Equilibrium in a Production Economy [*]

Maria B. Chiarolla[†] and Ulrich G. Haussmann[‡]

Abstract. Consider a closed production-consumption economy with multiple agents and multiple resources. The resources are used to produce the consumption good. The agents derive utility from holding resources as well as consuming the good produced. They aim to maximize their utility while the manager of the production facility aims to maximize profits. With the aid of a representative agent (who has a multivariable utility function) it is shown that an Arrow-Debreu equilibrium exists. In so doing we establish technical results that will be used to solve the stochastic dynamic problem (a case with infinite dimensional commodity space so the General Equilibrium Theory does not apply) elsewhere.

Key words: production-consumption economy, representative agent utility function, economic equilibrium.
AMS subject classification: 03H10, 62P20, 91B50
JEL subject classification: C62

---

[*] This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant 88051, and by the Program for Cultural and Scientific Cooperation between Università di Roma "La Sapienza" and the University of British Columbia.

[†] Dipartimento di Metodi e Modelli per l'Economia, il Territorio e la Finanza, Facoltà di Economia, Università di Roma "La Sapienza", via del Castro Laurenziano 9, 00161 Roma, Italy; maria.chiarolla@uniroma1.it

[‡] Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2; uhaus@math.ubc.ca

1 Introduction

Suppose one good is produced in an economy using N resources. Let $v(z)$ denote the amount of the commodity produced when $z \in \Re^N$ is the vector of amounts of resource used; then v is the production function. One unit of the ith resource costs y_i units of the good - we are doing our accounting in units of the good rather than money. The total amount of the ith resource in the economy is $Z_i \in (0, \infty)$. Resources can include raw materials, physical plant, labour and money. In the last case, Z_i is the money supply (in real terms). In the case of labour, each agent represents one unit of labour, so Z_i is the number of agents. Define

$$A^Z := \prod_{i=1}^{N} [0, Z_i]. \tag{1.1}$$

As elsewhere (Abel and Eberly 1997 [1]) the management of the production facility will set production levels to maximize profit, i.e. will use resources at level

$$\hat{z}(y) \in \operatorname{argmax}_{z \in A^Z} \{v(z) - y^\top z\}, \tag{1.2}$$

where $y^\top$ denotes the transpose of y. Note that this is a static problem, time plays no role. The production function v and the utility functions u^j, cf. below, are assumed to be smooth, strictly increasing and strictly concave.

There are J agents who consume the good and also hold the resources and production facility. Agent j derives utility $u^j(c^j, r^j)$ from consuming $c^j \geq 0$ units of the good and holding r_i^j units of the ith resource, $i = 1, \cdots, N$. We take $r^j \in R^j \subset \Re^N$, i.e. he may have no or limited interest in some of the resources. For example, in the case of labour, $r_i^j \in [0, 1]$ represents the amount of leisure enjoyed by the agent, i.e. the fraction of his unit of labour (his time) which he retains as leisure. Resources are initially held by the agents, agent j having an endowment of ρ_i^j units of the ith resource and a fraction φ^j of the production facility. He obtains a dividend, $\varphi^j\,[v(z) - y^\top z]$, from ownership of the facility. Then the value of his endowment, $\epsilon^j(y, z)$, valued in units of good, is a continuous function of the production input z and the price vector y. Moreover

$$\epsilon^j(y, z) := \sum_{i=1}^{N} \rho_i^j\, y_i + \varphi^j\,[v(z) - y^\top z], \qquad \sum_{j=1}^{J} \rho_i^j = Z_i, \qquad \sum_{j=1}^{J} \varphi^j = 1. \tag{1.3}$$

He chooses c^j and r^j in

$$A^j := [0, \infty) \times R^j$$

to maximize his utility subject to his budget constraint, i.e.

$$(\hat{c}^j, \hat{r}^j) \in \operatorname{argmax}_{(c,r) \in A^j} \{u^j(c, r) : c + y^\top r \leq \epsilon^j(y, z)\}. \tag{1.4}$$

The product structure of A^j is required to apply Chiarolla and Haussmann [4]. Since u^j is increasing, then the constraint reduces to an equality constraint and the problem can be transformed to the form (1.2) with the aid of a Lagrange multiplier η^j:

$$\sup_{x^j \in A^j} \{u^j(x^j) - \eta^j(\bar{y}^\top x^j - \epsilon^j(y, z))\}, \quad x^j := (c, r^\top)^\top, \ \bar{y} := (1, y^\top)^\top.$$

The equilibrium problem consists of finding a price vector y so that markets clear when agents and production managers maximize utilities and profits respectively, i.e.

$$\begin{cases} \sum_j \hat{c}^j &= v(\hat{z}), \\ \hat{z} + \sum_j \hat{r}^j &= Z. \end{cases} \tag{1.5}$$

Such a situation defines (static) equilibrium, similar to an Arrow-Debreu equilibrium but with production. We emphasize that the level of production is determined endogenously as the solution of a maximization problem. The more common case of a stochastic dynamic equilibrium without production, i.e. with exogenous endowment, is discussed in detail in Karatzas, Lehoczky and Shreve 1990 [11] and in Chiarolla and Haussmann 2001 [3]); the stochastic dynamic case with production is studied in Chiarolla and Haussmann 2009 [5].

Optimal consumption problems in a dynamic setting for a single agent with multivariate utility have been considered previously. We mention three such works. In Lakner 1988 [12] the multivariable utility can be reduced to a scalar function; in Bank and Riedel 2003 [2] a utility function of two goods is treated in an *ad hoc* fashion, and in Deelstra, Pham and Touzi 2001 [8] a more abstract approach is used to allow non-smooth utility functions. Unfortunately the assumptions imposed in the last are not always easy to verify. Indeed the point of our work - a static case - is to provide a relatively simple situation so that the effect of production, which is not considered in the above papers, can be included.

Existence of a solution to such problems with finite dimensional commodity space is established using topological methods in great generality by the General Equilibrium Theory (Debreu 1959 [7]; Florenzano 2003 [9]), but this requires a *survival condition* which does not always hold here because A^j may not be "comprehensive" (i.e. $A^j + \Re_+^{1+N} \subset A^j$) and u^j, v may not be strictly increasing at points of A^j, A^Z respectively where one component is zero. In any case our aim is not to make a significant contribution to the economics literature; rather we provide a solution to this problem based on the Negishi method (Negishi 1960 [13]) and an auxiliary maximization problem (maximization of the *utility of production*) interesting in its own right. In so doing we establish some technical results (cf. the Appendix) that are required to solve the stochastic dynamic equilibrium problem of [5] (Chiarolla and Haussmann 2009) where the General Equilibrium Theory does not apply because that problem is time dependent and stochastic, so not finite dimensional. Moreover a simple example demonstrates that the method allows numerical computation of the equilibrium. These results and the auxiliary maximization problem are the main contributions of the paper.

Let us consider the problem (1.2). It was shown that it has solution $\hat{z} = I^v(y)$ where I^v is an extension of $(\nabla v)^{-1}$ beyond $\nabla v(A^Z)$ (Chiarolla and Haussmann 2008 [4]). In the generic one-dimensional example with v strictly increasing and strictly concave and $A^Z = [0, \infty)$, the obvious solution is

$$I^v(y) = \begin{cases} (v_z)^{-1}(y) & \text{if } y \in (0, v_z(0)), \\ 0 & \text{if } y \geq v_z(0). \end{cases}$$

If the Inada condition holds, i.e. $v_z(0) = \infty$, $v_z(\infty) = 0$, then $I^v = (v_z)^{-1}$ on $(0, \infty)$. The problem (1.4) similarly has solution $(\hat{c}^j, \hat{r}^j) = I^{u^j}(\eta^j \bar{y})$.

