Equilibrium in a Stochastic Model with Consumption, Wages and Investment [*]

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Abstract. Existence of equilibrium is established in a continuous time economy where a finite number of agents derive utility from consuming both a good and leisure. They own and trade shares of the means of production of the good as well as financial assets. Share prices are modeled as Itô processes. The agents derive income from dividends paid out to shareholders of the productive asset and from wages. They choose a consumption policy, a leisure policy and an investment strategy which maximizes their utility. The manager of the production facility chooses employment rates so as to maximize profit. Labour is transformed into goods through a production function. The only exogenous quantities are the parameters of the financial market. Two simple examples are solved explicitly.

Key words. general equilibrium, asset pricing, consumption decisions, employment rates, wage rates, convex analysis, stochastic analysis

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1 Introduction

General equilibrium asset pricing models can be found in the literature going back to Merton [13], Cox, Ingersoll and Ross [5], Duffie and Huang [6], Huang [9], and Karatzas, Lehoczky and Shreve [11]. For example, in the model in [11] productive assets produce a consumable, tradable good. Agents have a random exogenous endowment stream of this good; they can hold positions in the productive assets (thus earning further amounts of the good as dividends) and can hold positions in financial securities to hedge risk, but their utility derives only from consumption. Asset and security prices are modeled as Itô processes. It is shown that under mild conditions, a

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unique equilibrium exists, i.e. a price for the consumable exists at which the individual agents, each acting to maximize her utility, will clear the goods market, the productive asset market as well as the financial market. The price of the productive assets is endogenously determined but the financial market dynamics are exogenous (in the moneyed model).

We may think of this result as a linking of macro- and microeconomics in that the actions of the individual agents determine the macroeconomic price variable. Of course it is far from complete even as a model for a closed economy - interest rates, money supply, employment rates and inflation are not considered. For an open economy, exchange rates and inter-economy trade must be considered. As financial markets are international, exchange rates cannot be ignored even in finance, and indeed they have received considerable attention recently by, among others, Dumas [7], Obstfeld and Rogoff [14], Rogers [16], and Basak and Gallmeyer [1].

The current work represents one step in an effort to produce a general macroeconomic equilibrium model derived from microeconomic considerations. Specifically, we specialize and extend the Karatzas et al. model to a situation where the endowment streams of the agents are wages and denominated in money, not goods, and are not exogenous. Now the labour provided by the agents produces the consumable good through a production function. The utility of each agent is derived from consumption of both goods and leisure, and there are now two “prices”, that of the good and that of leisure. The manager of the production facility is another decision maker: he sets the level of employment to maximize profit. This is a one economy model; a two economy generalization of the Karatzas et al. model can be found in [2]. We have not included capital or technology in the production end of the model in order to keep things simple. Capital investment in the productive asset would change the optimization problems in sections 4, 5 and 6 from static calculus of variations problems to dynamic control problems, hence raising the complexity considerably. This issue will be addressed in future.

As our model is closely linked to that of Karatzas et al., we use a generalization of their method. From the technical point of view, the main contribution of this work is the application of convex analysis to treat utility functions depending on more than one control variable, here these are consumption of the good and of leisure. This will be of importance in the two-country model when non-traded as well as traded goods have to be considered in addition to labour. We demonstrate existence of an equilibrium, but we are unable to establish uniqueness.

The model is described in detail in Section 2. Equilibrium is defined in Section 3 and the endogenous dynamics of the productive asset price are given. In Section 4 we solve the individual agent’s utility optimization problem and in Section 5 we introduce and solve the representative agent’s problem. This produces a first necessary condition for equilibrium. The problem of the manager of the productive asset is introduced and solved in Section 6. He must choose his labour input to maximize profits. This leads to the concept of utility of labour and to a more complete set of necessary conditions. Existence of an equilibrium is shown in Section 7. Section 8 contains two examples which can be solved explicitly. We conclude with the Appendix which contains technical results based on convex analysis.

2 Setting of The Model

The equilibrium model investigated in the present paper extends the classic KLS model (cf. [11]) so as to overcome the simplifying hypotheses of exogenous dividend and earning processes. It also introduces labour in an effort to capture further macroeconomic features.

The market we model is built on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with the filtration $\{\mathcal{F}_t : t \geq 0\}$ completed with respect to the filtration generated by an exogenous N-dimensional Brownian motion $\{W_t, t \geq 0\}$.

2.1 The Productive Asset

We assume that there is one productive asset producing a single kind of perishable consumption good which sells at price $p(t)$ units of local currency at time t. The asset uses labour which, at time t, costs $w(t)$ units of currency per unit of time. It can produce $R(t, L)$ units of the consumption good per unit of time when it employs L units of labour. Profits are distributed as dividends to the shareholders, hence at a rate

$$\delta(t) = p(t)R(t, L(t)) - w(t)L(t)$$

at time t. The manager of the asset chooses $L(t)$ so as to maximize the profit rate. There are J agents in the economy and they provide the labour, so $0 \leq L(t) \leq J$, but in fact we shall restrict labour to a smaller interval: $0 \leq L(t) \leq L_{\max}$ where $L_{\max}$ is chosen strictly smaller than J. It is the maximal employment rate in the economy, so full employment is not possible. (In the examples in Section 8, we choose $L_{\max}$ after solving the problem - once we see that $\sup_t L(t) < J$, we set $L_{\max}$ between these two.) Then

$$L(t) \in \arg \max_{0 \leq L \leq L_{\max}} \{p(t)R(t, L) - w(t)L\}.$$

For each time t, the *production function* $R(t, L)$ is assumed to be continuous on $[0, L_{\max}]$, strictly concave, increasing and continuously differentiable on $]0, L_{\max}[$, assuming values in $[0, \kappa_R]$ with $\kappa_R := \sup_t R(t, L_{\max}) < \infty$. In addition, we assume that for $L > 0$

$$\inf_t R(t, L) > 0, \qquad \lim_{L \downarrow 0} \inf_t R_L(t, L) = \infty. \tag{2.1}$$

Note that we use subscripts to denote partial derivatives, i.e. $R_L = \dfrac{\partial}{\partial L} R$. We also use k_x, κ_x to denote constants related to the variable x, but k and κ will be reserved for this purpose so no confusion arises. The assumption (2.1) can be replaced by assuming that $R(t, 0) \geq k_R$ where k_R may be arbitrarily small provided it is positive. R and all functions in the paper are assumed to be measurable.

The nonnegative, measurable, $\mathcal{F}_t$-adapted *wage process* $\{w(t) : t \in [0, T]\}$ and *price process* $\{p(t) : t \in [0, T]\}$ will be determined endogenously by equilibrium considerations.

The price per share of the asset (and there is a total of one divisible share), in units of the local currency, is the nonnegative $\mathcal{F}_t$-semimartingale

$$\begin{cases} dA(t) = \beta(t)\,dt + \alpha(t)\,dW(t), & t \in [0,T[, \\ A(T) = 0. \end{cases} \tag{2.2}$$

$A(t)$ represents the value of the productive asset at time t. The real-valued drift process β and the row vector diffusion process α will be determined endogenously by the equilibrium arguments so as to satisfy

$$\int_0^T \Big[|\beta(t)| + \|\alpha(t)\|^2\Big]dt < +\infty, \qquad \text{a.s.} \tag{2.3}$$

There is no difficulty in extending the model to several productive assets, but no new insight is gained in so doing, cf. [11].

2.2 The Financial Market

The J agents in the economy provide the labour, consume the good and own the productive asset. They will then need to select a personal consumption and labour (or as we prefer, leisure) strategy, and we assume that this is done by optimizing utility, cf. below. To allow the agents to *hedge* all the risk and to finance their consumption and labour/leisure strategies, we introduce a *complete financial market* consisting of one riskless asset, the money market, and N risky assets, tradable (in the local currency) at the prices $f_n,\ n = 0, \ldots, N$, with

$$\begin{cases} df_0(t) = r(t) f_0(t)\,dt, \qquad t \in [0,T], \qquad f_0(0) = 1; \\ df_n(t) = f_n(t)[b_n(t)\,dt + a_n(t)\,dW(t)], \qquad t \in [0,T], \quad n = 1, 2, \ldots, N, \end{cases} \tag{2.4}$$

where the $N \times N$ *volatility* matrix a, whose n-th row is $a_n = (a_{n1}, a_{n2}, \cdots, a_{nN})$, the vector of *mean rates of return* $b = (b_1, b_2, \cdots, b_N)^\top$ (here the superscript $\top$ stands for transpose), and the *nominal interest rate* process $r > 0$, are assumed to be measurable, $\mathcal{F}(t)$-adapted, bounded uniformly in $(t,\omega) \in [0,T] \times \Omega$, and exogenous to the model.

We make the *non-degeneracy assumption*

$$a(t)a(t)^\top \geq \varepsilon I \qquad \text{for some } \ \varepsilon > 0, \ \text{ a.e. } (t,\omega) \in [0,T] \times \Omega, \tag{2.5}$$

in order to ensure the invertibility of the matrix $a(t)$ for almost every $(t,\omega) \in [0,T] \times \Omega$, and hence to define the *market price of risk* (or *relative risk*) θ as the unique solution of

$$a(t)\theta(t) = b(t) - r(t)\mathbb{1}_N. \tag{2.6}$$

The assumption (2.5) together with the boundedness of r, b, a ensures that $\|\theta\|$ is bounded uniformly in $(t,\omega) \in [0,T] \times \Omega$, and hence the exponential process

$$Z(t) = \exp\Big[-\int_0^t \theta(s) \cdot dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\Big], \qquad t \in [0,T], \tag{2.7}$$

is a martingale by Novikov's theorem (cf. Proposition 3.5.12 in [12]). Hence the market is *standard* and *complete* (cf. [10]) and

$$\tilde{P}(A) := E\{Z(T)\mathbb{1}_A\}, \qquad A \in \mathcal{F}(T), \tag{2.8}$$

defines a probability measure equivalent (i.e., mutually absolutely continuous with respect) to P, with Radon-Nikodym derivative

$$\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = Z(t), \qquad t \in [0,T].$$

The process

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s)ds, \qquad t \in [0,T], \tag{2.9}$$

is a standard Brownian motion under $\tilde{P}$ (cf. [12]). The probability measure $\tilde{P}$ is the so-called *risk-neutral equivalent martingale measure* since under $\tilde{P}$ the appreciation rates of the assets are replaced by the interest rate r of the non-risky asset, and the asset-prices discounted by $\exp[-\int_0^t r(s)ds]$ become $\tilde{P}$-local martingales. Finally, since our market is assumed to be complete, the *state-price density* (or *deflator*)

$$\zeta(t) =: Z(t)\exp\Big[-\int_0^t r(s)ds\Big] \tag{2.10}$$

is the unique (up to scalar multiples) process such that $\zeta(t)Y(t)$ is a P-local martingale for every traded asset Y. Hence multiplication by $\zeta(t)$ converts expected wealth held at time t to the equivalent (discounted) amount of wealth at time zero.

2.3 The Agents' Problem

We can now clarify the problem facing the j'th agent. She

- has an initial endowment of ε_j shares of the productive asset with

$$\sum_{j=1}^{J} \varepsilon_j = 1; \tag{2.11}$$

- chooses her *leisure process* $\{l_j(t) : t \in [0,T]\}$, a measurable, $\mathcal{F}_t$-adapted process with

$$l_j(t) \in [0,1], \qquad t \in [0,T], \quad \text{a.s.}; \tag{2.12}$$

- chooses her *consumption process* $\{c_j(t) : t \in [0,T]\}$ measured in units of the commodity with

$$\inf_{t\in[0,T]} c_j(t) \geq 0, \qquad \sup_{t\in[0,T]} c_j(t) < +\infty, \quad \text{a.s.} \tag{2.13}$$

- chooses her *productive asset share process* $\{\pi_j(t) : t \in [0,T]\}$ (with $\pi_j(0) = \varepsilon_j$),
- chooses her *financial asset portfolio process* $\{\phi_j(t) = (\phi_{j,0}, \bar{\phi}_j)(t) : t \in [0,T]\}$ where $\bar{\phi}_j(t) = (\phi_{j,1}(t), \ldots, \phi_{j,N}(t))$ and $\phi_{j,k}(0) = 0$, $k = 0, \ldots, N$,

such that

$$\sup_{t\in[0,T]} |\pi_j(t)| < +\infty, \qquad \int_0^T \|\phi_j(t)\|^2 dt < +\infty, \quad \text{a.s.} \tag{2.14}$$

The components of the portfolio processes are measured in numbers of shares and may be either positive or negative, i.e. short selling and borrowing are allowed.

Note that $1 - l_j(t)$ denotes the intensity with which the agent works - the more intensely she works, the greater the stress and the more she will earn. Her *earnings process* $\{e_j(t) : t \in [0,T]\}$ is given by the measurable, $\mathcal{F}_t$-adapted, nonnegative, bounded process

$$e_j(t) = w(t)(1 - l_j(t)), \qquad t \in [0,T],$$

measured in units of the local currency. We shall see, cf. (4.1), that another interpretation is possible: the agent works at maximum intensity to produce a wage endowment stream, $w(t)$, and then "consumes" leisure at a rate $l_j(t)$, i.e. the agent buys-out the fraction l_j of her work effort. In this interpretation, there are now two consumables in the economy.

