each row \( \Rightarrow \) no free free var

\( \Rightarrow \) indep
3. a) (3 marks) Determine whether or not the vectors \( v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ b_2 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ b_3 - b_1 \end{bmatrix} \) span \( \mathbb{R}^3 \).

\[
\begin{pmatrix}
1 & 1 & 1 & b_1 \\
1 & 2 & -1 & b_2 \\
1 & 0 & -2 & b_3 - b_1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & -2 & b_2 - b_1 \\
0 & 1 & -1 & b_3 - b_1 \\
0 & 0 & 2 & b_2 - b_1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1 \\
-2
\end{pmatrix}
\]

Consistent for all \( b_1, b_2, b_3 \)

\( \Rightarrow \) yes they span \( \mathbb{R}^3 \).

b) (2 marks) With \( v_1, v_2, v_3 \) as above, determine whether or not \( w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \) is in the span of \( v_1 \) and \( v_2 \).

\[
\begin{pmatrix}
1 & 1 & 3 \\
1 & 2 & 2 \\
1 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 3 \\
0 & 0 & -2 \\
0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1 \\
-2
\end{pmatrix}
\]

Inconsistent \( \Rightarrow \) \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) not in \( \text{span} \ v_1, v_2 \).
4. The following question consists of a series of statements. For each statement, say whether it is true or false. Your answer should be either True or False. No explanation is required. Each part is worth two marks.

(a) Let \( A \) denote a \( 5 \times 5 \) matrix such that the echelon form of \( A \) has a pivot in every row. Then the equation \( Ax = b \) has a unique solution for every \( b \) in \( \mathbb{R}^5 \).

\[
\begin{pmatrix}
\ast & 0 & 0 & 0 & 0 \\
0 & \ast & 0 & 0 & 0 \\
0 & 0 & \ast & 0 & 0 \\
0 & 0 & 0 & \ast & 0 \\
0 & 0 & 0 & 0 & \ast
\end{pmatrix}
\begin{pmatrix}
\ast \\
\ast \\
\ast \\
\ast \\
\ast
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\Rightarrow \text{unique sol for all } b
\]

(b) Let \( A \) denote a \( 3 \times 5 \) matrix such that the echelon form of \( A \) has exactly two pivots. Let \( b \) denote a vector in \( \mathbb{R}^3 \). Then the equation \( Ax = b \) has infinitely many solutions for every choice of \( b \).

\[
\begin{pmatrix}
\ast & 0 & 0 & 0 & 0 \\
0 & \ast & 0 & 0 & 0 \\
0 & 0 & \ast & 0 & 0 \\
0 & 0 & 0 & \ast & 0 \\
0 & 0 & 0 & 0 & \ast
\end{pmatrix}
\begin{pmatrix}
\ast \\
\ast \\
\ast \\
\ast \\
\ast
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\Rightarrow \text{inconsistent}
\]

(c) Let \( A \) denote a \( 3 \times 5 \) matrix and let \( b \) denote a vector in \( \mathbb{R}^3 \). Then the matrix equation \( Ax = b \) represents a system of 3 linear equations in 5 variables.

\[
3
\begin{pmatrix}
\ast \\
\ast \\
\ast \\
x_1 \ x_2 \ x_3 \ x_4 \ x_5
\end{pmatrix}
= 
\begin{pmatrix}
\ast \\
\ast \\
\ast \\
\end{pmatrix}
\Rightarrow \text{T}
\]