If there is only one agent in the economy (with utility function u) then the equilibrium problem can be solved as follows. Take y such that $I^u(\eta\bar{y})^\top = (v(\hat{z}), (Z-\hat{z})^\top)$, i.e. $\eta\bar{y} = \nabla u(v(\hat{z}), Z-\hat{z}) = (u_c(v(\hat{z}), Z-\hat{z}), \nabla_r u(v(\hat{z}), Z-\hat{z})^\top)^\top$ where $\nabla_r u$ denotes the gradient with respect to the resources only. Recalling that $\hat{z}$ depends on y, we find that the equilibrium price vector solves a fixed point problem,

$$y = \frac{\nabla_r u(v(\hat{z}), Z-\hat{z})}{u_c(v(\hat{z}), Z-\hat{z})}. \tag{1.6}$$

Rather than solve this unpleasant fixed point problem for y we find another representation of $\hat{z}$ as the solution of an auxiliary maximization problem:

$$\hat{z} = \text{argmax}\ u(v(z), Z-z). \tag{1.7}$$

The solution of this problem gives $\hat{z}$ independent of y; the latter can then be found from (1.6). Then $(\hat{c}, \hat{r}^\top)^\top = I^u(\eta\bar{y})$ solves the agent's problem and y solves the firm's problem. Market clearing follows from $I^u(\eta\bar{y}) = (v(\hat{z}), (Z-\hat{z})^\top)^\top$.

Motivated by this result when $J=1$, we introduce a representative agent with a utility function, $u(c, r; \Lambda)$, which is an aggregation of the u^j and depends on a parameter $\Lambda = (\lambda_1, \cdots, \lambda_J)^\top$ with $\lambda_j > 0$, cf. (2.7). The properties of this function are developed elsewhere (Chiarolla and Haussmann 2008 [4]); they are sufficient to apply the above procedure, the well-known Negishi method, to solve the problem.

In Section 2 we summarize our notation and recall the results on the extension of the inverse of ∇u and the properties of the aggregated utility function established elsewhere (Chiarolla and Haussmann 2008 [4]). The economic equilibrium problem is solved in Section 3 and an example demonstrates how to compute equilibria numerically in a simple case. In the short Appendix we establish existence of the Lagrange multipliers and give the technical proof of some continuity.

2 Production Functions, Utility Functions and the Representative Agent

We first summarize some notation and results from convex analysis on $\Re^n$ (but in the context of *concave* functions); our reference is the classic text by Rockafellar (Rockafellar 1970 [14]). A set is *affine* if it is the translate of a subspace of $\Re^n$, possibly $\{0\}$ or $\Re^n$. $\text{int}(A)$ denotes the interior of the set A, $\text{cl}(A)$ denotes its closure, $\text{bdy}(A)$ denotes its boundary, $\text{aff}(A)$ denotes its affine hull (smallest affine set containing A), and $\text{ri}(A)$ denotes the relative interior of A, i.e. interior of A relative to $\text{aff}(A)$. If u is a function $\Re^n \mapsto [-\infty, \infty)$, then the (effective) domain of u is $\text{dom}(u) := \{x | u(x) > -\infty\}$ and $\text{im}(u) := u(\text{dom}(u))$. u is *non-decreasing* if $u(x) \leq u(x')$ whenever $x, x' \in \text{dom}(u)$ and $x \leq x'$ and where the latter inequality in $\Re^n$ is taken component wise. u is *strictly increasing* if $u(x) < u(x')$ for such x, x' with $x \neq x'$, i.e. u is strictly increasing in each of its n arguments. The function u is *(strictly) concave* if it is (strictly) concave on $\text{dom}(u)$ which we assume to be non-empty. This makes the function a proper, concave function in the terminology of convex analysis. The concave *conjugate* of the concave function $u : \Re^n \mapsto [-\infty, \infty)$ is

$$u^*(y) := \inf_{x \in \Re^n} \{x^\top y - u(x)\}.$$

A *supergradient* of u at $x \in \mathrm{dom}(u)$ is $y \in \Re^n$ such that for all $z \in \Re^n$

$$u(z) - u(x) \le (z - x)^\top y.$$

The set of all supergradients (called the supergradient set or the superdifferential) is denoted by $\partial u(x)$, and $\mathrm{dom}(\partial u) := \{x | \partial u(x) \ne \emptyset\}$ is the (effective) domain of ∂u. We set $\partial u(x) = \emptyset$ if $x \notin \mathrm{dom}(u)$. Note that if u is concave, then $-u$ is convex and $\partial u = -\partial(-u)$ where ∂ on the right side denotes subdifferential, so results for convex functions can be translated to concave functions. If u is differentiable on $\mathrm{int}(\mathrm{dom}(u))$, then $\partial u = \{\nabla u\}$ on this set and ∇u is *monotone* (i.e. $(\nabla u(x) - \nabla u(x'))^\top (x - x') \le 0$), and *strictly monotone* (i.e. < 0 for $x \ne x'$) on sets where u is strictly concave. Finally when $\partial u = \{\nabla u\}$, we will write $\partial u = \nabla u$.

The *normal cone* to the convex set A at $x \in A$, denoted by $\mathcal{N}_A(x)$, consists of the outward normals to A at x. It is empty for $x \notin A$ and is $\{0\}$ for $x \in \mathrm{int}(A)$. Note that $\mathrm{aff}(\mathcal{N}_A(x)) = \mathcal{N}_A(x) - \mathcal{N}_A(x)$ is a subspace . Define

$$\chi_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A; \end{cases}$$

it is the indicator function of A. Then $\partial \chi_A(x) = \mathcal{N}_A(x)$ for $x \in A$ and is empty for $x \notin A$. With $A \subset \Re^n$ define $u_A(x) := u(x) - \chi_A(x)$, $\Re^n_{++} := \{x \in \Re^n : x_i > 0,\ i = 1, \ldots, n\}$ and $\Re^n_+ := \{x \in \Re^n : x_i \ge 0,\ i = 1, \ldots, n\}$. Let $\vec{e}_i$ denote the ith standard basis vector.

It is convenient to define a production function v on all of $\Re^N$ but permitting the value $-\infty$, so we are really only interested in $v(z)$ for $z \in \mathrm{dom}(v) := \{z : v(z) > -\infty\}$. We shall certainly want v to be non-decreasing and concave. We add some regularity for technical reasons; thus we **assume**

$$\begin{array}{rl} \text{(i)} & v : \Re^N \mapsto [-\infty, \infty) \text{ is upper semicontinuous, concave, non-decreasing;} \\ \text{(ii)} & v \text{ is continuous on } \mathrm{dom}(v); \\ \text{(iii)} & v \text{ is strictly increasing, strictly concave and continuously differentiable} \\ & \text{on } \mathrm{int}(\mathrm{dom}(v)); \\ \text{(iv)} & A^Z \subset \mathrm{dom}(v) \text{ and } v(A^Z \backslash \{0\}) \subset (0, \infty); \\ \text{(v)} & \mathrm{dom}(\partial v_{A^Z}) \subset \mathrm{int}(\mathrm{dom}(v)) \cap A^Z; \\ \text{(vi)} & A^Z \backslash \{0\} \subset \mathrm{dom}(\partial v_{A^Z}). \end{array} \tag{2.1}$$