If agent j does not modify her initial productive asset portfolio, but sticks with her initial endowment (and we shall see that this is an optimal strategy), then her *income process* will simply be given by

$$\hat{e}_j(t) := e_j(t) + \varepsilon_j\delta(t) = w(t)(1 - l_j(t)) + \varepsilon_j\delta(t), \qquad t \in [0,T]. \tag{2.15}$$

The *aggregate income process* of the J agents in the market is defined as

$$\hat{e}(t) = \sum_{j=1}^{J} \hat{e}_j(t) = \sum_{j=1}^{J} e_j(t) + \delta(t). \qquad t \in [0,T], \tag{2.16}$$

If we write $f(t)$ for $(f_0(t), \bar{f}(t)^\top)^\top$ where $\bar{f}(t) = (f_1(t), \ldots, f_N(t))^\top$, then the wealth of agent j at time t is given by $X_j(t) := \pi_j(t)A(t) + \phi_j(t)f(t)$, so the wealth lies in the holdings of the productive asset and of the financial assets. This implies that the consumable cannot be stockpiled. We require that X_j satisfies the *budget equation*

$$\begin{aligned} X_j(t) \;&=\; \varepsilon_j A(0) - \int_0^t p(s)c_j(s)ds + \int_0^t e_j(s)ds \\ &\quad + \int_0^t \pi_j(s)\delta(s)ds + \int_0^t \pi_j(s)dA(s) + \int_0^t \phi_j(s)df(s), \qquad t \in [0,T], \quad \text{a.s.} \end{aligned}$$

In other words, present wealth is initial wealth minus cost of consumed plus wage income, plus dividend income, plus capital gains/losses generated by the productive asset (which has value zero at the terminal time), plus financial investment income. By writing $\phi_{j,0}(t)$ in terms of $X_j(t),\ \pi_j(t),\ A(t),\ \bar{\phi}_j(t)$ and $f(t)$, we can show that

$$\begin{aligned} X_j(t) \quad &= \quad \varepsilon_j A(0) - \int_0^t p(s)c_j(s)ds + \int_0^t w(s)(1-l_j(s))ds + \int_0^t r(s)X_j(s)ds \\ &\quad + \int_0^t \pi_j(s)\Big(\delta(s)+\beta(s)-r(s)A(s)-\alpha(s)\theta(s)\Big)ds \\ &\quad + \int_0^t \Big(\pi_j(s)\alpha(s)+\bar{\phi}_j(s)\mathrm{diag}[\bar{f}(s)]a(s)\Big)d\tilde{W}(s), \qquad t\in[0,T], \quad \text{a.s..} \end{aligned} \tag{2.17}$$

Definition 2.1 Given an interest rate r, a spot price p, a productive asset price A, a dividend rate δ and a wage rate w, a quadruple $\Big(c_j, l_j, \pi_j, \phi_j\Big)$ of consumption, leisure, productive asset share, and financial asset portfolio is *feasible for agent* j if (2.12), (2.13), (2.14), and (2.17) are satisfied, and her wealth X_j satisfies

$$\zeta(t)X_j(t) \geq -k(c_j, l_j, \pi_j, \phi_j), \qquad \forall t\in[0,T], \quad \text{a.s.,} \tag{2.18}$$

$$X_j(T) \geq 0, \quad \text{a.s.} \tag{2.19}$$

for some finite constant $k(c_j, l_j, \pi_j, \phi_j)$.

The condition (2.18) limits the debt the agent may incurr at any time, and (2.19) require that at the terminal time, when all debt must be liquidated, bankruptcy does not occur.

The *utility function* $U^j(t,c,l) : [0,T]\times[0,+\infty[\times[0,1] \to [-\infty,\infty[$ represents agent j's utility (at time t) of consumption at the rate $c \geq 0$ and of leisure at the rate $l \geq 0$. We assume that for each t, the function $U^j(t,\cdot,\cdot)$ is continuous, concave on its domain, i.e. where it is finite, and, on dom $\partial U^j(t,\cdot,\cdot)$ (i.e. where the subgradient set is non-empty, cf. the Appendix for convex analysis terminology), $U^j(t,\cdot,\cdot)$ is strictly concave, non-decreasing and twice continuously differentiable with

$$\lim_{c\to\infty} \sup_{\substack{0\leq l\leq 1\\ 0\leq t\leq T}} U_c^j(t,c,l) = 0. \tag{2.20}$$

At the boundaries we define derivatives by limits:

$$U_c^j(t,0,l) := \lim_{c\downarrow 0} U_c^j(t,c,l), \quad U_l^j(t,c,0) := \lim_{l\downarrow 0} U_l^j(t,c,l), \quad U_l^j(t,c,1) := \lim_{l\uparrow 1} U_l^j(t,c,l).$$

Furthermore for all $(c,l) \in]0,+\infty[\times]0,1]$ we require

$$\begin{aligned} \sup_{0\leq t\leq T} |U^j(t,c,l)| < \infty, \quad & \sup_{0\leq t\leq T} U_l^j(t,c,l) < \infty \\ \inf_{0\leq t\leq T} U_c^j(t,c,l) > 0, \quad & \sup_{0\leq t\leq T} U_c^j(t,c,l) < \infty \\ U_{c,l}^j(t,c,l) \geq 0. & \end{aligned} \tag{2.21}$$

The typical utility function we have in mind is of the Cobb-Douglas type, $U(t,c,l) = (c^p l^{1-p})^\gamma/\gamma,\ 0 < p < 1,\ 0 < \gamma < 1$, or of the separated type, $U = c^p + l^q$ with $p, q < 1$, possibly multiplied by a bounded positive function of t.

Definition 2.2 Given an interest rate r, a spot price p, a productive asset price A, a dividend rate δ and a wage rate w, a quadruple $\Big(c_j, l_j, \pi_j, \phi_j\Big)$ of consumption, leisure, productive asset portfolio, and financial asset portfolio is *optimal for agent* j if it is feasible and maximizes her expected total utility from consumption and leisure

$$E\Big\{\int_0^T U_j(t, c_j(t), l_j(t))dt\Big\}$$

over all feasible quadruples $\Big(c_j, l_j, \pi_j, \phi_j\Big)$ such that

$$E\Big\{\int_0^T (U_j)^-(t, c_j(t), l_j(t))dt\Big\} < +\infty. \tag{2.22}$$

The last condition is technical and states that the negative part of U, i.e. $\max\{-U, 0\}$, must be integrable.

3 Equilibrium

We now consider the market as a whole. The market maker, or just the market, must choose a price process p for the consumable, a wage process w, and a price process A for the productive asset. When the market is in equilibrium, certain relationships must hold between these quantities, the optimal labour process and the individual agents' optimal choices.

Definition 3.1 The market is in *equilibrium* if there exist a price process p, a wage process w, a productive asset price A and a dividends process δ, constants k_p, κ_p, κ_w, κ_δ, κ_A, depending on $p,\ w,\ \delta,\ A$, such that

$$0 < k_p < \zeta(t)p(t) < \kappa_p, \qquad \forall (t,\omega) \in [0,T]\times\Omega, \tag{3.1}$$

$$0 < \zeta(t)w(t) < \kappa_w, \qquad \forall (t,\omega) \in [0,T]\times\Omega, \tag{3.2}$$

$$0 \le \zeta(t)\delta(t) < \kappa_\delta, \qquad \forall (t,\omega) \in [0,T]\times\Omega, \tag{3.3}$$

$$0 \le \zeta(t)A(t) < \kappa_A, \qquad \forall (t,\omega) \in [0,T]\times\Omega. \tag{3.4}$$

There must also exist a labour process $\hat{L}$, and a family of quadruples $\Big(\hat{c}_j, \hat{l}_j, \hat{\pi}_j, \hat{\phi}_j\Big)_{j=1,\dots,J}$ such that

$$\Big(\hat{c}_j, \hat{l}_j, \hat{\pi}_j, \hat{\phi}_j\Big) \quad \text{is optimal for agent } j \text{ relative to } p, A, w, \delta, \quad j = 1, \dots, J; \tag{3.5}$$

$$\hat{L}(\cdot\,) \in \arg\max\Big\{E\Big\{\int_0^T \exp\Big[-\int_0^t r(s)ds\Big]\Big(p(t)R(t, L(t)) - w(t)L(t)\Big)dt\Big\} : \\ L \text{ measurable}, L(t) \in [0, L_{\max}]\Big\}; \tag{3.6}$$

$$\delta(t) = p(t)R(t, \hat{L}(t)) - w(t)\hat{L}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.}; \tag{3.7}$$

$$\sum_{j=1}^{J} \hat{c}_j(t) = R(t, \hat{L}(t)), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.}; \tag{3.8}$$

$$\sum_{j=1}^{J} \hat{l}_j(t) = J - \hat{L}(t), \quad \text{a.e. } t \in [0,T], \quad \text{a.s.}; \tag{3.9}$$

$$\sum_{j=1}^{J} \hat{\pi}_j(t) = 1, \quad \text{a.e. } t \in [0,T], \quad \text{a.s.}; \tag{3.10}$$

$$\sum_{j=1}^{J} \hat{\phi}_j(t) = \bar{0}_{N+1}^{\top}, \quad \text{a.e. } t \in [0,T], \quad \text{a.s.}; \tag{3.11}$$

where $\bar{0}_{N+1}$ is the $N+1$-dimensional vector with all components equal to 0.

In this definition, (3.5) requires that each agent acts optimally in equilibrium, (3.6) ensures that labour requirements are chosen to maximize the expected total discounted value of output, and (3.7) requires profits to be distributed as dividends. The remaining conditions are market clearing conditions. (3.8) requires clearing of the goods market, (3.9) of the labour market, (3.10) of the capital asset market, and (3.11) of the stock market.

By arguments similar to those used in [11], section 8, we can show the following.

Proposition 3.2 *(a) If A is the equilibrium price of the productive asset and (2.2), (2.3), (3.3), (3.4) hold, then*

$$\delta(t) + \beta(t) - r(t)A(t) - \alpha(t)\theta(t) = 0, \qquad \textit{Lebesgue} \times P - \textit{a.e.} \tag{3.12}$$

and

$$A(t) = \tilde{E}\Big\{ \int_t^T \exp\Big[- \int_t^s r(u)du \Big] \delta(s) ds \Big| \mathcal{F}_t \Big\} = \frac{1}{\zeta(t)} E\Big\{ \int_t^T \zeta(s)\delta(s) ds \Big| \mathcal{F}_t \Big\}, \quad t > 0, \tag{3.13}$$

i.e. $A(t)$ is the expected value under the risk-neutral probability of the discounted future dividend stream.

(b) If A is defined by (3.13) and the dividend process satisfies (3.3), then A satisfies (2.2), (2.3), (3.4) with

$$\begin{cases} \beta(t) = r(t)A(t) - \delta(t) + \exp\Big[\int_0^t r(s)ds \Big] H(t)\theta(t) \\ \alpha(t) = \exp\Big[\int_0^t r(s)ds \Big] H(t). \end{cases}$$

Here H is derived from the martingale representation

$$\tilde{E}\Big\{ \int_0^T \exp\Big[- \int_0^s r(u)du \Big] \delta(s) ds \Big| \mathcal{F}_t \Big\} = A(0) + \int_0^t H(s)\, d\tilde{W}(s).$$

Therefore, we conclude that once we find an equilibrium dividend process $\delta(\cdot)$, the productive asset price A as defined by (3.13) meets all the equilibrium requirements.

4 Solution of the Individual Agent Optimization Problem

Each individual agent $j = 1, \ldots, J$ takes as given the spot price p satisfying (3.1), the wage process w satisfying (3.2), the dividend process δ satisfying (3.3), and the productive asset price A eventually defined by (3.13), which are set by the market. Agent j only aims to maximizes her expected total utility from consumption and leisure (cf. Definition 2.2)

$$E\Big\{ \int_0^T U^j(t, c_j(t), l_j(t))dt \Big\}$$

over all feasible quadruples $\Big(c_j, l_j, \pi_j, \phi_j\Big)$.

Arguing as in Lemma 9.1 and Theorem 9.2 of [11] we can show

Lemma 4.1 *Assume (3.1) - (3.3) and (3.13).*

(a) If $\Big(c_j, l_j, \pi_j, \phi_j\Big)$ *is a feasible quadruple for agent j, then*

$$E\Big\{ \int_0^T \zeta(t)\Big[p(t)c_j(t) + w(t)l_j(t)\Big]dt \Big\} \leq E\Big\{ \int_0^T \zeta(t)\Big(\varepsilon_j\delta(t) + w(t)\Big)dt \Big\}. \tag{4.1}$$

(b) Conversely, if for agent j there exist a consumption process c_j satisfying (2.13) and a leisure process l_j satisfying (2.12) such that (4.1) holds, then it is possible to find a financial asset portfolio ϕ_j such that the quadruple $(c_j, l_j, \pi_j, \phi_j)$ is feasible with the productive asset share π_j constant, given by $\pi_j = \varepsilon_j$.

Proof: Recall that the wealth X_j of agent j, under the risk-neutral equivalent martingale probability measure $\tilde{P}$, satisfies the budget equation (2.17)

$$\begin{aligned} X_j(t) &= \varepsilon_j A(0) - \int_0^t p(s)c_j(s)ds + \int_0^t w(s)(1 - l_j(s))ds + \int_0^t r(s)X_j(s)ds \\ &\quad + \int_0^t \pi_j(s)\Big(\delta(s) + \beta(s) - r(s)A(s) - \alpha(s)\theta(s)\Big)ds \\ &\quad + \int_0^t \Big(\pi_j(s)\alpha(s) + \bar{\phi}_j(s)\mathrm{diag}[\bar{f}(s)]a(s)\Big)d\tilde{W}(s), \qquad t \in [0,T], \quad \text{a.s.}; \end{aligned}$$

that is (using Proposition 3.2),

$$\begin{aligned} X_j(t) &= \exp\Big[\int_0^t r(s)ds\Big]\Big\{\varepsilon_j A(0) + \int_0^t \exp\Big[-\int_0^s r(u)du\Big]\Big(w(s)(1 - l_j(s)) - p(s)c_j(s)\Big)ds \\ &\quad + \int_0^t \exp\Big[-\int_0^s r(u)du\Big]\Big(\pi_j(s)\alpha(s) + \bar{\phi}_j(s)\mathrm{diag}[\bar{f}(s)]a(s)\Big)d\tilde{W}(s)\Big\}, \qquad t > 0, \end{aligned} \tag{4.2}$$

and by using localization as in [11], Lemma 9.1, and (2.18), (2.19), i.e. feasibility, it follows that

$$E\Big\{ \int_0^T \zeta(t)p(t)c_j(t)dt \Big\} \leq \varepsilon_j A(0) + E\Big\{ \int_0^T \zeta(t)w(t)(1 - l_j(t))dt \Big\}.$$

But

$$A(0) = E\Big\{\int_0^T \zeta(t)\delta(t)dt\Big\}, \tag{4.3}$$

as follows from (3.13) with $t = 0$, and hence (4.1) is proved.

From (3.2), (3.3)and (4.1) it follows that

$$Q_j \equiv \int_0^T \exp\Big[-\int_0^s r(u)du\Big]\Big(w(s)(1-l_j(s)) - p(s)c_j(s)\Big)ds$$

is $\tilde{P}$-integrable. As in [11], Theorem 9.2, the martingale representation result gives a process $\tilde{H}$ such that

$$\tilde{E}(Q_j|\mathcal{F}_t) = \tilde{E}(Q_j) + \int_0^t \tilde{H}(s)\,d\tilde{W}(s).$$

Now set

$$\begin{aligned}\bar{\phi}_j(t) &= -\Big\{\exp\Big[\int_0^t r(u)du\Big]\tilde{H}(t) + \varepsilon_j\alpha(t)\Big\}a(t)^{-1}(\mathrm{diag}[\bar{f}(t)])^{-1} \\ \phi_{j,0}(t) &= [X_j(t) - \varepsilon_j A(t) - \bar{\phi}_j(t)\bar{f}(t)]/f_0(t).\end{aligned} \tag{4.4}$$

Also from (4.3) and (4.1) it follows that $\varepsilon_j A(0) + \tilde{E}Q_j \geq 0$. Manipulating (4.2) as in Theorem 9.2 of [11], allows us to use this result to show feasibility, cf. Definition 2.1. Hence (b) holds. □

It follows that the optimization problem of agent j may be reformulated as a maximization problem not involving the productive asset, that is,

$$\begin{cases}\max E\Big\{\int_0^T U^j(t, c_j(t), l_j(t))dt\Big\} \\ \textit{subject to } (2.12), (2.13), (2.22), \textit{and } (4.1).\end{cases} \tag{4.5}$$

Notice that the financial parameters enter this new formulation only through the state-price density ζ in the constraint (4.1). As we shall see in Corollary 4.4, the optimal holdings of real assets is $\pi_j(t) = \varepsilon_j$, and of financial assets it is $\phi_j(t)$ given by (4.4) when the optimal consumption and leisure strategies are used in the determination of Q_j, i.e. of $\tilde{H}$. Then the holdings of the risky asset relate to the excess of wage income over consumption expenditure and to the growth rate of the real asset held by the agent.