Then v is a closed, proper, concave function, cf. (i), (iv), (i.e. $v < +\infty$ always, $v > -\infty$ somewhere and v is upper semicontinuous), and $\partial v = \nabla v$ on $\mathrm{int}(\mathrm{dom}(v))$. Note that the first part of (iv) implies that $\mathrm{ri}(\mathrm{dom}(v)) = \mathrm{int}(\mathrm{dom}(v))$ since $Z_i > 0$, and the second implies that for $z \in A^Z$, $v(z) = 0$ implies $z = 0$. Since $\mathrm{ri}(A^Z) \cap \mathrm{ri}(\mathrm{dom}(v)) = \mathrm{int}(A^Z) \cap \mathrm{int}(\mathrm{dom}(v)) \ne \emptyset$, then $\partial v_{A^Z}(z) = \partial v(z) - \mathcal{N}_{A^Z}(z)$ (Rockafellar 1970 [14], Theorem 23.8.) $= \nabla v(z) - \mathcal{N}_{A^Z}(z)$ on

$\mathrm{int}(\mathrm{dom}(v))$, and $\mathrm{dom}(\partial v_{A^Z}) = A^Z \cap \mathrm{dom}(\partial v)$. Part (vi) simply means that $\|\nabla v(z)\| < \infty$ for $z \neq 0$. Certainly the assumptions imply that $\mathrm{dom}(\partial v_{A^Z}) = A^Z$ or $A^Z \backslash \{0\}$.

The above set-up has a drawback: it does not cover the case when v is of Cobb-Douglas form (i.e. $v(z) = \prod_{i=1}^N z^{\alpha_i}, \alpha_i > 0, \sum_{i=1}^N \alpha_i < 1$), since then $v(z) = 0$ for $z \in \mathrm{bdy}(\Re^N_{++})$. It may happen that $v = 0$ on a part of $\mathrm{bdy}(\Re^N_{++})$, but then such points cannot lie in $\mathrm{int}(\mathrm{dom}(v))$ by (2.1)(iii), so by (2.1)(v) such points are not in $\mathrm{dom}(\partial v)$ and (2.1)(vi) fails. To remediate this, we will sometimes modify these conditions by replacing (iv) and (vi) by

(iv)′ $A^Z \subset \mathrm{dom}(v)$ and $v(A^Z) \subset [0, \infty)$;

(vi)′ there exist closed sets G_i^v, with disjoint interiors such that $A^Z = \bigcup_{i=1}^N G_i^v$, and for $i \in \mathcal{I}^v := \Big\{ i \in \{1, \ldots, N\} : \{z \in A^Z : z_i = 0\} \cap \mathrm{dom}(\partial v) = \emptyset \Big\}$, $\lim_{\delta \to 0} \inf_{z \in G_i^v(\delta)} v_{z_i}(z) = \infty$, where $G_i^v(\delta) := G_i^v \cap \{z_i \leq \delta\}$.

We observe that if $i \in \mathcal{I}^v$ then at the boundary of A^Z where $z_i = 0$, $\|\nabla v(z)\| = \infty$, i.e. ∇v does not exist. Note that if $\mathcal{I}^v = \emptyset$ then $A^Z \backslash \{0\} \subset \mathrm{dom}(\partial v)$ so v strictly increasing implies that we are in the previous case, hence we may assume $\mathcal{I}^v \neq \emptyset$. **Note that conditions (2.1)(i)-(iii),(iv)′,(v),(vi)′ will be referred to as (2.1)′.**

The utility function u^j of the jth agent should also be defined on all of $\Re^n$ with the value $-\infty$ allowed, but again we are really only interested in $u^j(x)$ for $x \in \mathrm{dom}(u^j) := \{x : u^j(x) > -\infty\}$; in fact we shall further restrict the domain to a set A^j, so we consider $u^j_{A^j} = u^j - \chi_{A^j}$.

Write $x = (x_0, x_1, \ldots, x_N)^\top$ for $(c, r^\top)^\top$ and set $n = 1 + N$. Then for $j = 1, \ldots, J$, we **assume**

(i) $u^j : \Re^n \mapsto [-\infty, \infty)$ is upper semicontinuous, concave, non-decreasing;

(ii) u^j is continuous on $\mathrm{dom}(u^j)$;

(iii) u^j is twice continuously differentiable on $\mathrm{int}(\mathrm{dom}(u^j))$ and $u_{x_i x_k} \geq 0$ for $i \neq k$;

(iv) $\Re^n_{++} \subset \mathrm{dom}(\partial u^j)$;

(v) for $i = 0, 1, \cdots, N$, $D_i^j := \begin{cases} [0,\infty) \text{ or } (0,\infty) & \text{if } i \in \mathcal{M}^j, \\ [0,1] \text{ or } (0,1] & \text{if } i \in \tilde{\mathcal{M}}^j, \\ \{0\} & \text{if } i \in \mathcal{M}^j_o, \end{cases}$
with $\mathcal{M}^j \cup \tilde{\mathcal{M}}^j \cup \mathcal{M}^j_o = \{0, 1, \cdots, N\}$, $0 \in \mathcal{M}^j$, and $A^j := \prod_{i=0}^N D_i^j \subset \mathrm{dom}(u^j)$, A^j closed in $\mathrm{dom}(u^j)$; (2.2)

(vi) $\mathrm{dom}(\partial u^j_{A^j}) \subset \mathrm{int}(\mathrm{dom}(u^j)) \cap A^j$;

(vii) u^j is strictly increasing, strictly concave on $\mathrm{dom}(\partial u^j_{A^j})$;

(viii) there exists $\mu^j \in 0^+ A^j := \{\sum_{i \in \mathcal{M}^j} \lambda_i \vec{e}_i : \lambda_i \geq 0\}$ such that for all $x \in 0^+ A^j$,
$$\inf_{z \in \mathrm{dom}(\partial u^j_{A^j})} \nabla u^j(z)^\top x \leq x^\top \mu^j.$$

The condition (iii) means that the resources as well as consumption are ALEP complementary, cf. Kannai 1980 [10]. We rewrite $A^j = D_0^j \times R^j$ with $R^j = \prod_{i \geq 1} D_i^j$. Part (iv) of (2.2) states that $\nabla u^j(x)$ is locally bounded on $\Re^n_{++}$. Moreover $\mathrm{ri}(\mathrm{dom}(u^j)) = \mathrm{int}(\mathrm{dom}(u^j))$ since $\Re^n_{++} \subset \mathrm{dom}(\partial u^j) \subset \mathrm{dom}(u^j)$. Hence $\mathrm{int}(\mathrm{dom}(u^j)) \subset \mathrm{dom}(\partial u^j)$. The fact that A^j is closed in $\mathrm{dom}(u^j)$ determines the inclusion or exclusion of 0 at the left end of D_i^j. Since

$\mathrm{ri}(A^j) = \mathrm{ri}(\mathrm{dom}(u^j_{A^j})) \subset \mathrm{dom}(\partial u^j_{A^j}) \subset \mathrm{int}(\mathrm{dom}(u^j))$, cf. (vi), then again a result of Rockafellar (Rockafellar 1970 [14], Theorem 23.8) implies that $\partial u^j_{A^j} = \partial u^j - \mathcal{N}_{A^j}$, hence $\mathrm{dom}(\partial u^j_{A^j}) = \mathrm{dom}(\partial u^j) \cap A^j \subset \mathrm{int}(\mathrm{dom}(u^j)) \cap A^j$ by part (vi) and we have equality throughout.