We now wish to study the inverse of $\nabla_{c,l}U^j$. Let $\mathcal{D} :=]0,\infty[\times]0,1[\subset \mathrm{dom}\,\partial U^j(t,\cdot,\cdot)$. Strict concavity of U^j implies that the mapping $(c,l) \to \nabla_{c,l}U^j(t,c,l)$ is invertible on its image $\mathcal{R}^j \subset \mathbb{R}^2_+ \equiv \{y \in \mathbb{R}^2 : y_i > 0,\ i = 1,2\}$. We write $\bar{\mathcal{D}}$ for $[0,\infty[\times[0,1]$, the closure of $\mathcal{D}$. $U^j(t,\cdot,\cdot)$ is defined on this set although it may possibly assume the value $-\infty$, eg. $U^j(t,c,l) = \log c + \log l$. According to Lemma 9.1 in the Appendix, there exists a continuous function $I^j(t,\cdot) : \mathbb{R}^2_+ \mapsto \bar{\mathcal{D}}$ which extends $(\nabla_{c,l}U^j(t,\cdot,\cdot))^{-1}$. It is continuously differentiable on $\mathcal{R}^j$.

A little thought shows that $\mathcal{R}^j$ is the closed (relative to $\mathbb{R}^2_+$) set bounded by the parametric curves (recall that t is fixed)

$$\mathcal{C}_1^j := \{y = \nabla_{c,l}U^j(t,c,1) : c \geq 0\}, \qquad \mathcal{C}_2^j := \{y = \nabla_{c,l}U^j(t,0,l) : 0 \leq l \leq 1\},$$

$$\mathcal{C}_3^j := \{y = \nabla_{c,l} U^j(t,c,0) : c \geq 0\}.$$

Moreover, since $U^j_{c,l} \geq 0$, then $\mathcal{C}_i^j$ can be represented as $y_2 = \psi_i(y_1),\ i = 1,3$ and $y_1 = \psi_2(y_2)$ with ψ_i non-increasing. Then $\mathbb{R}^2_+$ decomposes into six open sets plus their boundaries, $\mathcal{C}_i^j$, as indicated in Figure 1. Of course, the decomposition may degenerate into fewer sets if $U^j_c(t,0,l) = \infty$ or $U^j_l(t,c,0) = \infty$.

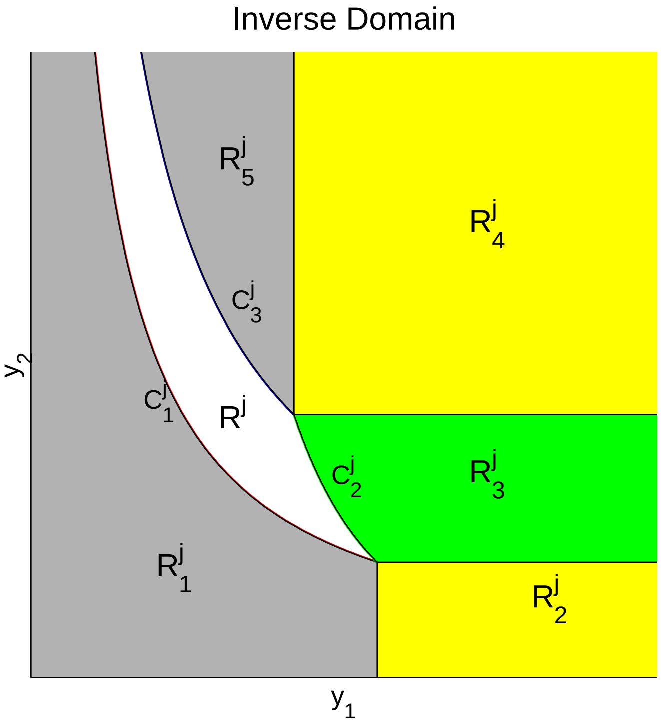

Figure 1

Let $\mathcal{I}_1^j(t,y_1,l)$ be the inverse of $c \mapsto U_c^j(t,c,l)$ for fixed $t,\ l$. This function is decreasing on $[0, U_c^j(t,0,l)]$ for l such that $(\mathcal{I}_1^j(t,y_1,l), l) \in \operatorname{dom} \partial U^j(t,\cdot,\cdot)$, i.e. for $y \in \mathcal{R}_1^j \cup \mathcal{R}_5^j \cup \mathcal{R}^j$, cf. Figure 1. Also let $\mathcal{I}_2^j(t,y_2)$ be the inverse of $l \mapsto U_l^j(t,0,l)$ for fixed t. This function is also decreasing on $[U_l^j(t,0,1), U_l^j(t,0,0)]$ for $(0, \mathcal{I}_2^j(t,y_2) \in \operatorname{dom} \partial U^j(t,\cdot,\cdot)$, i.e. for $y \in \mathcal{R}_3^j \cup \mathcal{R}^j$.

Corollary 9.2 tells us that $y \to I^j(t,y)$ is differentiable on each $\mathcal{R}_i^j$, and moreover

$$I^j(t,y) = \begin{cases} (\mathcal{I}_1^j(t,y_1,1),1) & y \in \mathcal{R}_1^j \\ (0,1) & y \in \mathcal{R}_2^j \\ (0,\mathcal{I}_2^j(t,y_2)) & y \in \mathcal{R}_3^j \\ (0,0) & y \in \mathcal{R}_4^j \\ (\mathcal{I}_1^j(t,y_1,0),0) & y \in \mathcal{R}_5^j \\ (\nabla_{c,l} U^j(t,\cdot,\cdot))^{-1}(y) & y \in \mathcal{R}^j \end{cases} \tag{4.6}$$

If we now consider problem (4.5) without the constraints (2.12), (2.13), and (2.22) for the time being, then we maximize

$$E\Big\{ \int_0^T U^j(t, x(t))dt \Big\}$$

subject to (4.1), i.e.

$$E\Big\{ \int_0^T \zeta(t)(p(t), w(t))\cdot x(t)dt \Big\} \leq \xi_j$$

where

$$\xi_j := E\Big\{ \int_0^T \zeta(t)[\varepsilon_j \delta(t) + w(t)]dt \Big\}. \tag{4.7}$$

If $\xi_j = 0$, then $w = 0$ on a set of full measure and this contradicts (3.2), so $\xi_j > 0$.

A Lagrange multiplier argument indicates that the solution is given by $(c(t), l(t)) = I^j\Big(t, \eta_j(\zeta(t)p(t), \zeta(t)w(t))\Big)$ where the multiplier η_j must satisfy

$$\mathcal{X}_j(\eta) := E\left\{ \int_0^T \zeta(t)(p(t), w(t)) \cdot I^j(t, \eta\zeta(t)(p(t), w(t))))\, dt \right\} = \xi_j. \tag{4.8}$$

It is clear from (3.3) and (3.2) that ξ_j is finite. To obtain the same for $\mathcal{X}_j(\cdot\,)$ we interpose a Lemma.

Lemma 4.2 *For any $k < \infty$ there exists a constant $\bar{c}(k)$ such that for $t \in [0, T],\ y_1 \geq k,\ y_2 > 0$*

$$I_1^j(t, y) \leq \bar{c}(k).$$

Proof: Concavity of $(c, l) \mapsto U^j(t, c, l)$ implies that $U_c^j(t, c, l)$ is non-increasing in c. By (2.20) there exists $\bar{c}$ such that $0 \leq c \leq \bar{c}$ if $U_c^j(t, c, l) \geq k$. Observe that $I^j(t, y) = (c, l)$ if $y = \nabla_{c,l} U^j(t, c, l)$, so for $k \leq y_1 = U_c^j(t, c, l)$ we have $I_1^j(t, y) = c \leq \bar{c}$ and the result follows. □

With $y(t) = (\zeta(t)p(t), \zeta(t)w(t))$, we have $\eta y_1(t) \geq \eta k_p$, cf. (3.1), so by the above Lemma, $I_1^j(t, \eta y(t)) \leq \bar{c}(\eta k_p)$. (Here numeric subscripts denote components.) Since $I_2^j(t, \eta y(t)) \in [0, 1]$, then for η fixed, $I^j(t, \eta y(t))$ is bounded uniformly in t. Since also $y(t)$ is bounded, cf. (3.1) and (3.2), then the finiteness of $\mathcal{X}_j$ follows.

Now we must show that (4.8) has a solution. From Corollary 9.3 we know that $\eta \rightarrow y(t) \cdot I^j(t, \eta y(t))$ is continuous, non-increasing on $]0, \infty[$ and decreasing on $]0, \eta_y(t)[\cap\{\eta : \eta y(t) \notin \mathcal{R}_2^j\}$ where

$$\eta_y(t) = \sup\{\eta : y(t) \cdot I^j(t, \eta y(t)) > 0\},$$

and $\lim_{\eta\downarrow 0} y \cdot I^j(t, \eta y(t)) = \infty$. This implies, as in [11], Lemma 9.3, the following technical result.

Lemma 4.3 *$\mathcal{X}_j$ maps $]0, \infty[$ into $]0, \infty[$, is continuous and non-increasing. Let $\bar{\eta}_j := \sup\{\eta > 0 : \mathcal{X}_j(\eta) > 0\}$. Then $\mathcal{X}_j$ is decreasing on $]0, \bar{\eta}_j[\cap\{\eta : \mathcal{X}_j(\eta) \neq E\int_0^T \zeta(t)w(t)\, dt\}$ and*

$$\lim_{\eta\downarrow 0} \mathcal{X}_j(\eta) = \infty, \qquad \lim_{\eta\uparrow\bar{\eta}_j} \mathcal{X}_j(\eta) = 0.$$

Hence there exists an η_j satisfying (4.8) which moreover is unique unless $\varepsilon_j\delta(t) = 0$ a.e., a.s. and

$$\frac{w(t)}{p(t)} < \frac{U_l^j(t,0,1)}{U_c^j(t,0,1)} \quad \text{a.e, a.s.}$$

The economic interpretation of the last case is that if the real wage is is less than the ratio of the agents marginal utilities of leisure to that of consumption at zero consumption and total leisure, and if the agent has no investment income, then the multiplier may be non-unique.

Corollary 4.4 *Assume (3.1) - (3.3), (3.13). Then there exists a unique optimal consumption-leisure strategy for agent j given by*

$$(\hat{c}_j(t), \hat{l}_j(t)) = I^j(t, \eta_j\zeta(t)p(t), \eta_j\zeta(t)w(t)), \qquad t \in [0,T]. \tag{4.9}$$

The corresponding productive asset share and financial asset portfolio are given by Lemma 4.1 (b).

Proof: The above discussion shows that (4.9) solves (4.5) once we establish that (2.12), (2.13) and (2.22) hold. But (2.13) is satisfied thanks to Lemma 4.2 and (3.1), and (2.12) follows from the construction of I^j. (2.22) also holds by comparison with a constant consumption policy, cf. [11], Theorem 9.4, for details. □

Remark 4.5 Note that in case η_j is non-unique, $(\hat{c}_j(t), \hat{l}_j(t)) = (0,1)$, hence is still unique.

5 The Representative Agent's Optimization Problem

The actions of the individual agents can be aggregated and represented as the action of a single fictitious *representative agent.* His utility function must opportunely weight the utility functions of all the agents in the economy; the factor Λ below will accomplish this. It will be arbitrary to begin with, but in the end it will be chosen so as to produce an equilibrium.

For $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J) \in (0,\infty)^J$ let us define the function $U(t,c,l;\Lambda)$ as follows.

$$U(t,c,l;\Lambda) := \sup_{\substack{c_j \geq 0,\ \sum_{j=1}^J c_j = c \\ l_j \in [0,1], \sum_{j=1}^J l_j = l}} \sum_{j=1}^J \lambda_j U^j(t,c_j,l_j). \tag{5.1}$$

Due to the finite nature of the interval on which l lies we must add a hypothesis to guarantee that we have a good representation of $\nabla_{c,l}U^{-1}$. We **assume** that $U_l^j(t,c,0) = +\infty$ for all j,t,c. In Lemma 9.4 some properties of U are established which we recall here. Let $\mathcal{D}_o =]0,\infty[\times]0,J[$.

For each $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J) \in (0,\infty)^J$, $U(\cdot\,;\Lambda) : [0,T] \times [0,\infty[\times[0,J] \mapsto \mathbb{R}$ is measurable, *increasing, strictly concave, continuously differentiable with respect to* (c,l) *on* $\mathcal{D}_o$*. For each* t, $\nabla U(t,\cdot\,;\Lambda)$ *is* $\mathbb{R}_+^2$*-valued and its inverse exists and can be extended as*

$$I^\Lambda(t,y)=\sum_j I^j(t,\frac{y}{\lambda_j}),$$

a continuous mapping with $\mathbb{R}_+^2 \subset \operatorname{dom} I^\Lambda(t,\cdot\,)\subset \bar{\mathbb{R}}_+^2$ *and with image* $= \operatorname{dom}\partial U(t,\cdot;\Lambda)\supset \mathcal{D}_o$*. Then for* y *such that* $I^\Lambda(t,y)\in\mathcal{D}_o$

$$\nabla U(t,I^\Lambda(t,y);\Lambda)=y. \tag{5.2}$$

Moreover, for each $(c,l)\in\operatorname{dom}\partial U(t,\cdot;\Lambda)$*, there exist* $(\bar{c}_j,\bar{l}_j)\in\operatorname{dom}\partial U^j(t,\cdot)$ *such that*

$$U(t,c,l;\Lambda)=\sum_{j=1}^J \lambda_j U^j(t,\bar{c}_j,\bar{l}_j),$$

and

$$\begin{cases} U_c(t,c,l;\Lambda)=\lambda_j U_c^j(t,\bar{c}_j,\bar{l}_j)+r_3^j & j=1,\dots,J,\\ U_l(t,c,l;\Lambda)=\lambda_j U_l^j(t,\bar{c}_j,\bar{l}_j)-r_1^j & j=1,\dots,J,\end{cases} \tag{5.3}$$

where $r_i^j\ge 0,\ (1-\bar{l}_j)r_1^j=0,\ \bar{c}_j r_3^j=0$*. Finally* $U_l(t,c,0;\Lambda)=+\infty$*.*

We can think of the representative agent as an agent whose income process is given by the *aggregate income process* $\hat{e}(t)=w(t)(J-l(t))+\delta(t)$ of (2.16). Then his wealth is

$$\begin{aligned} X(t) &= \sum_{j=1}^J \pi_j(t)A(t)+\sum_{j=1}^J \phi_j(t)f(t)\\ &= A(0)-\int_0^t p(s)c(s)ds+\int_0^t \Big(w(s)(J-l(s))+r(s)X(s)\Big)\,ds \\ &\quad+\int_0^t \Big(\sum_j \pi_j(s)\alpha(s)+\sum_j \bar{\phi}_j(s)\operatorname{diag}[\bar{f}(s)]a(s)\Big)d\tilde{W}(s), \qquad t\in[0,T], \quad \text{a.s.}\end{aligned} \tag{5.4}$$

where c and l are this agent's consumption and leisure processes. We are assuming (3.1) - (3.3) and (3.13). We also require

$$\sup_{0\le t\le T} c(t)<\infty, \tag{5.5}$$

$$E\left\{\int_0^T U^-(t,c(t),l(t);\Lambda)dt\right\}<\infty, \tag{5.6}$$

Then, by analogy with Section 4, the agent's optimization problem may be formulated as follows,

$$\begin{cases} \displaystyle\max_{c\ge 0,\ l\in[0,J]} E\Big\{\int_0^T U(t,c(t),l(t);\Lambda)dt\Big\} \\ \textit{subject to } (5.5),\ (5.6) \textit{ and } (5.8)\end{cases} \tag{5.7}$$

where

$$E\Big\{\int_0^T \zeta(t)\Big[p(t)c(t)+w(t)l(t)\Big]dt\Big\} \le E\Big\{\int_0^T \zeta(t)\Big(\delta(t)+Jw(t)\Big)dt\Big\}. \tag{5.8}$$

We like to think that $X(t)=\sum_j X_j(t)$ and $(c(t),l(t))=\sum_j(c_j(t),l_j(t))$, but this cannot be assumed à priori.

Problem (5.7) is of the form (4.5), and can be solved in the same way. Note that

$$\begin{aligned}
\mathcal{X}(\eta) &= E\left\{\int_0^T \zeta(t)(p(t),w(t))\cdot I^\Lambda(t,\eta\zeta(t)p(t),\eta\zeta(t)w(t))\,dt\right\}\\
&= \sum_j E\left\{\int_0^T \zeta(t)(p(t),w(t))\cdot I^j(t,\eta\zeta(t)p(t)/\lambda_j,\eta\zeta(t)w(t)/\lambda_j;\Lambda)\,dt\right\}\\
&= \sum_j \mathcal{X}_j(\frac{\eta}{\lambda_j}).
\end{aligned}$$

Note also that $\delta\neq 0$, cf.Remark 6.2, so the multiplier will be unique. Then from Lemma 4.3 it follows that $\mathcal{X}$ maps $]0,\infty[$ onto $[0,\infty[$, is continuous and decreasing. Hence there is a solution $\eta(\Lambda)$ of

$$\mathcal{X}(\eta)=E\Big\{\int_0^T \zeta(t)\Big(\delta(t)+Jw(t)\Big)dt\Big\} \tag{5.9}$$

and it is unique. We may conclude that there exists a unique optimal consumption-leisure strategy for the central planner given by

$$\begin{aligned}
(\hat{c}(t;\Lambda),\hat{l}(t;\Lambda)) &= I^\Lambda(t,\eta(\Lambda)\zeta(t)p(t),\eta(\Lambda)\zeta(t)w(t))\\
&= \sum_{j=1}^J I^j\Big(t,\frac{1}{\lambda_j}\eta(\Lambda)\zeta(t)p(t),\frac{1}{\lambda_j}\eta(\Lambda)\zeta(t)w(t)\Big), \quad t\in[0,T],
\end{aligned} \tag{5.10}$$

with $\eta(\Lambda)$ uniquely defined by

$$\begin{aligned}
&E\Big\{\int_0^T (\zeta(t)p(t),\zeta(t)w(t))\cdot I^\Lambda(t,\eta\zeta(t)p(t),\eta\zeta(t)w(t))\,dt\Big\}\\
&\qquad = E\Big\{\int_0^T \zeta(t)\Big(\delta(t)+Jw(t)\Big)dt\Big\}.
\end{aligned} \tag{5.11}$$

We observe that if η_j is a solution of (4.8), $j=1,\ldots,J$ and if we set $\Lambda=(\lambda_1,\ldots,\lambda_J)=(\eta_1^{-1},\ldots,\eta_J^{-1})$, then $\eta(\Lambda)=1$ is the unique solution of (5.9). This leads us to the first *necessary condition* for the existence of an equilibrium.

Proposition 5.1 *If* $\Big[p,A,w,\delta,\hat{L},\Big(\hat{c}_j,\hat{l}_j,\hat{\pi}_j,\hat{\phi}_j\Big)_{j=1,2,\ldots,J}\Big]$ *is an equilibrium, if* η_j *is defined by (cf. (4.8))*

$$E\Big\{\int_0^T \Big(\zeta(t)p(t),\zeta(t)w(t)\Big)\cdot I_j\Big(t,\eta_j\zeta(t)p(t),\eta_j\zeta(t)w(t)\Big)dt\Big\} = E\Big\{\int_0^T \zeta(t)\Big(\varepsilon_j\delta(t)+w(t)\Big)dt\Big\}$$

for $j = 1, 2, \ldots, J$, and if $\Lambda \in (0,\infty)^J$ is defined as $\Lambda = (1/\eta_1, 1/\eta_2, \ldots, 1/\eta_J)$, then necessarily the equilibrium spot price and wage rate are given by

$$(p(t), w(t)) = \frac{1}{\zeta(t)} \nabla_{c,l} U\Big(t, R(t, \hat{L}(t)), J - \hat{L}(t); \Lambda\Big), \quad t \in [0,T]. \tag{5.12}$$

Proof: (3.8) and (3.9), (4.9) and Lemma 9.4 imply

$$\begin{aligned} (R(t,\hat{L}(t)), J - \hat{L}(t)) &= \sum_{j=1}^{J} (\hat{c}_j, \hat{l}_j) = \sum_{j=1}^{J} I^j(t, \eta_j \zeta(t) p(t), \eta_j \zeta(t) w(t)) \\ &= I^{\Lambda}(t, \zeta(t)p(t), \zeta(t)w(t)). \end{aligned} \tag{5.13}$$

Since $(p(t), w(t) \in \mathbb{R}_+^2$, then $\hat{L}(t) = I^L(t, w(t)/p(t)) > 0$, cf. (6.1) below, hence $R(t, \hat{L}(t)) > 0$ and $J - \hat{L}(t) \in]J - L_{\max}, J[$. So $(R(t, \hat{L}(t)), J - \hat{L}(t)) \in \mathcal{D}_o$ and (5.2) implies (5.12). □

6 The Manager's Optimization Problem

The manager of the productive asset aims to maximize the profits. Specifically, he will determine the labour requirement at each time t so as to maximize the net value per unit time of production, i.e.

$$\sup_{0 \le L \le L_{\max}} \{p(t)R(t,L) - w(t)L\}. \tag{6.1}$$

Note that $J - l \in [0, J]$ represents the supply of labour and $L \in [0, L_{\max}]$ the demand for labour.

We can define $I^L(t, \cdot\,)$, an extension of the inverse function of $R_L(t, \cdot\,)$, as was done with I^j, cf. Lemma 9.1. This is a continuous, non-increasing function, defined on $\mathbb{R}_+$, taking values in $]0, L_{\max}]$, decreasing and differentiable on $]R_L(t, L_{\max}), R_L(t, 0)[$. Then we have

Lemma 6.1 *The* sup *in (6.1) is attained at*

$$\hat{L}(t) = I^L\Big(t, \frac{w(t)}{p(t)}\Big), \qquad t \in [0,T]. \tag{6.2}$$

Proof: From (3.1) and (3.2) it follows that for each t,

$$\frac{w(t)}{p(t)} \in \mathbb{R}_+$$

so the result follows from elementary calculus. □

Remark 6.2 The strict concavity of R implies that $\delta > 0$. In fact

$$\delta = p[R(\hat{L}) - \frac{w}{p}\hat{L}] > pR(0) \ge 0.$$

Remark 6.3 This result allows a stochastic equivalent to a classical result. Assume that $R_L(t, L(t))$ is a semi-martingale and p satisfies

$$\frac{dp}{p} = \Pi\, dt + \sigma_p\, dW(t)$$

(Π is the inflation rate). Then $R_L(t, L(t)) = w(t)/p(t)$ and stochastic calculus implies

$$\frac{dw}{w} - \Pi\, dt = \frac{dR_L}{R_L} + \frac{d < R_L, p >}{R_L p} + \sigma_p\, dW(t).$$

In other words, the growth rate of real wages is equal to the growth rate of marginal productivity of labour plus a stochastic correction plus a zero-mean martingale. The deterministic equivalent, where the stochastic correction and the martingale are zero, can be found in Claassen [3], p. 172.

We can now establish a link between this problem and that of the representative agent at least in equilibrium with Λ as in Proposition 5.1. From (5.13) and (5.10) it follows that

$$\begin{aligned}(R(t, \hat{L}(t)), J - \hat{L}(t)) &= I^{c,l}\Big(t, \nabla_{c,l} U(t, \hat{c}(t; \Lambda), \hat{l}(t; \Lambda); \Lambda); \Lambda\Big) \\ &= \Big(\hat{c}(t; \Lambda), \hat{l}(t; \Lambda)\Big),\end{aligned}$$

so that the optimal utility of the representative agent is

$$U(t, \hat{c}(t; \Lambda), \hat{l}(t; \Lambda); \Lambda) = U(t, R(t, \hat{L}(t)), J - \hat{L}(t); \Lambda).$$

We shall think of this as the *utility of labour* and write it as

$$\hat{U}(t, L; \Lambda) := U(t, R(t, L), J - L; \Lambda).$$

Lemma 6.4 *$\hat{U}(t, L; \Lambda)$ is a strictly concave function of L on $[0, L_{\max}]$. Moreover*

$$\sup_{0 \le L \le L_{\max}} \hat{U}(t, L; \Lambda)$$

is attained at

$$L(t; \Lambda) = \begin{cases} 0 & \text{if } \hat{U}_L(t, 0; \Lambda) \le 0 \\ L_{\max} & \text{if } \hat{U}_L(t, L_{\max}; \Lambda) \ge 0 \\ \tilde{L}(t) & \text{otherwise} \end{cases} \qquad t \in [0, T] \tag{6.3}$$

where $\tilde{L}(t)$ is the unique solution of $\hat{U}_L(t, L; \Lambda) = 0$.

Proof: A differentiable function $f(x)$ is strictly concave if and only if for all $x, x' \in \text{dom } f$ with $x \neq x'$

$$f(x') - f(x) < (x' - x) \cdot \nabla f(x),$$

cf. 9.1). From this representation follows easily that strict concavity of U, concavity of R and positivity of the partial U_c imply strict concavity of $\hat{U}$.

Now (6.3) follows. □

Remark 6.5 The function $\tilde{L}(t)$, hence $L(t;\Lambda)$, is measurable in t since U is, cf. Lemma 9.6.

Corollary 6.6 *If (5.12) holds with $\hat{L}$ defined by (6.2), then the solution of the manager's problem maximizes the utility of labour, i.e. $\hat{L}(t) = L(t;\Lambda)$.*

Proof: We first show that L as defined by (6.3) is the unique solution of (6.4) below. Fix t and let ∂ denote subgradient with respect to L, cf. the Appendix. Let $N_C(x)$ denote the normal cone of $[0, L_{\max}]$ at x, and $R_o(t,\cdot\,)$ denote $R(t,\cdot\,)$ extended as $-\infty$ off $[0, L_{\max}]$. Since $L(t;\Lambda)$ maximizes $\hat{U}$ over $[0, L_{\max}]$, then we know from convex analysis that

$$\begin{aligned} 0 \in\ & \partial\hat{U}(t, L(t;\Lambda)) + N_C(L(t;\Lambda)) = \\ & U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)\partial R_o(t, L(t;\Lambda)) - U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda), \end{aligned}$$

cf. [4], Theorem 2.3.9(i) Corollary 2.4.3, Proposition 2.4.4, since $U_c\partial R + N_C = U_c\partial R_o$. Hence

$$\frac{U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)}{U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)} \in \partial R_o(t, L(t;\Lambda))$$

or equivalently (since $v \in \partial R_o(t, L)$ if and only if $L \in \partial R_o^*(t, v) = I^L(t, v)$, cf. the proof of Lemma 9.1)

$$L(t;\Lambda) = I^L\Big(t, \frac{U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)}{U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)}\Big). \tag{6.4}$$

In other words, (6.3) gives (uniquely) the solution of (6.4).

It follows from (5.12) and (6.2) that $\hat{L}(t)$ also satisfies (6.4), hence $\hat{L}(t) = L(t;\Lambda)$. □

This gives us more complete *necessary conditions* for the existence of an equilibrium.

Proposition 6.7 *If $\Big[p, A, w, \delta, \hat{L}, \Big(\hat{c}_j, \hat{l}_j, \hat{\pi}_j, \hat{\phi}_j\Big)_{j=1,2,\ldots,J}\Big]$ is an equilibrium, if η_j is defined by (cf. (4.8))*

$$E\Big\{\int_0^T \Big(\zeta(t)p(t), \zeta(t)w(t)\Big)\cdot I_j\Big(t, \eta_j\zeta(t)p(t), \eta_j\zeta(t)w(t)\Big)dt\Big\} = E\Big\{\int_0^T \zeta(t)\Big(\varepsilon_j\delta(t) + w(t)\Big)dt\Big\}$$

for $j = 1, 2, \ldots, J$, and if $\Lambda \in (0,\infty)^J$ is defined as $\Lambda = (1/\eta_1, 1/\eta_2, \ldots, 1/\eta_J)$, then necessarily

- *the equilibrium labour rate $\hat{L}(t)$ is given uniquely by (6.3)*
- *the equilibrium spot price and wage rate are given by*

$$(p(t), w(t)) = \frac{1}{\zeta(t)}\nabla_{c,l}U\Big(t, R(t, \hat{L}(t)), J - \hat{L}(t);\Lambda\Big), \quad t \in [0,T]. \tag{6.5}$$

Remark 6.8 In this model, the market is in equilibrium if at each time t

- the representative agent's utility is maximized, cf. (5.7), which is equivalent to maximization of each agent's utility,

- the utility of labour is maximized, cf. Corollary 6.6,
- the value of the productive asset is maximized.