Condition (viii) restricts the behaviour of $u^j_{A^j}$ at ∞ and helps to determine $\mathrm{dom}(I^{u^j_{A^j}})$. In the scalar case, $n = 1$, part of the Inada condition demands "$u^j_x = 0$ at ∞"; we have relaxed this to (viii), i.e. $u^j_x \leq \mu^j$ for some $\mu^j \geq 0$.

Remark 2.1 Notice that the other half of the Inada condition requires $\lim_{x \to 0} u^j_x(x) = \infty$ in the scalar case; here we only require that *if* $\|\nabla u^j(x)\| \to \infty$ as x approaches $\mathrm{bdy}(\Re^n_{++})$, then it does so uniformly, cf. (2.4) below.

If we use $(2.1)'$ we add to (2.2):

(ix) for all j and $i \in \mathcal{I}^v$, $u^j_{r_i}(c, r)/u^j_c(c, r)$ is independent of r_k, $k \neq i$.

Note that (2.2) augmented by (ix) will be referred to as $(2.2)'$.

We now add more assumptions in three steps.

- The boundary of A^j decomposes into a finite number of relatively open sets, $\mathcal{C}^j_k$, i.e. corners, edges, faces, etc. To eliminate some pathologies we **assume** for any j and k

$$\mathcal{C}^j_k \cap \mathrm{dom}(\partial u^j) \neq \emptyset \Rightarrow \mathcal{C}^j_k \subset \mathrm{dom}(\partial u^j). \tag{2.3}$$

- Turning to behaviour at $\mathrm{bdy}(A^j) \cap (\mathrm{dom}(\partial u^j))^c$ where ^{c} denotes complement, define

$$\mathcal{I}^j := \{i \in \mathcal{M}^j \cup \tilde{\mathcal{M}}^j : \{x \in A^j : x_i = 0\} \cap \mathrm{dom}(\partial u^j) = \emptyset\}.$$

The continuity of ∇u^j implies that for any convergent sequence $\{x^k\} \subset A^j$, $\lim_k \|\nabla u^j(x^k)\| = \infty$ only if $\mathcal{I}^j \neq \emptyset$. We **assume** for each j

> there exist sets G^j_i, closed in A^j, with disjoint relative interiors such that $A^j = \bigcup_{i=0}^N G^j_i$, and for $i \in \mathcal{I}^j$, $\lim_{\delta \to 0} \inf_{x \in G^j_i(\delta)} u^j_{x_i}(x) = \infty$, where $G^j_i(\delta) := G^j_i \cap \{x_i \leq \delta\}$. Conversely, $x^k_i \to 0$ for any bounded sequence $\{x^k\} \subset A^j$ such that $u^j_{x_i}(x^k) \to \infty$ and the rate of convergence of x^k_i depends on the sequence only through $\sup_k \|x^k\|$. (2.4)

The first part of (2.4) implies that for $x \in \mathrm{bdy}(A^j)$ but $x \notin \mathrm{dom}(\partial u^j)$, $\lim_{x^k \to x} u^j_{x_i}(x^k) = \infty$ uniformly in G^j_i; it is void if ∇u^j is finite on $A^j \cap \mathrm{bdy}(\Re^n_{++})$. This part is used to establish the existence of an equilibrium, cf. Lemma 4.3. The second part implies that if $u^j_{x_i}(x^k) \to \infty$ then $i \in \mathcal{I}^j$ and is used to obtain existence of the Lagrange multipliers, cf. Corollary 4.2, and regularity of the representative agent's utility function, cf. Theorem 2.6.

We point out that in the case $n = 1$ the Inada conditions imply (2.4) and (2.2)(viii) with $\mu^j = 0$, although the latter are weaker.

- Finally we link the u^j's and Z. With $\tilde{J}_i := \text{card}\{j : i \in \tilde{\mathcal{M}}^j\}$ we **assume**

  $$\textstyle\bigcap_{j=1}^J \mathcal{M}_o^j = \emptyset, \quad \text{for } i \notin \bigcup_{j=1}^J \mathcal{M}^j,\ Z_i \le \tilde{J}_i. \tag{2.5}$$

  This condition states first that each resource is coveted by at least one agent and second that for resources for which the maximum demand from every agent is finite (i.e. 0 or 1), the supply can never exceed this maximal demand, hence positive production is at the cost of unmet demand for resources by the agents. An example would be labour/leisure. Let it be resource one. Moreover $1 \in \tilde{\mathcal{M}}^j \cup \mathcal{M}_o^j$, i.e. the jth agent can provide one unit of labour ($1 \in \tilde{\mathcal{M}}^j$) or none ($1 \in \mathcal{M}_o^j$), but in the former case he reduces this by his leisure choice, $x_1^j \in [0, 1]$. Then the maximum amount of labour available is $Z_1 = \tilde{J}_1$, i.e. the second part of (2.5) holds.

Note that (2.1) - (2.5) are our **standard assumptions**; when they are modified by replacing (2.1) by (2.1)$'$ and (2.2) by (2.2)$'$ we refer to them as the **alternate assumptions**. One or the other is always assumed to hold.

Example 2.2 Here are some examples.

(i) $n = 1$.

(a)

$$u^j(x) := \begin{cases} \ln(x+\varepsilon) & \text{if } x > -\varepsilon, \\ -\infty & \text{otherwise.} \end{cases}$$

Then $\text{dom}(u^j) = \text{dom}(\partial u^j) = (-\varepsilon, \infty)$. If $\varepsilon > 0$ then we can take $A^j = [0, \infty)$ or $[0, 1]$, but if $\varepsilon = 0$ then we must take $A^j = (0, \infty)$ or $(0, 1]$ in order to satisfy (2.2). (2.2)(viii) holds with $\mu = 0$. Since $\text{bdy}(A^j)$ consists of $\{0\}$ and possibly $\{1\}$, then (2.3) holds. If $\varepsilon > 0$ then (2.4) is empty since $\mathcal{I}^j = \emptyset$, and for $\varepsilon = 0$ it is satisfied with $G_i^j = A^j$.

(b) $v(z)$ of the same form with $\varepsilon > 0$ satisfies (2.1) for $A^Z = [0, Z]$.

(c)

$$u^j(x) := \begin{cases} \sqrt{x+1} & \text{if } x \ge 0, \\ -\infty & \text{otherwise} \end{cases}$$

and $A^j = [0, \infty)$. To satisfy (2.2)(vi) we redefine $u^j(x) = \sqrt{x+1}$ on $x > -1$. This produces no change in $u^j_{A^j}$ and now (2.2) - (2.4) hold.

(d) For $\mu \ge 0$

$$u^j(x) := \begin{cases} \mu x + \frac{x}{1+x} & \text{if } x > -1, \\ -\infty & \text{otherwise.} \end{cases}$$

Then $\text{dom}(u^j) = \text{dom}(\partial u^j) = (-1, \infty)$. Let $A^j = [0, \infty)$. Then (2.2)-(2.4) are satisfied with $\mu^j = \mu$. In fact (2.4) is empty.

(ii) $n = 2$ and

$$u^j(x_0, x_1) := \begin{cases} (x_0 + \varepsilon_0)^{\gamma_0}(x_1 + \varepsilon_1)^{\gamma_1} & \text{if } x_i \geq -\varepsilon_i,\ i = 0, 1, \\ -\infty & \text{if either } x_i < -\varepsilon_i, \end{cases}$$

with $\gamma_i > 0,\ \gamma_0 + \gamma_1 < 1$. Then $\mathrm{dom}(u^j) = -(\varepsilon_0, \varepsilon_1) + \Re^2_+$ and $\mathrm{dom}(\partial u^j) = -(\varepsilon_0, \varepsilon_1) + \Re^2_{++} = \mathrm{int}(\mathrm{dom}(u^j))$. Take $\varepsilon_i \geq 0$.

(a) If $A^j = [0, \infty) \times [0, 1]$, then (2.2) is satisfied with $\mu^j = 0$. Note that $\mathrm{dom}(\partial u^j_{A^j}) = A^j$ if both $\varepsilon_i > 0$ so (2.4) is empty, but $\mathrm{dom}(\partial u^j_{A^j}) = (0, \infty) \times [0, 1]$ if $\varepsilon_0 = 0, \varepsilon_1 > 0$. Then $\mathcal{I}^j = \{0\}$ and we take $G^j_0 = A^j, G^j_1 = \emptyset$. If both $\varepsilon_i = 0$, then $\mathcal{I}^j = \{0, 1\}$ and $G^j_0 = A^j \cap \{x_1 \geq x_0\}$, $G^j_1 = A^j \cap \{x_1 \leq x_0\}$. Then (2.4) holds. Moreover $\mathrm{bdy}(A^j)$ is composed of $(0, \infty) \times \{0\}$, $\{(0, 0)\}$, $\{0\} \times (0, 1)$, $\{(0, 1)\}$, $(0, \infty) \times \{1\}$ so (2.3) holds.

(b) If $A^j = [0, \infty) \times \{0\}$ and $\varepsilon_1 > 0$ then $\mathrm{dom}(\partial u^j_{A^j}) = A^j$ if $\varepsilon_0 > 0$, but $\mathrm{dom}(\partial u^j_{A^j}) = A^j \cap \{x_0 > 0\}$ if $\varepsilon_0 = 0$. As above it follows that (2.2)-(2.4) are satisfied with $\mu^j = 0$.

(c) For $A^j = [0, \infty) \times \{0\}$, we may equivalently consider the case $\gamma_1 = 0$, i.e.

$$u^j(x_0, x_1) := \begin{cases} (x_0 + \varepsilon_0)^{\gamma_0} & \text{if } x_0 \geq -\varepsilon_0, \\ -\infty & \text{if } x_0 < -\varepsilon_0, \end{cases}$$

with $0 < \gamma_0 < 1$, so $\mathrm{dom}(\partial u^j) = (-\varepsilon_0, \infty) \times \Re = \mathrm{int}(\mathrm{dom}(u^j))$. (2.2)-(2.4) are satisfied with $\mu^j = 0$. Observe that $\mathcal{N}_{A^j}(x) = \{0\} \times \Re$ for $x \in \mathrm{ri}(A^j)$, $\mathcal{N}_{A^j}(x) = (-\infty, 0] \times \Re$ for $x = (0, 0)$ and is empty otherwise. Moreover $\mathrm{dom}(\partial u^j_{A^j}) = A^j$ for $\varepsilon_0 > 0$, $\mathrm{dom}(\partial u^j_{A^j}) = (0, \infty) \times \{0\}$ for $\varepsilon_0 = 0$, so for $x \in \mathrm{ri}(A^j)$, $\partial u^j_{A^j}(x) = \nabla u^j(x) - \mathcal{N}_{A^j}(x) = \{u^j_{x_0}(x)\} \times \Re$.

(iii) $N = 2$ and (Cobb-Douglas)

$$v(z_1, z_2) := \begin{cases} z_1^{\gamma_1} z_2^{\gamma_2} & \text{if } z_i \geq 0,\ i = 1, 2, \\ -\infty & \text{if either } z_i < 0, \end{cases}$$

with $\gamma_i > 0,\ \gamma_1 + \gamma_2 < 1$. Then $\mathrm{dom}(v) = \Re^2_+$ and $\mathrm{dom}(\partial v) = \Re^2_{++} = \mathrm{int}(\mathrm{dom}(v))$. $A^Z = [0, Z_1] \times [0, Z_2]$. Then (2.1) fails since $v(0, Z_1) = 0 \notin (0, \infty)$ and $\|\nabla v(0, Z_1)\| = \infty$. In fact $\mathcal{I}^v = \{1, 2\}$. However we can define G^v_i analogously to G^j_i in (ii)(a) so that $(2.1)'$ holds. Moreover $(2.2)'$(ix) holds if for example for all j, $u^j(c, r_1, r_2) = c^{\alpha^j_0} r_1^{\alpha^j_1} r_2^{\alpha^j_2}$ with $\alpha^j_i > 0$, $\alpha^j_0 + \alpha^j_1 + \alpha^j_2 < 1$ because $u^j_{r_i}(c, r)/u^j_c(c, r) = (\alpha^j_i\, c)/(\alpha^j_0\, r_i)$.

To solve (1.4) we require an extension of the inverse of $\mathrm{gr}_{A^j} u^j(x) := \nabla u^j(x)$ when $x \in A^j$, $= -\infty$ otherwise; such a result is established by the authors (Chiarolla and Haussmann 2008 [4], Proposition 3.2). We recall it below, cf. Theprem 2.4. For the moment let *the pair* (u, A) *stand for* (u^j, A^j) *or* (v, A^Z) and let m stand for $m^j = \mathrm{card}(\mathcal{M}^j)$, μ for μ^j with $m = 0$, $\mu = 0$ in the case of v. Define

$$\mathcal{R}_{u_A} := \nabla u(\mathrm{dom}(\partial u_A)) = \nabla u(A \cap \mathrm{int}(\mathrm{dom}(u))). \tag{2.6}$$

Notice that as u is strictly increasing on $\mathrm{dom}(\partial u_A)$, then $\mathcal{R}_{u_A}$ lies in the positive orthant of $\mathrm{aff}(A)$, i.e. in $\Re^n_{++}$ if $\mathcal{M}_o = \emptyset$.

Definition 2.3 Since A is a polyhedron then $\mathrm{bdy}(A)$ (which may be all of A) decomposes into a finite number of disjoint convex sets, $\mathcal{C}_k,\ k = 1, \ldots, K$, each being a corner, edge, face or other lower dimensional set such that $\mathcal{C}_k = \mathrm{ri}(\mathcal{C}_k)$. Set $\mathcal{S}_k := \nabla u(\mathcal{C}_k)$, $\mathcal{S}_0 := \mathrm{int}(\mathcal{R}_{u_A})$ and $\mathcal{C}_0 := \mathrm{int}(A)$.

Note that $\mathcal{S}_k = \emptyset$ if $\nabla u(\cdot)$ is not finite on $\mathcal{C}_k$, i.e. $\mathcal{C}_k \cap \mathrm{dom}(\partial u) = \emptyset$.

Theorem 2.4 *(a) There exists a continuous, monotone function* $I^{u_A} : \mathrm{int}(\mathrm{dom}(u^*_A)) \to \mathrm{dom}(\partial u_A) = \mathrm{dom}(\partial u) \cap A$ *which extends* $(\mathrm{gr}_A u)^{-1}$. I^{u_A} *is strictly monotone on* $\mathcal{R}_{u_A}$. *Moreover* $\mathrm{dom}(I^{u_A})$ *is an open set such that* $(\mu + \Re^m_{++}) \oplus \Re^{n-m} \subset \mathrm{dom}(I^{u_A}) \subset \Re^m_{++} \oplus \Re^{n-m}$.