Only the last statement requires substantiation. Observe that (6.1) is equivalent to maximizing for each t or just for $t=0$ the expression

$$\tilde{E}\left\{ \int_t^T e^{-\int_t^s r(u)\,du}[p(s)R(s,L(s)) - w(s)L(s)]\,ds \middle| \mathcal{F}_t \right\}$$

Now (3.7) and (3.13) imply that the central planner maximizes $A(t)$, the price or value of the productive asset, at each time.

Remark 6.9 Although we have not done so, there is no difficulty in allowing randomness to enter into R. We simply assume that for each L the process $R(t,L)$ is $\mathcal{F}$-adapted and $R(t,L)$ satisfies all the earlier assumptions a.s. The reason for not doing so is that the randomness in the financial markets is of a much higher order than that in the rest of the economy. Nevertheless, innovation in technology is a random event and should be considered at some point.

7 Existence of Equilibria

We can now establish existence of equilibrium in the economy. Recall that $a,\ b,\ r$ hence θ and ζ are given exogenously. Moreover, once δ is known, then A can be defined by (3.13) and $\alpha,\ \beta$ from Proposition 3.2. For the rest we have the following sufficient condition.

Proposition 7.1 [cf. [11], Thm. 10.2, p.106] *For $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_J) \in (0,\infty)^J$ define $L(t;\Lambda)$ by (6.3) , and define a spot price process $p(t;\Lambda)$ by*

$$p(t;\Lambda) := \frac{1}{\zeta(t)} U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda), \tag{7.1}$$

a wage process $w(t;\Lambda)$ by

$$w(t;\Lambda) := \frac{1}{\zeta(t)} U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda), \tag{7.2}$$

and a dividends process $\delta(t;\Lambda)$ by

$$\delta(t;\Lambda) := p(t;\Lambda) R(t, L(t;\Lambda)) - w(t;\Lambda) L(t;\Lambda). \tag{7.3}$$

For $j = 1, 2, \dots, J$, let $\eta_j(\Lambda)$ be defined by (cf. (4.8))

$$\begin{aligned} & E\Big\{ \int_0^T \nabla_{c,l} U\Big(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda\Big) \cdot \\ & \qquad I^j\Big(t, \eta_j(\Lambda) \nabla_{c,l} U(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda); \Lambda\Big) dt \Big\} \qquad (7.4) \\ = \quad & E\Big\{ \int_0^T \zeta(t)(\delta(t;\Lambda)\varepsilon_j + w(t;\Lambda)) dt \Big\} \end{aligned}$$

and $(\hat{c}_j(t;\Lambda), \hat{l}_j(t;\Lambda))$ *by (cf. (4.9))*

$$(\hat{c}_j(t;\Lambda), \hat{l}_j(t;\Lambda)) := I^j\Big(t, \eta_j(\Lambda)\nabla_{c,l}U(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda); \Lambda\Big). \tag{7.5}$$

If

$$\lambda_j\eta_j(\Lambda) = 1 \quad \forall j = 1, 2, \ldots, J, \tag{7.6}$$

then $\Big[p, A, w, \delta, L, \Big(\hat{c}_j, \hat{l}_j, \hat{\pi}_j, \hat{\phi}_j\Big)_{j=1,\ldots,J}\Big]$ *is an equilibrium with*

$$\begin{cases} A(t;\Lambda) = \dfrac{1}{\zeta(t)}E\Big\{ \displaystyle\int_t^T \zeta(s)\delta(s;\Lambda)ds\Big|\mathcal{F}(t)\Big\}, \\ \hat{\pi}_j(t) = \varepsilon_j, \\ \hat{\phi}_j(t) \quad \textit{defined by (4.4)} \end{cases} \tag{7.7}$$

for $j = 1, 2, \ldots, J$, *for almost every* $t \in [0, T]$, *almost surely.*

Proof: First we show that (3.1)-(3.4) hold. For each j, $\bar{c}_j \le \kappa_R$ and $\sum_j \bar{l}_j = J - L(t;\Lambda) \ge J - L_{\max}$, so

$$\begin{aligned} \zeta(t)p(t;\Lambda) &= U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda) \\ &\ge \lambda_j U_c^j(t, \bar{c}_j, \bar{l}_j) \quad \text{for all } j, \text{ cf. } (5.3) \\ &\ge \lambda_j U_c^j(t, \kappa_R, \bar{l}_j) \quad \text{for all } j \\ &\ge \lambda_j U_c^j\Big(t, \kappa_R, 1 - \frac{L_{\max}}{J}\Big) \quad \text{for at least one } j \\ &\ge \inf_{t,j} \lambda_j U_c^j\Big(t, \kappa_R, 1 - \frac{L_{\max}}{J}\Big) \\ &:= k_p > 0. \end{aligned}$$

Similarly, for at least one j, $\bar{l}_j \ge 1 - L_{\max}/J > 0$ so $r_2^j = 0$,

$$\begin{aligned} \zeta(t)w(t;\Lambda) &= U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda); \Lambda) \\ &\le \lambda_j U_l^j(t, \bar{c}_j, \bar{l}_j) + r_2^j \quad \text{for all } j, \text{ cf. } (5.3) \\ &\le \lambda_j U_l^j\Big(t, \kappa_R, 1 - \frac{L_{\max}}{J}\Big) \quad \text{for at least one } j \\ &\le \sup_{t,j} \lambda_j U_l^j\Big(t, \kappa_R, 1 - \frac{L_{\max}}{J}\Big) \\ &:= \kappa_w. \end{aligned} \tag{7.8}$$

(2.1) implies that there exists $L_o > 0$ such that if $L < L_o$ then $R_L(t, L) > \kappa_w/k_p$, i.e. $L < I^L(t, \kappa_w/k_p)$ for all t. Hence

$$L(t;\Lambda) = I^L\left(t, \frac{w(t)}{p(t)}\right)$$

$$
\begin{aligned}
&\geq \; I^L\left(t, \frac{\kappa_w}{k_p}\right) \\
&\geq \; L_o > 0.
\end{aligned}
$$

Then $R(t, L(t;\Lambda)) \geq \inf_t R(t, L_o) := k_R > 0$ (again using (2.1)), so $\sum_j \bar{c}_j \geq k_R > 0$ and then

$$
\begin{aligned}
\zeta(t)p(t;\Lambda) \;&=\; U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda) \\
&=\; \lambda_j U_c^j(t, \bar{c}_j, \bar{l}_j) + r_3^j \quad \text{for all } j, \text{ cf. (5.3)} \\
&\leq\; \lambda_j U_c^j\Big(t, \frac{k_R}{J}, \bar{l}_j\Big) \quad \text{for at least one } j \\
&\leq\; \lambda_j U_c^j\Big(t, \frac{k_R}{J}, 1\Big) \quad \text{since } U_{c,l}^j \geq 0 \\
&\leq\; \sup_{t,j} \lambda_j U_c^j\Big(t, \frac{k_R}{J}, 1\Big) \\
&:=\; \kappa_p,
\end{aligned}
\tag{7.9}
$$

cf. (2.21).

Also $\zeta(t)w(t;\Lambda) = U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda) > 0$, so (3.1) and (3.2) hold. It follows readily that $\zeta(t)\delta(t) \leq \kappa_p \kappa_R$. Moreover using (6.3) we get

$$
\begin{aligned}
\zeta(t)\delta(t) \;&=\; U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda));\Lambda)R(t, L(t;\Lambda))) \\
&\quad\; - U_l(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)L(t;\Lambda) \\
&\geq\; U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)\Big(R(t, L(t;\Lambda)) - R_L(t, L(t;\Lambda))L(t;\Lambda)\Big) \\
&\geq\; U_c(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda)R(t, 0) \geq 0
\end{aligned}
$$

because R is concave. Hence (3.3) holds. It is easy to see that $\zeta(t)A(t;\Lambda) \in]0, T\kappa_\delta[$ so (3.4) also holds.

(7.4), (4.9) and Corollary 4.4 imply (3.5). (7.1), (7.2) and Corollary 6.6 imply (3.6). (3.7) holds by definition, cf.(7.3). It remains to establish the market clearing conditions.

From (7.5), (7.6) and the definition of $I^{c,l}$ follow (3.8) and (3.9). Also from (2.11) follows (3.10). Finally (3.11) can be established much as in [11], Theorem 10.2. □

We now prove the existence of an equilibrium.

Theorem 7.2 [cf. [11], Thm. 11.1, p.109] *There exists a vector* $\Lambda \in (0, \infty)^J$ *such that if* $\eta_j(\Lambda)$, $j = 1, 2, \ldots, J$, *is defined by (7.4) as in Proposition 7.1, then*

$$
\lambda_j \, \eta_j(\Lambda) = 1, \qquad j = 1, 2, \ldots, J,
\tag{7.10}
$$

and hence there exists an equilibrium.

Proof: Because of Proposition 7.1 it suffices to show that there exists $\Lambda \in (0, \infty)^J$ such that

$$
E\Big\{ \int_0^T \nabla_{c,l} U\Big(t, R(t, L(t;\Lambda)), J - L(t;\Lambda);\Lambda\Big) \cdot
$$

$$\Big[I^j\Big(t,\frac{1}{\lambda_j}\nabla_{c,l}U(t,R(t,L(t;\Lambda)),J-L(t;\Lambda);\Lambda);\Lambda\Big)-\Big(\varepsilon_jR(t,L(t;\Lambda)),1-\varepsilon_jL(t;\Lambda)\Big)^{\top}\Big]dt\Big\}$$

$$= 0 \tag{7.11}$$

for $j=1,2,\ldots,J$ where $L(\cdot\,;\Lambda)$ is given by of (6.3).

To prove (7.11) we argue as in the proof of [11], Thm. 11.1, p.109. Let $e_1,e_2,\ldots,e_J$ be the elementary vectors of $\mathbb{R}^J$. Set $\mathcal{A}=\{1,2,\ldots,J\}$ and, for every $B\subset\mathcal{A}$, denote by $\mathcal{C}_B$ the convex hull of $\{e_h; h\in B\}$, that is

$$\mathcal{C}_B:=\Big\{\sum_{h\in B}\lambda_he_h:\lambda_h\ge 0,\ \sum_{h\in B}\lambda_h=1\Big\};$$

finally define

$$\mathcal{C}_{\mathcal{A}}^{+}:=\Big\{\sum_{h\in A}\lambda_he_h:\lambda_h>0,\ \sum_{h\in\mathcal{A}}\lambda_h=1\Big\}.$$

For every $\Lambda\in\mathcal{C}_{\mathcal{A}}^{+}$ and $j=1,2,\ldots,J$, denote by $\mathcal{K}_j(\Lambda)$, the left-hand side of (7.11). According to Lemma 9.7 $\mathcal{K}_j$ extends to $\mathcal{C}_{\mathcal{A}}$ as a continuous function.

It follows that for $\lambda_j=0$,

$$\begin{aligned}
&\mathcal{K}_j(\Lambda)\\
&= -E\Big\{\int_0^T\Big[\nabla_{c,l}U\Big(t,R(t,L(t;\Lambda)),J-L(t;\Lambda);\Lambda\Big)\cdot\Big(\varepsilon_jR(t,L(t;\Lambda)),1-\varepsilon_jL(t;\Lambda)\Big)^{\top}\Big]dt\Big\}\\
&= -E\Big\{\int_0^T\Big[U_l(t,R(t,L(t;\Lambda)),J-L(t;\Lambda);\Lambda)+\varepsilon_j\hat{U}_L(t,L(t;\Lambda);\Lambda)L(t;\Lambda)\\
&\qquad +\varepsilon_jU_c(t,R(t,L(t;\Lambda)),J-L(t;\Lambda);\Lambda)\Big(R(t,L(t;\Lambda))-R_L(t,L(t;\Lambda))L(t;\Lambda)\Big)\Big]dt\Big\}.
\end{aligned}$$

But $U_l\ge 0$, $U_c>0$, and $R-R_LL>0$ since R is strictly concave. Moreover, from the definition of $L(t;\Lambda)$, cf. (6.3), it follows that $\hat{U}_LL\ge 0$; thus for $\lambda_j=0$ we must have $\mathcal{K}_j(\Lambda)<0$.

Finally we observe that for any $\Lambda\in\mathcal{C}_{\mathcal{A}}$

$$\begin{aligned}
&\sum_{j=1}^{J}\mathcal{K}_j(\Lambda)\\
&= E\Big\{\int_0^T\Big[\nabla_{c,l}U\Big(t,R(t,L(t;\Lambda)),J-L(t;\Lambda);\Lambda\Big)\cdot\\
&\qquad \Big(R(t,L(t;\Lambda))-\sum_{j=1}^{J}\varepsilon_jR(t,L(t;\Lambda)),J-L(t;\Lambda)-(J-\sum_{j=1}^{J}\varepsilon_jL(t;\Lambda))\Big)\Big]dt\Big\}\\
&= 0
\end{aligned}$$

since $\sum_{j=1}^{J}\varepsilon_j=1$. We are now in the same setting as [11], (12.4), and by similar arguments we obtain the existence of $\Lambda\in(0,\infty)^J$ such that (7.11) holds. □

8 Examples

8.1 Logarithmic Utility

Let us consider the simple case of agents with identical preferences specified by discounted logarithmic utilities, i.e. $U^j(t,c,l) = e^{-\rho t}(\log c + \mu \log l)$, $j = 1,\dots,J$ with $\rho > 0$, $\mu > 0$. Also, assume a technology function of the form $R(t,L) = f(t)\sqrt{L}$ for some function f with $\inf_t f(t) > 0$. Then R and U^j satisfy (2.1), (2.20), (2.21). Moreover

$$I^L(t,z) = \min\left\{\frac{f(t)^2}{(2z)^2}, L_{\max}\right\}, \quad I^j(t,y_1,y_2) = e^{-\rho t}\left(\frac{1}{y_1}, \min\left\{\frac{\mu}{y_2}, e^{\rho t}\right\}\right)$$

and

$$(\hat{c}_j(t), \hat{l}_j(t)) = e^{-\rho t}\left(\frac{1}{\eta_j\zeta(t)p(t)}, \min\left\{\frac{\mu}{\eta_j\zeta(t)w(t)}, e^{\rho t}\right\}\right). \tag{8.1}$$