(b) There exists a projection $\mathcal{P}_{u_A} : \mathrm{dom}(I^{u_A}) \mapsto \mathcal{R}_{u_A}$ *such that*

$$I^{u_A}(y) = (\nabla u)^{-1}(\mathcal{P}_{u_A}(y)),$$

i.e. $y = \nabla u(x) - \vec{n}$ *with* $x = I^{u_A}(y)$, $\vec{n} \in \mathcal{N}_A(x)$ *and* $\mathcal{P}_{u_A}(y) := \nabla u(x)$.

(c) If ∇u *is* p *times continuously differentiable, then* I^{u_A} *and* $\mathcal{P}_{u_A}$ *are* p *times continuously differentiable on* $\mathrm{int}(\mathcal{S}_k)$ *for each* k. *Moreover for each* k, I^{u_A} *has a continuously differentiable extension to* $\mathrm{cl}(\mathcal{S}_k)$.

Since A is a polyhedral set, then $\mathcal{N}_A(x)$ is constant on each face, so the projection is particularly simple to identify (Chiarolla and Haussmann 2008 [4], Figure 2). The boundary points of $\mathcal{R}_{u_A}$ are either points of $\mathrm{bdy}(\mathrm{dom}(I^{u_A}))$ (hence not in $\mathcal{R}_{u_A}$) or are points in $\mathcal{R}_{u_A}$ across which ∇u_A can be extended (Chiarolla and Haussmann 2008 [4], Figure 1). Note that the extension of the derivative of I^{u_A} is given by (3.7) of Chiarolla and Haussmann 2008 [4].

Example 2.5 We give two examples. Take $n = 3$, $x = (c, r_1, r_2)^\top$.

(i) Let $u^1(x) = c^{\frac{1}{3}} r_1^{\frac{1}{3}} r_2^{\frac{1}{6}}$ and $A^1 = [0, \infty) \times [0, 1] \times [0, \infty)$. Then $\mathrm{dom}(\partial u^1_{A^1}) = (0, \infty) \times (0, 1] \times (0, \infty)$ and on $\nabla u^1(\mathrm{dom}(\partial u^1_{A^1})) = \{y \in \Re^3_{++} : y_1 \geq u^1_{r_1}(c, 1, r_2),\ (c, r_2) \in \Re^2_{++}\}$ we find

$$I^{u^1_{A^1}}(y_0, y_1, y_2) = \frac{1}{4 \times 3^6}\Big(\frac{2}{y_0^3\, y_1^2\, y_2},\ \frac{2}{y_0^2\, y_1^3\, y_2},\ \frac{1}{y_0^2\, y_1^2\, y_2^2}\Big).$$

This function extends to $(0, \infty) \times (-\infty, \infty) \times (0, \infty)$ by projection parallel to the y_1-axis, i.e. if the above formula gives $r_1 = I_1^{u^1_{A^1}}(y) \notin (0, 1]$ then increase y_1, specifically if

$$y_1 < \frac{1}{2^{\frac{1}{3}} \times 9\ y_0^{\frac{2}{3}}\ y_2^{\frac{1}{3}}}$$

then replace it by this quantity when computing $I_1^{u^1_{A^1}}(y)$.

(ii) Let $u^2(x) = c^{\frac{1}{3}} r_1^{\frac{1}{3}}$ and $A^2 = [0, \infty) \times [0, 1] \times \{0\}$. Then $\mathrm{dom}(\partial u^2_{A^2}) = (0, \infty) \times (0, 1] \times \{0\}$ and on $\nabla u^2(\mathrm{dom}(\partial u^2_{A^2})) = \{y \in \Re^2_{++} \oplus \{0\} : y_1 \geq u^2_{r_1}(c, 1),\ c \in (0, \infty)\}$ we find

$$I^{u^2_{A^2}}(y_0, y_1, y_2) = \frac{1}{27}\Big(\frac{1}{y_0^2\, y_1},\ \frac{1}{y_0\, y_1^2},\ 0\Big).$$

As above, this function extends to $\Re^2_{++} \oplus \Re^1$ by projection parallel to the y_1 and y_2-axes, i.e. if $y_1 < \frac{1}{\sqrt{27\, y_0}}$ then replace it by this quantity, and set $y_2 = 0$.

As we shall use the Negishi method to establish existence, we introduce a representative agent; his utility function is an aggregation of the $u^j_{A^j}$ defined as follows. For $\Lambda = (\lambda_1, \cdots, \lambda_J) \in \Re^J_{++}$ define $u(x;\Lambda)$ as a supremal convolution on $\Re^n$:

$$u(x;\Lambda) := \sup_{\sum_j x^j = x} \sum_{j=1}^J \lambda_j u^j_{A^j}(x^j), \tag{2.7}$$

where $x^j = (x^j_0, \cdots, x^j_N)^\top \in \Re^n$. Recall $n = N+1$. Define

$$I^u(y;\Lambda) := \sum_{j=1}^J I^{u^j_{A^j}}\Big(\frac{y}{\lambda_j}\Big) \qquad \tilde{A} := \sum_j \mathrm{dom}(\partial u^j_{A^j}), \qquad \tilde{\mathcal{R}}(\Lambda) := \nabla u(\tilde{A};\Lambda). \tag{2.8}$$

and set $\mathcal{D}(\Lambda) := \mathrm{dom}(I^u(\cdot\,;\Lambda)) = \bigcap_j \lambda_j \mathrm{dom}(I^{u^j_{A^j}})$. Then

$$\bigcap_j \Big(\lambda_j \mu^j + \Re^{m^j}_{++} \oplus \Re^{n-m^j}\Big) \subset \mathcal{D}(\Lambda) \subset \bigcap_j \Big(\Re^{m^j}_{++} \oplus \Re^{n-m^j}\Big) \tag{2.9}$$

Observe that if $\mu^j = 0$ for all j, then $\mathcal{D}(\Lambda) = \bigcap_j \Big(\Re^{m^j}_{++} \oplus \Re^{n-m^j}\Big)$. Combining Theorem 4.3 and Corollary 4.4 of Chiarolla and Haussmann 2008 [4] yields

Theorem 2.6 *Assume either the standard assumptions or the alternate ones. Then the following hold.*

(i) For each $\Lambda \in \Re^J_{++}$, $u(\cdot\,;\Lambda) : \Re^n \mapsto [-\infty, \infty)$ *is a closed, proper, concave, non-decreasing function on* $\Re^n$ *with* $\mathrm{dom}(u(\cdot\,;\Lambda)) = A := \sum_{j=1}^J A^j$. *For each* $x \in A$ *there exist* $\hat{x}^j \in A^j$ *such that*

$$x = \sum_j \hat{x}^j, \quad u(x;\Lambda) = \sum_j \lambda_j u^j_{A^j}(\hat{x}^j). \tag{2.10}$$

Moreover $I^u(\cdot\,;\Lambda)$ *is the inverse of* $\partial u(\cdot\,;\Lambda)$ *and is monotone.* $u(\cdot\,;\Lambda)$ *is strictly concave on* $\mathrm{dom}(\partial u(\cdot\,;\Lambda)) = \mathrm{im}(I^u(\cdot\,;\Lambda))$.