Here η_j satisfies (cf. (4.8))

$$\frac{1}{\eta_j}E\int_0^T (e^{-\rho t} + \min\{\eta_j\zeta(t)w(t), \mu e^{-\rho t}\})\,dt = \xi_j =: E\int_0^T \zeta(t)\bar{e}_j(t)\,dt$$

where $\bar{e}_j(t) := \varepsilon_j\delta(t) + w(t)$ is the endowment stream of agent j with zero leisure. (Recall our interpretation that the agent generates her endowment by working at maximum rate and then "buys" leisure.) We assume for the time being that the constraints are inactive, i.e. that $\mu \le e^{\rho t}\eta_j\zeta(t)w(t)$ for all t. Then

$$(\hat{c}_j(t), \hat{l}_j(t)) = \frac{\rho\xi_j}{(1-e^{-\rho T})(1+\mu)}\left(\frac{e^{-\rho t}}{\zeta(t)p(t)}, \frac{\mu e^{-\rho t}}{\zeta(t)w(t)}\right). \tag{8.2}$$

Market clearing requires

$$\frac{\rho\sum_j \xi_j}{(1-e^{-\rho T})(1+\mu)}\left(\frac{e^{-\rho t}}{\zeta(t)p(t)}, \frac{\mu e^{-\rho t}}{\zeta(t)w(t)}\right) = (R(t,L(t)), J - L(t)), \tag{8.3}$$

so that

$$\frac{w(t)}{p(t)} = \frac{\mu R(t,L(t))}{J - L(t)}.$$

Substituting this into $L(t) = I^L(t, w(t)/p(t))$ gives

$$L(t) = L_e := \frac{J}{1+2\mu}, \quad \delta(t) = \frac{Jw(t)}{1+2\mu},$$

and

$$\bar{e}(t) := \sum_j \bar{e}_j(t) = \delta(t) + Jw(t) = Jw(t)\frac{2+2\mu}{1+2\mu} = 2(1+\mu)w(t)L_e = (1+\mu)p(t)R(t,L_e).$$

If we set

$$\gamma := \frac{\rho \sum_j \xi_j}{(1 - e^{-\rho T})(1+\mu)} = e^{\rho t}\zeta(t)p(t)R(t, L_e)$$

then $(1+\mu)\gamma = e^{\rho t}\zeta(t)\bar{e}(t)$. This, the definition of ξ_j and (8.2), (8.3) imply

$$\begin{aligned}
(\hat{c}_j(t), \hat{l}_j(t)) &= \frac{\rho \xi_j}{(1-e^{-\rho T})(1+\mu)\gamma}(R(t,L_e), J - L_e) \\
&= \frac{\rho}{(1-e^{-\rho T})} E \int_0^T \frac{\bar{e}_j(t)}{\bar{e}(t)} e^{-\rho t}\, dt \left(R(t, L_e), J - L_e\right) \\
&:= \lambda_j \Big(R(t, L_e), J - L_e\Big) \\
&= \lambda_j \left(f(t)\sqrt{L_e}, 2\mu L_e\right)
\end{aligned}$$

where λ_j is a weighted average relative endowment (with zero leisure) of the jth agent in the market, i.e. it is a measure of her importance in the market.

To see that the above $\Lambda = (\lambda_1, \ldots, \lambda_J)$ gives an equilibrium, we need to show that $\mathcal{K}_j(\Lambda) = 0$ for all j, cf (7.11). But $\nabla_{c,l}U(t,c,l) = e^{-\rho t}(1/c, \mu/l)$ and $I^j(t, \nabla U(t, R, J-L)/\lambda_j) = (\hat{c}_j, \hat{l}_j)$ so

$$\begin{aligned}
\mathcal{K}_j(\Lambda) &= \frac{(1-e^{-\rho T})}{\rho}\lambda_j(1+\mu) - E\int_0^T \left[\varepsilon_j + \frac{\mu}{J - L_e}(1 - \varepsilon_j L_e)\right] e^{-\rho t}\, dt \\
&= \frac{(1-e^{-\rho T})}{\rho}\lambda_j(1+\mu) - E\int_0^T (1+\mu)\frac{\bar{e}_j(t)}{\bar{e}(t)} e^{-\rho t}\, dt \\
&= 0.
\end{aligned}$$

Note that

$$\lambda_j = \frac{\rho}{(1-e^{-\rho T})} E\int_0^T \frac{\bar{e}_j(t)}{\bar{e}(t)} e^{-\rho t}\, dt = \frac{\varepsilon_j}{2(1+\mu)} + \frac{1+2\mu}{2J(1+\mu)}$$

so

$$\frac{\mu e^{-\rho t}}{\eta_j \zeta(t) w(t)} = \frac{\lambda_j \mu e^{-\rho t}}{U_l(t, R(t, L_e), J - L_e)} = \hat{l}_j(t) = \frac{\mu}{1+\mu}\left(\frac{\varepsilon_j J}{1+2\mu} + 1\right) \leq 1$$

if $0 < \mu \leq 1/(J-2)$. Under this restriction the constraints are inactive for all possible ε_j as assumed above. The problem seems analytically intractable for larger μ. Note also that any choice of $L_{\max} \in [J/(1+2\mu), J[$ will work here.

Remark 8.1 We observe that the optimal labour policy L_e and the leisure policies $\hat{l}_j$ are constant but the optimal consumption is proportional to f, i.e. to the total output of the economy. Similarly, wage rates $w(t)$ are inversely proportional to $e^{\rho t}\zeta(t)$ but prices $p(t)$ are inversely proportional to $e^{\rho t}\zeta(t)f(t)$. Hence for example cycles in the production function are transmitted to prices and consumption but not to wages and leisure rates.

8.2 Power Utility

Now assume that $R(t,L) = L^\rho$, $U^j(t,c,l) = c^\mu l^\nu$, where $0 < \rho, \mu, \nu < 1$, $\mu + \nu < 1$. Then

$$\begin{aligned} I^L(t,z) &= \min\left\{ \left(\frac{\rho}{z}\right)^{\frac{1}{1-\rho}}, L_{\max} \right\}, \\ I^j(t,y_1,y_2) &= \left(\left(\frac{\mu}{y_1}\right)^{\frac{1-\nu}{1-\mu-\nu}} \left(\frac{\nu}{y_2}\right)^{\frac{\nu}{1-\mu-\nu}}, \min\left\{ \left(\frac{\mu}{y_1}\right)^{\frac{\mu}{1-\mu-\nu}} \left(\frac{\nu}{y_2}\right)^{\frac{1-\mu}{1-\mu-\nu}}, 1 \right\} \right), \end{aligned}$$

and

$$\begin{aligned} &(\hat{c}_j(t), \hat{l}_j(t)) \\ &= \left(\left(\frac{\mu}{\eta_j\zeta(t)p(t)}\right)^{\frac{1-\nu}{1-\mu-\nu}} \left(\frac{\nu}{\eta_j\zeta(t)w(t)}\right)^{\frac{\nu}{1-\mu-\nu}}, \right. \\ &\qquad \left. \min\left\{ \left(\frac{\mu}{\eta_j\zeta(t)p(t)}\right)^{\frac{\mu}{1-\mu-\nu}} \left(\frac{\nu}{\eta_j\zeta(t)w(t)}\right)^{\frac{1-\mu}{1-\mu-\nu}}, 1 \right\} \right) \\ &= \left(\frac{\mu}{\eta_j\zeta(t)p(t)}\right)^{\frac{\mu}{1-\mu-\nu}} \left(\frac{\nu}{\eta_j\zeta(t)w(t)}\right)^{\frac{\nu}{1-\mu-\nu}} \left(\frac{\mu}{\eta_j\zeta(t)p(t)}, \frac{\nu}{\eta_j\zeta(t)w(t)} \right) \end{aligned} \tag{8.4}$$

where again we assume that the min is **not** assumed at 1. Here η_j is chosen to satisfy

$$\frac{\mu+\nu}{\eta_j^{\frac{1}{1-\mu-\nu}}} E\int_0^T \left(\frac{\mu}{\zeta(t)p(t)}\right)^{\frac{\mu}{1-\mu-\nu}} \left(\frac{\nu}{\zeta(t)w(t)}\right)^{\frac{\nu}{1-\mu-\nu}} dt = \xi_j.$$

If we set

$$M(t) := \left(\frac{\mu}{\zeta(t)p(t)}\right)^{\frac{\mu}{1-\mu-\nu}} \left(\frac{\nu}{\zeta(t)w(t)}\right)^{\frac{\nu}{1-\mu-\nu}},$$

then

$$(\hat{c}_j(t), \hat{l}_j(t)) = \frac{\xi_j}{\mu+\nu} \left(E\int_0^T M(t)\,dt \right)^{-1} M(t) \left(\frac{\mu}{\zeta(t)p(t)}, \frac{\nu}{\zeta(t)w(t)} \right) \tag{8.5}$$

Market clearing requires that

$$\frac{\sum_j \xi_j}{\mu+\nu} \left(E\int_0^T M(t)\,dt \right)^{-1} M(t) \left(\frac{\mu}{\zeta(t)p(t)}, \frac{\nu}{\zeta(t)w(t)} \right) = (R(t,L(t)), J - L(t)), \tag{8.6}$$

so we conclude that

$$\frac{w(t)}{p(t)} = \frac{\nu R(t,L(t))}{\mu(J-L(t))}$$

and hence

$$L(t) = L_e := \frac{J}{1+\tilde{\rho}}, \qquad \delta(t) = \frac{\mu\tilde{\rho}-\nu}{\nu(1+\tilde{\rho})}Jw(t), \qquad \bar{e}(t) = \frac{\mu+\nu}{\mu}p(t)R(t,L_e)$$

where $\tilde{\rho} := \rho\mu/\nu$.

We define

$$\gamma := \frac{\sum_j \xi_j}{\mu+\nu}\left(E\int_0^T M(t)\,dt\right)^{-1}.$$

Then we can solve (8.6) to obtain

$$M(t) = R(t,L_e)^{\mu}(J-L_e)^{\nu}\gamma^{-\mu-\nu}.$$

It follows that

$$\zeta(t)\bar{e}(t) = \gamma M(t) = (\mu+\nu)R(t,L_e)^{\mu}(J-L_e)^{\nu}\gamma^{1-\mu-\nu}$$

and so from (8.5)

$$\begin{aligned}
(\hat{c}_j(t),\hat{l}_j(t)) &= E\int_0^T \zeta(t)\bar{e}_j(t)\,dt\left((\mu+\nu)\gamma E\int_0^T M(t)\,dt\right)^{-1}(R(t,L_e),J-L_e)\\
&= E\int_0^T \gamma^{\mu+\nu-1}\zeta(t)\bar{e}_j(t)\,dt\left((\mu+\nu)E\int_0^T \gamma^{\mu+\nu}M(t)\,dt\right)^{-1}(R(t,L_e),J-L_e) \qquad (8.7)\\
&= \lambda_j(L_e^{\rho},\tilde{\rho}L_e),
\end{aligned}$$

where

$$\begin{aligned}
\lambda_j &= \left(\frac{E\int_0^T R(t,L_e)^{\mu}(J-L_e)^{\nu}\bar{e}(t)^{-1}\bar{e}_j(t)\,dt}{E\int_0^T R(t,L_e)^{\mu}(J-L_e)^{\nu}\,dt}\right)\\
&= E\frac{1}{T}\int_0^T \frac{\bar{e}_j(t)}{\bar{e}(t)}\,dt.
\end{aligned}$$

Again each agent receives a fraction of the total good/leisure available depending on her relative importance as measured by the endowment, and the total available depends on the total labour available, but this too is constant.

It can be shown that

$$\nabla_{c,l}U(t,c,l) = \left(\sum_j \lambda_j^{\frac{1}{1-\mu-\nu}}\right)^{1-\mu-\nu}\left(\mu c^{\mu-1}l^{\nu},\nu c^{\mu}l^{\nu-1}\right) = \left(\mu c^{\mu-1}l^{\nu},\nu c^{\mu}l^{\nu-1}\right).$$

Since $R_e := R(t,L_e)$ and L_e are constant, this implies

$$\begin{aligned}
\mathcal{K}_j(\Lambda) &= T\lambda_j(\mu+\nu)R_e^{\mu}(J-L_e)^{\nu} - E\int_0^T R_e^{\mu}(J-L_e)^{\nu}\left(\varepsilon_j\mu + \nu\left[\frac{1-\varepsilon_j L_e}{J-L_e}\right]\right)dt\\
&= TR_e^{\mu}(J-L_e)^{\nu}\left[\lambda_j(\mu+\nu) - \varepsilon_j(\mu-\frac{\nu}{\tilde{\rho}}) - \frac{\nu}{\tilde{\rho}L_e}\right]\\
&= 0
\end{aligned}$$

where the last equality follows by direct computation of $\bar{e}_j/\bar{e}$. Hence the given Λ is a fixed-point of $\mathcal{K}$ and we have an equilibrium. Note that

$$\hat{l}_j = \frac{1}{\tau+1}\left[1+\varepsilon_j J\left(\frac{\rho\tau^2-1}{\rho\tau+1}\right)\right] \le 1$$

if

$$\tau := \frac{\mu}{\nu} \le \frac{1+\sqrt{1+4J(J-1)\rho}}{2(J-1)\rho}.$$

In this case the above solution gives an equilibrium; otherwise an equilibrium exists but it is not given by (8.7). Any value of $L_{\max} \in [J/(1+\tilde{\rho}), J[$ will do. In the usual case of $J >> 1$ the upper bound on τ reduces to $\tau < 1/\sqrt{\rho}$.

Remark 8.2 It is not surprising that the solutions are time independent because the production function and the utilities are. We observe also that the individual agents' optimal consumption policy of good and leisure is the same in both examples: a fraction of the total available in the economy, the fraction being the relative endowment of the agent.

9 Appendix

We summarize here some results from convex analysis (but in the context of *concave* functions); a reference is [15]. If f is a function $\mathbb{R}^n \mapsto [-\infty, \infty[$, then dom $f := \{x | f(x) > -\infty\}$. f is *non-decreasing* if $f(x) \le f(x')$ whenever $x, x' \in$ dom f and $x \le x'$ and where the latter inequality in $\mathbb{R}^n$ is taken componentwise. f is *increasing* if $f(x) < f(x')$ for such x, x' with $x \ne x'$. The function f is *(strictly) concave* if it is upper semi-continuous and (strictly) concave on dom f which is assumed to be non-empty. This makes the function a closed, proper, concave function in the terminology of convex analysis. Its *subgradients* at x are all $y \in \mathbb{R}^n$ such that

$$f(z) - f(x) \le (z-x)\cdot y \quad \forall z \in \mathbb{R}^n.$$

The set of all subgradients is denoted by $\partial f(x)$, and dom $\partial f := \{x | \partial f(x) \ne \emptyset\}$. It follows that f is concave if and only if for all $x \ne x'$ and all $y \in \partial f(x),\ y' \in \partial f(x')$, we have

$$(x-x')\cdot y \le f(x) - f(x') \le (x-x')\cdot y' \tag{9.1}$$

with strict inequality in case f is strictly concave. If in addition f is differentiable on int(dom f), then $\partial f = \{\nabla f\}$ and ∇f is non-increasing, and decreasing in the case of strict concavity. Here int(A) denotes the interior of the set A and non-increasing and decreasing are defined as follows. For $D \subset \mathbb{R}^n$ the function $g : D \mapsto \mathbb{R}^n$ is *non-increasing* if $(x-x')\cdot(g(x)-g(x')) \le 0$ for all $x, x' \in D$. It is decreasing if $(x-x')\cdot(g(x)-g(x')) < 0$.

We now study the inverse of the utility function U^j. For t fixed and (c,l) in $\bar{\mathcal{D}} = [0,\infty[\times[0,1]$, we write $u(c,l)$ for $U^j(t,c,l)$. Our convention regarding the definition of ∇U^j at the boundary of $\mathcal{D}$ implies that $\nabla u(x) \in \partial u(x)$ for $x \in$ dom ∂u.

Lemma 9.1 *There exists a continuous function* $F : \mathbb{R}^2_+ \mapsto \bar{\mathcal{D}}$ *which extends* $(\nabla u)^{-1}$. *It is continuously differentiable on* $\mathcal{R}^j$.

Proof: [1] We extend u to $\mathbb{R}^2$ as

$$u_o(c, l) = \begin{cases} u(c, l) & \text{if } (c, l) \in \bar{\mathcal{D}} \\ -\infty & \text{otherwise.} \end{cases}$$

u_o is a closed, proper, concave function, strictly concave on $\mathcal{D}$. The assumptions on U^j imply that

$$\mathcal{D} = \text{ri}(\text{dom } u_o) \subset]0, \infty[\times]0, 1] \subset \text{dom } u_o = \bar{\mathcal{D}} \backslash \{x : u_o(x) = -\infty\}$$

where ri denotes the relative interior (which is the interior here) and dom u_o is the set where $u_o > -\infty$. The conjugate concave function of u_o is defined as

$$u_o^*(y) := \inf_{x \in \mathbb{R}^2} \{x \cdot y - u_o(x)\} = \inf_{x \in \bar{\mathcal{D}}} \{x \cdot y - u(x)\} = \inf_{x \in \mathcal{D}} \{x \cdot y - u(x)\}. \tag{9.2}$$

From (9.1) it follows that $\mathcal{R}^j \subset \text{dom}\, u_o^*$.

We now set

$$F(y) = \arg \min_{x \in \mathbb{R}^2} \{x \cdot y - u_o(x)\} = \{x : \; y \in \partial u_o(x)\}$$

where $\partial u_o(x)$ is the subgradient set of $u_o(x)$, cf. [15], Theorem 23.5. The strict concavity of u_o on dom ∂u_o and (9.1) imply that F is single-valued.

Intuitively, $F = (\nabla u_o)^{-1}$, but we need to make this precise. [15], Corollary 23.5.1 states that

$$x \in \partial u_o^*(y) \Leftrightarrow y \in \partial u_o(x), \tag{9.3}$$

so $F = \partial u_o^*$. Repeating the process with u_o^* rather than u_o and calling the function corresponding to F, S, gives $S = \partial u_o^{**} = \partial u_o$ since $u_o^{**} = u_o$. Then dom $S = \{x : S(x) \neq \emptyset\} \subset$ dom u_o. It now follows from (9.3) that $F = S^{-1}$, and since $S = \nabla u$ on $\mathcal{D}$, we have that F extends the inverse of ∇u beyond $\mathcal{R}^j$.

We now want to know how far this extension goes. Since $\mathcal{D}$ lies in $\mathbb{R}^2_+$, it follows from (9.2) that u_o^* is non-decreasing. Hence for any $\varepsilon > 0$, for $y \in [\varepsilon, \infty[\times[0, \infty[$

$$u_o^*(y) \geq u_o^*((\varepsilon, 0)) = -\sup_{x \in \mathcal{D}} \{u(x) - \varepsilon x_1\} > -\infty$$

because $\lim_{x_1 \to \infty} u(x)/x_1 = 0$, cf (2.20), and because u is continuous. From [15], Theorem 23.4, it follows that $\partial u_o^* = F$ is then defined on $\mathbb{R}^2_+$. Since it is single-valued, then $F = \nabla u_o^*$, and so u_o^* is differentiable on $\mathbb{R}^2_+$. Since it is proper and concave, then F is continuous on this set, cf. [15], Theorem 25.5. The image of F is the domain of $S \subset$ dom u_o, hence lies in $\bar{\mathcal{D}}$.

Finally, since ∇u is continuously differentiable then the inverse function theorem implies that F is continuously differentiable on $\mathcal{R}^j$. Hence if $H(x)$ is the Hessian matrix of u at x, then the derivative of F at $\nabla u(x)$ is $H(x)^{-1}$. □

Again we suppress the fixed t in our notation.

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[1] We thank R. T. Rockafellar for assistance with the proof.

Corollary 9.2 *F is differentiable on each $\mathcal{R}_i^j$, and moreover*

$$F(y) = \begin{cases} (\mathcal{I}_1^j(t, y_1, 1), 1) & y \in \mathcal{R}_1^j \\ (0,1) & y \in \mathcal{R}_2^j \\ (0, \mathcal{I}_2^j(t, y_2)) & y \in \mathcal{R}_3^j \\ (0,0) & y \in \mathcal{R}_4^j \\ (\mathcal{I}_1^j(t, y_1, 0), 0) & y \in \mathcal{R}_5^j \\ (\nabla_{c,l} U^j(t, \cdot\,, \cdot))^{-1}(y) & y \in \mathcal{R}^j \end{cases} \tag{9.4}$$

Proof: For $y \in \mathcal{R}_1^j$ let $\tilde{y}$ be the point of intersection of $\mathcal{C}_1^j$ with the vertical line through y and let $c > 0$ be the parameter value which gives this point on $\mathcal{C}_1^j$, i.e. $y_1 = \tilde{y}_1,\ y_2 < \tilde{y}_2$ and $\tilde{y} = \nabla u(c, 1)$. From (9.2) it follows that $F(y) = \arg\min_{x \in \bar{\mathcal{D}}}\{x \cdot y - u(x)\}$. We shall show that $x \cdot y - u(x)$ achieves a minimum at $(c,1)$ for all $y_2 \le \tilde{y}_2$. This will establish the first part of (9.4) because $c = \mathcal{I}_1^j(t, y_1, 1)$, i.e. $y_1 = \tilde{y}_1 = u_{x_1}(c,1)$. Consider the directional derivative $\delta\Psi((c,1); v)$ of $\Psi(x) = x \cdot y - u(x)$ in the direction v with $v_2 < 0$, i.e. the inward direction for $\mathcal{D}$ at $(c,1)$. Then (recall the convention of defining partial derivatives at the boundary by taking limits from inside $\mathcal{D}$)

$$\delta\Psi((c,1); v) = (y - \nabla u(c,1)) \cdot v \ge (\tilde{y} - \nabla u(c,1)) \cdot v = 0$$

since $\tilde{y}$ lies on $\mathcal{C}_1^j$. This establishes that the min is attained at $(c,1)$.

The cases $y \in \mathcal{R}_i^j,\ i = 2, \ldots, 5$ follow similarly and the other case comes from the Lemma.

The differentiability of F follows from (9.4) and the differentiability of $\mathcal{I}_k^j$ for arguments not corresponding to points in the boundaries of $\mathcal{R}_i^j$. □

For $y \in \mathbb{R}_+^2$ it is convenient to define $h(\eta; y) := y \cdot F(\eta y)$.

Corollary 9.3 *$h(\cdot\,; y)$ is continuous, non-increasing on $]0, \infty[$ and decreasing on $]0, \eta^y[\cap \{\eta y \notin \mathcal{R}_2^j\}$ where*

$$\eta^y := \sup\{\eta : h(\eta; y) > 0\}.$$

Moreover $\lim_{\eta \downarrow 0} h(\eta; y) = \infty$.

Proof: Continuity of h follows from that of F. Define $g(\eta) = u_o^*(\eta y)$ for $\eta \in]0, \infty[$. Then g is concave and $g'(\eta) = h(\eta; y)$. Hence $h(\eta; y)$ is non-increasing.

As η varies from 0 to ∞, ηy runs along a ray from 0 out, and $F(\eta y) = F(\tilde{z})$ where typically $\tilde{z}$ runs along $\mathcal{C}_1^j$ (as the vertical projection onto $\mathcal{C}_1^j$) until it hits the ray, follows the ray through $\mathcal{R}^j$, then moves up $\mathcal{C}_2^j$ (as the horizontal projection of ηy onto $\mathcal{C}_2^j$), or down $\mathcal{C}_3^j$ (as the vertical projection of ηy onto $\mathcal{C}_3^j$), until it stops at $\nabla_{c,l} U(t, 0, 0)$. While the ray passes through $\mathcal{R}_2^j$ (if it does), $\tilde{z}$ remains at $\nabla_{c,l} U(t, 0, 1)$.

Strict concavity of u on dom ∂u implies

$$(x^1 - x^2) \cdot (\nabla u(x^1) - \nabla u(x^2)) < 0, \quad \text{for } x^1 \ne x^2 \in \text{dom } \partial u$$

or

$$(F(y^1) - F(y^2)) \cdot (y^1 - y^2) < 0, \quad \text{for } y^1 \ne y^2 \in \mathcal{R}^j. \tag{9.5}$$

For $\eta_1, \eta_2 > 0$ such that $y^1 := \eta_1 y,\ y^2 := \eta_2 y \in \mathcal{R}^j$ it follows from (9.5) that

$$(\eta_1 - \eta_2)(h(\eta_1; y) - h(\eta_2; y)) < 0,$$

i.e. $h(\cdot\,; y)$ is strictly decreasing, provided $\eta y \in \mathcal{R}^j$.

For $\eta_1, \eta_2 > 0$ such that $y^1 := \eta_1 y,\ y^2 := \eta_2 y \in \mathcal{R}^j_1$, i.e. $\tilde{z}^1,\ \tilde{z}^2 \in \bar{\mathcal{R}}^j$, we have $\tilde{z}^i_1 = y^i_1$, and it follows from (9.4),(9.5) that

$$(\eta_1 - \eta_2)(h(\eta_1; y) - h(\eta_2; y)) = (\tilde{z}^1 - \tilde{z}^2)\cdot (F(\tilde{z}^1) - F(\tilde{z}^2)) < 0,$$

i.e. $h(\cdot\,; y)$ is strictly decreasing, provided $\eta y \in \mathcal{R}^j_1$. The cases of $\mathcal{R}^j_3$ and $\mathcal{R}^j_5$ are treated similarly. Note that $h = 0$ if and only if $\eta y \in \mathcal{R}^j_4$.

Finally

$$\lim_{\eta \downarrow 0} h(\eta; y) = \lim_{\eta \downarrow 0} y_1 \mathcal{I}^j_1(t, \eta y_1, 1) + y_2 = \infty.$$

□

We turn now to the utility function of the representative agent. Observe that with the added assumption that $U^j_l(t, c, 0) = +\infty$, the curve $\mathcal{C}^j_3$ as well as the sets $\mathcal{R}^j_4$ and $\mathcal{R}^j_5$ in Figure 1 disappear and the curve $\mathcal{C}^j_2$ no longer has finite length.

Lemma 9.4 *For each $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J) \in (0,\infty)^J$, $U(\cdot\,; \Lambda) : [0,T] \times [0,\infty[\times [0,J] \mapsto \mathbb{R}$ is measurable, and for each t fixed, it is increasing, strictly concave, continuously differentiable with respect to (c,l) on $\mathcal{D}_o$. For each t, $\nabla U(t, \cdot\,; \Lambda)$ is $\mathbb{R}^2_+$-valued and its inverse exists and can be extended as*

$$I^\Lambda(t, y) := \sum_j I^j(t, \frac{y}{\lambda_j}),$$

a continuous mapping with $\mathbb{R}^2_+ \subset \operatorname{dom} I^\Lambda(t, \cdot\,) \subset \overline{\mathbb{R}^2_+}$ and with image $\operatorname{dom} \partial U(t, \cdot; \Lambda) \supset \mathcal{D}_o$. Then for y such that $I^\Lambda(t, y) \in \mathcal{D}_o$

$$\nabla U(t, I^\Lambda(t, y); \Lambda) = y. \tag{9.6}$$

Moreover, for each $(c,l) \in \operatorname{dom} \partial U(t, \cdot; \Lambda)$, there exist $(\bar{c}_j, \bar{l}_j) \in \operatorname{dom} \partial U^j(t, \cdot)$ such that

$$U(t, c, l; \Lambda) = \sum_{j=1}^J \lambda_j U^j(t, \bar{c}_j, \bar{l}_j), \tag{9.7}$$

and

$$\begin{cases} U_c(t, c, l; \Lambda) = \lambda_j U^j_c(t, \bar{c}_j, \bar{l}_j) + r^j_3 & j = 1, \ldots, J, \\ U_l(t, c, l; \Lambda) = \lambda_j U^j_l(t, \bar{c}_j, \bar{l}_j) - r^j_1 & j = 1, \ldots, J, \end{cases} \tag{9.8}$$

where $r^j_i \geq 0,\ (1 - \bar{l}_j) r^j_1 = 0,\ \bar{c}_j r^j_3 = 0$. Finally $U_l(t, c, 0; \Lambda) = +\infty$.

Proof: We write x for (c,l), $U^{j,\Lambda}(x)$ for $\lambda_j U^j(t,c,l)$ extended as $-\infty$ off $\bar{\mathcal{D}}$ (and t, constant, suppressed). Similarly we write $U(x)$ for $U(t,c,l;\Lambda)$. Then

$$U(x) = \sup_{\sum z^j = x} \sum U^{j,\Lambda}(z^j).$$

Since dom $U^{j,\Lambda} \subset \bar{\mathcal{D}}$ does not contain any subspace of $\mathbb{R}^2$, then U is a closed, proper, concave function and the sup is attained at say $\bar{z}^j$, $j = 1, \cdots, J$, cf [15], Corollary 9.2.1, i.e. (9.7) holds. Furthermore, [15], Theorem 16.4 implies that

$$U^*(y) = \sum_j (U^{j,\Lambda})^*(y) = \sum_j \lambda_j U^{j*}(\frac{y}{\lambda_j}).$$

Now

$$\bigcap_j \text{ri}(\text{dom } (U^{j,\Lambda})^*) \supset \mathbb{R}^2_+,$$

so by [15], Theorem 23.8, the subdifferential of U^* is

$$\partial U^*(y) = \sum_j \partial (U^{j,\Lambda})^*(y) = \sum_j \partial U^{j*}(\frac{y}{\lambda_j}) = \sum_j I^j(t, \frac{y}{\lambda_j}) = I^\Lambda(t,y), \tag{9.9}$$

so ∂U^* is single-valued at least on $\mathbb{R}^2_+$. We can now conclude that the concave function U^* is continuously differentiable on $\mathbb{R}^2_+$, and as usual $\partial U^* = (\partial U)^{-1}$. Since the domain of $I^\Lambda(t,\cdot\,)$, i.e. of each $I^j(t,\cdot\,)$, is contained in the non-negative quadrant, then the subgradients of U are also in this set, so U is non-decreasing. Since ∂U^* is single-valued, then $\partial U(x) \cap \partial U(x') = \emptyset$ for $x \neq x'$. As in the proof of [15], Theorem 26.3, this implies that U is essentially strictly concave, hence also increasing on dom ∂U.