(ii) $\mathrm{im}(I^u(\cdot\,;\Lambda)) = \tilde{A}$ *is convex. For* $x \in \tilde{A}$, *there exists* $y \in (I^u(\cdot\,;\Lambda))^{-1}(x)$ *such that the* $\hat{x}^j$ *of (i) can be represented as*

$$\hat{x}^j = I^{u^j_{A^j}}(\tfrac{y}{\lambda_j}). \tag{2.11}$$

(iii) $u(\cdot;\Lambda)$ *is continuously differentiable on* $\tilde{A}$. *This convex set is dense in* A *hence* A *and* $\tilde{A}$ *have the same interior. Moreover* $I^u(\cdot;\Lambda) = (\nabla u(\cdot;\Lambda))^{-1}$ *on* $\tilde{\mathcal{R}}(\Lambda)$, *so* $I^u(y;\Lambda)$ *is a continuous, monotone extension of* $(\nabla u(\cdot;\Lambda))^{-1}$, *strictly monotone on* $\mathrm{cl}(\tilde{\mathcal{R}}(\Lambda)) \cap \mathcal{D}(\Lambda)$. $u(\cdot\,;\Lambda)$ *is strictly increasing on* $\tilde{A}$. *For* $y \in \tilde{\mathcal{R}}(\Lambda)$, *we have*

$$\nabla u(I^u(y;\Lambda);\Lambda) = y. \tag{2.12}$$

(iv) For each $x \in \tilde{A}$,

$$\nabla u(x;\Lambda) = \lambda_j \nabla u^j(\hat{x}^j) - \vec{n}^j(\hat{x}^j), \quad j = 1, \cdots, J, \tag{2.13}$$

where $\vec{n}^j(\hat{x}^j) \in \mathcal{N}_{A^j}(\hat{x}^j)$ *and for each* $i = 1, \ldots, n$ *there exists* $j(i)$ *such that* $[\vec{n}^{j(i)}(\hat{x}^{j(i)})]_i = 0$.

(v) $\nabla u(x; \Lambda)$ *is piecewise continuously differentiable.*

Remark 2.7 Corollary 4.4 of Chiarolla and Haussmann 2008 [4] allows us to replace the awkward (open) set

$$\mathcal{R}_o(\Lambda) := \Big\{ y \in \mathcal{D}(\Lambda) : \bigcap_j \mathrm{aff}\Big(\mathcal{N}_{A^j}\Big(I^{u^j_{A^j}}\big(\frac{y}{\lambda_j}\big)\Big)\Big) = \{0\} \Big\} \tag{2.14}$$

of Chiarolla and Haussmann 2008 [4], Theorem 4.3, by $\tilde{\mathcal{R}}(\Lambda) = \nabla u(\tilde{A}; \Lambda)$, a subset of $\mathrm{cl}(\mathcal{R}_o(\Lambda))$.

Remark 2.8 Since $\tilde{A} = \sum \mathrm{dom}(\partial u^j_{A^j})$ and $\mathrm{dom}(\partial u^j_{A^j})$ is closed except possibly on some of the $\mathcal{C}^j_k \subset \mathrm{bdy}(\Re^n_{++})$, then $\tilde{A}$ has the same structure. In fact a face, edge etc. (in $\mathrm{bdy}(\Re^n_{++})$) of $\tilde{A}$ is contained in $\tilde{A}$ if and only if the corresponding face etc. of each $\mathrm{dom}(\partial u^j_{A^j})$ is contained in $\mathrm{dom}(\partial u^j_{A^j})$. Let $\mathcal{I} := \bigcup_j \mathcal{I}^j$, cf. (2.4) for $\mathcal{I}^j$. The elements of $\mathcal{I}$ identify the coordinate hyperplanes on which $\partial u^j = \emptyset$ for at least one j. In fact (2.2) implies that $\mathrm{dom}(\partial u^j_{A^j}) = A^j \backslash \bigcup_{i \in \mathcal{I}^j} \{x_i = 0\}$ so $\tilde{A} = A \backslash \bigcup_{i \in \mathcal{I}} \{x \in A : x_i = 0\}$.

Remark 2.9 In Chiarolla and Haussmann 2008 [4], Example 4.7, we have shown that if u^1, u^2, A^1, A^2 are as defined in Example 2.5, i.e.

$$\begin{aligned}
u^1(c, r_1, r_2) &= c^{\frac{1}{3}} r_1^{\frac{1}{3}} r_2^{\frac{1}{6}}, \quad A^1 = [0,\infty) \times [0,1] \times [0,\infty), \\
u^2(c, r_1, r_2) &= c^{\frac{1}{3}} r_1^{\frac{1}{3}}, \qquad A^2 = [0,\infty) \times [0,1] \times \{0\},
\end{aligned}$$

then

$$A = [0,\infty) \times [0,2] \times [0,\infty), \quad \tilde{A} = (0,\infty) \times (0,2] \times (0,\infty),$$

and

$$\begin{aligned}
u_{x_0}(x; \Lambda) &= \lambda_1 u^1_{x_0}(\hat{x}^1) &&= \lambda_2 u^2_{x_0}(\hat{x}^2), \\
u_{x_1}(x; \Lambda) &= \lambda_1 u^1_{x_1}(\hat{x}^1) - n^1_1 &&= \lambda_2 u^2_{x_1}(\hat{x}^2) - n^2_1, \\
u_{x_2}(x; \Lambda) &= \lambda_1 u^1_{x_2}(\hat{x}^1)
\end{aligned}$$

with $n^j_1 > 0$ only if $\hat{x}^j_1 = 1$; otherwise $n^j_1 = 0$. Note that $\mathcal{M}^1 = \{0, 2\}$ $\mathcal{M}^2 = \{0\}$, $\tilde{\mathcal{M}}^j = \{1\}$, $\mathcal{M}^1_o = \emptyset$, $\mathcal{M}^2_o = \{2\}$, $\mathcal{I}^1 = \{0, 1, 2\}$, $\mathcal{I}^2 = \{0, 1\}$. In fact we can solve for $u(c, r_1, r_2; \Lambda)$. We may take $\Lambda = (1, \lambda)$ as u is linear in Λ, cf. (2.7). We can solve the latter. The set $\tilde{A}$ decomposes into $\tilde{A} = \tilde{A}_0 \cup \tilde{A}_1 \cup \tilde{A}_2$ with

$$\begin{aligned}
\tilde{A}_1 &= \{(c, r_1, r_2) : c > 0,\ r_1 \in (1, 2],\ r_2 \geq \lambda^6 (r_1 - 1)^{-2}\}, \\
\tilde{A}_2 &= \{(c, r_1, r_2) : c > 0,\ r_1 \in (1, 2],\ r_2 \leq \lambda^6 (r_1 - 1)^{2}\}, \\
\tilde{A}_0 &= \tilde{A} \backslash (\tilde{A}_1 \cup \tilde{A}_2).
\end{aligned}$$