We will now show that $I^\Lambda(t,\cdot\,)$ is an extension of the inverse of ∇U. Fix t, Λ. Define

$$\mathcal{R}_o := \bigcap_{k=1}^{2} \bigcup_{i,j} \left\{ y \;:\; y_k \mapsto I_i^j\left(t, \frac{y}{\lambda_j}\right) \text{ is decreasing} \right\} = \bigcup_j \lambda_j \mathcal{R}^j \cup \left[\bigcup_j \lambda_j \overline{\mathcal{R}_1^j} \cap \bigcup_j \lambda_j \overline{\mathcal{R}_3^j} \right].$$

Since I_i^j is non-increasing then $I^\Lambda(t,\cdot\,)$ is one-to-one on this set. If we did not assume that $U_l^j(t,c,0) = +\infty$ then for some Λ there would be vertical line segments with endpoints in $\mathcal{R}_o$ on which I^Λ would constant hence I^Λ would not be one-to-one. To aid in the visualization we note that $\lambda_j \mathcal{R}^j$ looks like $\mathcal{R}^j$ so we can think of superimposing J copies of Figure 1 with the set $\mathcal{R}^j$ placed randomly in each copy.

Next we show that $I^\Lambda(t,\cdot\,)$ maps onto dom ∂U. For $x \in$ dom ∂U, we have (9.7) with $\bar{z}^j = (\bar{c}_j, \bar{l}_j) \in$ dom ∂U^j. The value $U(x)$ is then the result of a constrained maximization problem over $z^j \in \mathbb{R}^2$, $j = 1, \ldots, J$, such that

$$\sum_{j=1}^{J} z^j = x,$$

$$g_1(z^j) := z_2^j - 1 \leq 0, \quad g_2(z^j) := -z_2^j \leq 0, \quad g_3(z^j) := -z_1^j \leq 0, \quad j = 1, \ldots, J.$$

Then [15], Corollary 28.2.2, implies that there are multipliers $y \in \mathbb{R}^2$, $r^j \in \mathbb{R}^3$ with $r_i^j \geq 0$, $r^j \cdot g(\bar{z}^j) = 0, \quad j = 1, \ldots, J$, such that

$$-\nabla U^{j,\Lambda}(\bar{z}^j) + y + \hat{r}^j = 0, \quad j = 1, \ldots, J, \tag{9.10}$$

where

$$\hat{r}^j = \begin{pmatrix} -r_3^j \\ r_1^j - r_2^j \end{pmatrix}.$$

Since $U_l^j(t, c, 0) = +\infty$ it now follows that $r_2^j = 0$ for all j. Then $\bar{z}^j = (\nabla_x U^j(t, \cdot\,))^{-1}(\frac{y + \hat{r}_j}{\lambda_j})$. In fact, $\hat{r}_j$ compensates y so that $\tilde{y}_j = (y + \hat{r}_j)/\lambda_j \in \mathcal{R}^j$ and $\bar{z}^j = I^j(t, \frac{y}{\lambda_j})$. If $y \notin \mathcal{R}_o$ then we may redefine it so that it is. For example, if $y \in \bigcup_j \lambda_j \overline{\mathcal{R}_1^j} \backslash \mathcal{R}_o$ we set $r_1^m := \min\{r_1^j > 0 : j = 1, \ldots, J\}$. By adding r_1^m to y_2, we obtain a new $y \in \mathcal{R}_o$ with non-negative multipliers. The general strategy is as follows. For $y \in \bigcup_j \lambda_j \overline{\mathcal{R}_1^j} \backslash \mathcal{R}_o$, project it vertically upward onto $\mathcal{R}_o$ - this is the new y. For $y \in \bigcup_j \lambda_j \overline{\mathcal{R}_3^j} \backslash \mathcal{R}_o$, project it horizontally to the left onto $\mathcal{R}_o$. For $y \in \bigcap_j \lambda_j \overline{\mathcal{R}_2^j}$, the new y is the upper left corner of this set. This process will produce a new value for y (and new, consistent values for r_1^j and r_3^j) still satisfying (9.10) for the same $\bar{z}^j$ but now with $y \in \mathcal{R}_o$.

Then

$$I^\Lambda(t, y) = \sum_j I^j(t, \frac{y}{\lambda_j}) = \sum_j \bar{z}^j = x. \tag{9.11}$$

Thus $I^\Lambda(t, \cdot\,)$ is onto dom ∂U and so $(I^\Lambda)^{-1}$ exists on dom ∂U.

It follows that ∂U is single-valued on int(dom $\partial U) = \mathcal{D}_o$, hence U is differentiable there and in fact continuously differentiable since it is concave. Moreover $I^\Lambda = (\nabla U)^{-1}$ and (9.6) follows.

From (9.10) and (9.11) follows

$$\nabla_x U(t, x; \Lambda) = \Big(I^\Lambda(t, \cdot\,)\Big)^{-1}(x) = y = \lambda_j \nabla_x U^j(t, \bar{z}^j) - \hat{r}_j, \tag{9.12}$$

and (9.8) now follows.

To finish we must show that $\lim_{l \downarrow 0} U_l(t, c, l; \Lambda) = \infty$. But this follows by continuity of $l \mapsto U_l$ and (9.8) since $\bar{l}_j \leq l$ so $r_1^j = 0$ when $l < 1$. □

Remark 9.5 We point out that Lipschitz continuity in t, c, l of the $\nabla_{c,l} U^j$ and in t, L of R_L can be used to remove the assumption that $L_{\max} < J$. We only need this for $\hat{L}$ and since $U_l^j(t, c, 0) = \infty$ implies $\hat{L}(t) < J$, it suffices that $\hat{L}$ be continuous when it is near J, i.e. the solution of $\hat{U}_L(t, L; \Lambda) = 0$ be continuous. The implicit function theorem yields this if $\hat{U}_L$ is locally Lipschitz (since $\hat{U}_{LL} < 0$). The methods of [11], sec. 13, can be used to show that

$$\nabla_{c,l}(\bar{c}_j, \bar{l}_j) = \nabla_{c,l} I^j(t, \nabla U(t, c, l; \Lambda)/\lambda_j) = H_j(I^j(t, \nabla U(t, c, l; \Lambda)/\lambda_j)^{-1} < 0$$

where H_j denotes the Hessian of U^j. Since for each (c, l) in (9.8), $r_1^j = 0$ for some j, cf. the construction of y, and also $r_3^j = 0$ for some (possibly different) j, then the conclusion follows.

Let us now prove the measurability of $\tilde{L}$, cf. Remark 6.5.

Lemma 9.6 *The function $\tilde{L}(t)$ is measurable.*

Proof: For any $\varepsilon > 0$, the separability of the continuous functions on $[\varepsilon, L_{\max}]$ allows us to smooth $\hat{U}_L(t, L; \Lambda)$ in t to a continuous function $u^n(t, L)$ with

$$\lim_{n\to\infty} \int_0^T \sup_{\varepsilon \le L \le L_{\max}} |u^n(t, L) - \hat{U}_L(t, L; \Lambda)|\, dt = 0.$$

The set $\{(t, L) \in [0, T] \times]0, L_{\max}[: u^n(t, L) = 0\}$ is σ-compact so there is a measurable function, $L_n(t)$, such that $u^n(t, L_n(t)) = 0$ a.e., cf, [8], Lemma B, p. 199. Since a subsequence of u^n converges almost everywhere in t, uniformly in $L \ge \varepsilon$, then for any convergent subsequence of $L_n(t)$ with limit $L_\infty \ge L_\varepsilon$ we must have $\hat{U}_L(t, L_\infty; \Lambda) = 0$ *a.e.*. Hence $L_\infty = \tilde{L}(t)$ by the uniqueness of the solution. It now follows that $L_n(t)$ converges a.e. to $\tilde{L}(t)$, hence the latter function is measurable. □

We conclude with a result used in the proof of Theorem 7.2.

Lemma 9.7 *$\mathcal{K}_j$ can be extended as a continuous function to $\mathcal{C}_{\mathcal{A}}$.*

Proof: We can extend $\lambda \mapsto I^j(t, y/\lambda)$ to $[0, \infty[$ by setting it to $(0, 0)$ for $\lambda = 0$. This remains a continuous map because for $\lambda \to 0$, $y/\lambda \in \mathcal{R}_4^j$ eventually. It follows that $(y, \Lambda) \mapsto I^\Lambda(t, y)$ is continuous on $\mathbb{R}_+^2 \times \mathcal{C}_{\mathcal{A}}$.

We now claim that $(x, \Lambda) \mapsto \nabla U(t, x; \Lambda)$ is continuous on $\mathcal{D}_o \times \mathcal{C}_{\mathcal{A}}$. In fact, suppose $(x_n, \Lambda_n) \to (x_o, \Lambda_o)$ and $y_n := \nabla U(t, x_n; \Lambda_n)$. We set $\Lambda_n = (\lambda_1^n, \cdots, \lambda_J^n)$. The boundedness of $\{(x_n, \Lambda_n)\}_n$ and (9.8) imply

$$\begin{aligned} U_c(t, x_n; \Lambda_n) &\le \lambda_i^n U_c^i(t, \bar{z}_n^i) + r_3^i \quad \text{for all } i \\ &\le \lambda_j^n U_c^j\Big(t, ((x_n)_1/J, (x_n)_2)\Big) \quad \text{for some } j \\ &\le \sup_{i,n} \lambda_i^n U_c^i\Big(t, ((x_n)_1/J, (x_n)_2)\Big). \end{aligned}$$

Note that since $(\bar{z}_n^j)_1 \ge (x_n)_1/J > 0$ then $r_3^j = 0$. Similarly for the other partial derivative; hence $\{y_n\}_n$ is bounded.

Define $\mathcal{R}_o^n$ in the same manner as $\mathcal{R}_o$ in the proof of Lemma 9.4 but with λ_j replaced by λ_j^n. Then $\mathcal{R}_o^n \to \mathcal{R}_o$. Hence if $\{y_{n'}\}$ is any convergent subsequence with limit y_o, then $y_o \in \mathcal{R}_o$. Since $I^{\Lambda_{n'}}(t, y_{n'}) = x_{n'}$ then, by taking limits, $I^{\Lambda_o}(t, y_o) = x_o$. But this defines $y_o \in \mathcal{R}_o$ uniquely; hence the full sequence must converge to y_o. This establishes the claim.

The continuity of L is a little more difficult. If $\Lambda_n \to \Lambda_o$ and we set $L_n := L(t; \Lambda_n)$, then L_n lies in the compact set $[0, L_{\max}]$. For any convergent subsequence $L_{n'} \to L_o$ we have

$$\begin{aligned} \hat{U}_L(t, L_{n'}; \Lambda_{n'}) &= \nabla U\Big(t, R(t, L_{n'}), J - L_{n'}; \Lambda_{n'}\Big) \cdot \Big(R_L(t, L_{n'}), -1\Big) \\ &\to \nabla U\Big(t, R(t, L_o), J - L_o; \Lambda_o\Big) \cdot \Big(R_L(t, L_o), -1\Big) \qquad (9.13) \\ &= \hat{U}_L(t, L_o; \Lambda_o). \end{aligned}$$

Now suppose that $\|z - z_o\|$ is so small that $|U(t,z;\Lambda_o) - U(t,z_o;\Lambda_o)| < \varepsilon$ and $|U^j(t,z) - U^j(t,z_o)| < \varepsilon,\ j = 1,\ldots,J$, and $\|\Lambda_{n'} - \Lambda_o\| < \varepsilon$. Then

$$\begin{aligned} U(t,z;\Lambda_{n'}) &= \sup_{\sum_j z^j = z} \sum_j \lambda_j^{n'} U^j(t,z^j) \\ &\leq U(t,z;\Lambda_o) + \varepsilon \sum_j U^j(t,z^j) \\ &\leq U(t,z_o;\Lambda_o) + \varepsilon \sum_j U^j(t,z) + \varepsilon \\ &\leq U(t,z_o;\Lambda_o) + \varepsilon \sum_j U^j(t,z_o) + J\varepsilon^2 + \varepsilon, \end{aligned}$$

where we used the fact that U^j is increasing. The same inequality holds with $\hat{U}(t,z;\Lambda_{n'})$ and $\hat{U}(t,z_o;\Lambda_o)$ interchanged. Setting $z = (R(t,L_{n'}), J - L_{n'})$ and $z = (R(t,0),J)$ and using the continuity of $U(t,\cdot\,)$ gives

$$\hat{U}(t,L_{n'};\Lambda_{n'}) \to \hat{U}(t,L_o;\Lambda_o) \qquad \hat{U}(t,0;\Lambda_{n'}) \to \hat{U}(t,0;\Lambda_o). \tag{9.14}$$

We must now show that L_o is uniquely defined by (6.3) as $L(t;\Lambda_o)$. This will establish the continuity of $L(t;\cdot\,)$. From (9.13) it follows readily that $L_{n'} \to L(t;\Lambda_o)$ when $\hat{U}_L(t,0;\Lambda_o) \neq 0$ and $\hat{U}_L(t,L_{\max};\Lambda_o) \neq 0$. If $\hat{U}_L(t,0;\Lambda_o) = 0$, the only troublesome case occurs when $\hat{U}_L(t,0;\Lambda_{n'}) > 0$ for infinitely many n', so we may assume that the sup of $\hat{U}_L(t,\cdot\,;\Lambda_{n'})$ occurs in the interior of $[0,L_{\max}]$. If the limit $L_o \neq 0$, then $\hat{U}(t,L_o;\Lambda_o) < \hat{U}(t,0;\Lambda_o)$ by strict concavity. But now (9.14) implies that for n' sufficiently large $\hat{U}(t,L_{n'};\Lambda_{n'}) < \hat{U}(t,0;\Lambda_{n'})$, contradicting the maximality of $L_{n'}$. It follows that in this case also $L_o = 0 = L(t;\Lambda_o)$. The other endpoint is treated similarly. This proves the continuity of $L(t;\cdot\,)$ for each t.

The result now follows from the bounded convergence theorem if we recall (7.8), (7.9) and that$I^j = (\bar{c}_j,\bar{l}_j) \leq (\kappa_R,1)$. □

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