The constraint $\hat{x}_1^1 = 1$ is active on $\tilde{A}_1$ and $\hat{x}_1^2 = 1$ is active on $\tilde{A}_2$. On $\tilde{A}_0$ we have

$$\begin{aligned} u(c, r_1, r_2; \Lambda) &= c^{\frac{1}{3}}\, r_1^{\frac{1}{3}}\, (\sqrt{r_2} + \lambda^3)^{\frac{1}{3}}, \\ \hat{x}^1 &= \Big(\frac{c\,\sqrt{r_2}}{\sqrt{r_2} + \lambda^3},\, \frac{r_1\,\sqrt{r_2}}{\sqrt{r_2} + \lambda^3},\, r_2 \Big), \\ \hat{x}^2 &= \Big(\frac{c\,\lambda^3}{\sqrt{r_2} + \lambda^3},\, \frac{r_1\,\lambda^3}{\sqrt{r_2} + \lambda^3},\, 0 \Big); \end{aligned}$$

on $\tilde{A}_1$

$$\begin{aligned} u(c, r_1, r_2; \Lambda) &= c^{\frac{1}{3}}\, (r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}(r_1 - 1)^{\frac{1}{2}})^{\frac{2}{3}}, \\ \hat{x}^1 &= \Big(\frac{c\, r_2^{\frac{1}{4}}}{(r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}(r_1 - 1)^{\frac{1}{2}})},\, 1,\, r_2 \Big), \\ \hat{x}^2 &= \Big(\frac{c\,\lambda^{\frac{3}{2}}(r_1 - 1)^{\frac{1}{2}}}{(r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}(r_1 - 1)^{\frac{1}{2}})},\, r_1 - 1,\, 0 \Big); \end{aligned}$$

and on $\tilde{A}_2$

$$\begin{aligned} u(c, r_1, r_2; \Lambda) &= c^{\frac{1}{3}}\, \Big((r_1 - 1)^{\frac{1}{2}} r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}} \Big)^{\frac{2}{3}}, \\ \hat{x}^1 &= \Big(\frac{c\,(r_1 - 1)^{\frac{1}{2}}\, r_2^{\frac{1}{4}}}{(r_1 - 1)^{\frac{1}{2}}\, r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}},\, r_1 - 1,\, r_2 \Big), \\ \hat{x}^2 &= \Big(\frac{c\,\lambda^{\frac{3}{2}}}{(r_1 - 1)^{\frac{1}{2}}\, r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}},\, 1,\, 0 \Big). \end{aligned}$$

Recall $m^j = \mathrm{card}(\mathcal{M}^j)$, so $m^1 = 2$, $m^2 = 1$; since also $\mu^j = 0$ then we obtain from (2.9) that $\mathcal{D}(\Lambda) = \mathrm{dom}(I^u(\cdot\,; \Lambda)) = [(0, \infty) \times (-\infty, \infty) \times (0, \infty)] \cap [(0, \infty) \times (-\infty, \infty)^2] = (0, \infty) \times (-\infty, \infty) \times (0, \infty)$.

3 The Equilibrium Problem

The standard or alternate assumptions are made. We consider the implications of equilibrium, beginning with (1.2). Recall that y is the price vector. The proof of Theorem 2.4, i.e. Proposition 3.2 of Chiarolla and Haussmann 2008 [4], implies that

$$\hat{z}(y) := I^{v_{A}z}(y) \tag{3.1}$$

is the unique element of $\partial v^*_{Az}(y)$, hence Rockafellar 2008 [14], Theorem 23.5(a*), implies that $\hat{z}(y)$ is the unique solution of (1.2).

Recall that the budget constraint in problem (1.4) is binding since u^j is increasing. After introducing the Lagrange multiplier η_j, we find as above, that

$$\hat{x}^j = \begin{pmatrix} \hat{c}^j \\ \hat{r}^j \end{pmatrix} := I^{u^j_{A^j}}(\eta_j \bar{y}), \qquad \bar{y} =: \begin{pmatrix} 1 \\ y \end{pmatrix}, \tag{3.2}$$

is the solution of (1.4) provided a multiplier exists. The latter must satisfy

$$\bar{y}^\top I^{u^j_{A^j}}(\eta_j \bar{y}) = \epsilon^j(y, \hat{z}(y)). \tag{3.3}$$

In the Appendix, Corollary 4.2, we show that at least one η_j exists and that despite the possible non-uniqueness, the value of $\hat{x}^j = I^{u^j_{A^j}}(\eta_j \bar{y})$ is unique!

If there exists $\eta > 0$ such that $\lambda_j \eta_j = \eta$ for all j, then in equilibrium (cf. (1.5) and (3.2)) (2.8) implies that

$$\begin{pmatrix} v(\hat{z}) \\ Z - \hat{z} \end{pmatrix} = I^u(\eta \bar{y}; \Lambda), \tag{3.4}$$

so according to (2.12), if $\eta \bar{y} \in \tilde{\mathcal{R}}(\Lambda)$, then

$$\nabla u(v(\hat{z}), Z - \hat{z}; \Lambda) = \eta \bar{y}.$$

If we write $\nabla u(c, r; \Lambda)$ as $(u_c(c, r; \Lambda), \nabla_r u(c, r; \Lambda)^\top)^\top$ then the previous equation can be solved for y as $y = \Big(u_c(v(\hat{z}), z - \hat{z}; \Lambda)\Big)^{-1} \nabla_r u(v(\hat{z}), Z - \hat{z}; \Lambda)$, hence

$$\hat{z} = I^{v_{A^Z}}\Big(\frac{\nabla_r u(v(\hat{z}), Z - \hat{z}; \Lambda)}{u_c(v(\hat{z}), Z - \hat{z}; \Lambda)}\Big) \tag{3.5}$$

from (3.1).

With a view to establishing existence of equilibrium, we can solve the unpleasant fixed point problem (3.5) uniquely, hence identify $\hat{z}$, by turning it into an auxiliary maximization problem. Note that this process will not require knowledge of y! Define the *utility of production* by $\hat{u}(\cdot\,; \Lambda)$ with

$$\begin{aligned} \hat{u}(z; \Lambda) &:= u(v(z), Z - z; \Lambda), \\ \hat{u}_1(z; \Lambda) &:= u_c(v(z), Z - z; \Lambda), \\ \hat{u}_2(z; \Lambda) &:= \nabla_r u(v(z), Z - z; \Lambda), \end{aligned} \tag{3.6}$$

and define $A^\star := \sum_{j=1}^J \prod_{i=1}^N D_i^j$, $A_0 := \sum_{j=1}^N D_0^j$ so A_0 is the positive half-line, closed at 0 if and only if $D_0^j = [0, \infty)$ for all j. Moreover $A^\star = \prod_{i=1}^N D_i^\star$ where

$$D_i^\star = \begin{cases} (0, \infty) \text{ or } [0, \infty) & \text{if } i \in \cup_j \mathcal{M}^j, \\ (0, \tilde{J}_i] \text{ or } [0, \tilde{J}_i] & \text{if } i \in \Big(\cup_j \mathcal{M}^j\Big)^c \cap \Big(\cup_j \tilde{\mathcal{M}}^j\Big) \end{cases}$$

if we recall from (2.5) that $\cap_j \mathcal{M}_o^j = \emptyset$. Again from (2.5) it follows that

$$\mathrm{int}(A^Z) \subset A^\star \cap (Z - A^\star) \subset A^Z. \tag{3.7}$$

Then

$$\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda)) = \{z : (v(z), Z - z) \in \mathrm{dom}(u(\,\cdot\,,\cdot\,;\Lambda))\} = v^{-1}(A_0) \cap (Z - A^\star).$$

Lemma 3.1 *$\hat{u}(\,\cdot\,;\Lambda)$ is strictly concave on $A^\star \cap \mathrm{dom}(\partial\hat{u}(\,\cdot\,;\Lambda))$. Hence*

$$(P_\Lambda) \qquad \sup_{z\in A^Z} \hat{u}(z;\Lambda)$$

is attained uniquely at a point z^Λ. Moreover $x^\Lambda := (v(z^\Lambda), (Z - z^\Lambda)^\top)^\top \in \tilde{A}$, and $\nabla\hat{u}(z^\Lambda;\Lambda)$ is normal to A^Z at z^Λ.

Proof:

The strict concavity of $\hat{u}$ follows readily from strict concavity of u on $\tilde{A} = \mathrm{dom}(\partial u(\,\cdot\,;\Lambda))$, the concavity of v and the fact that u is non-decreasing. As $z \mapsto \hat{u}(z;\Lambda)$ is continuous on A^Z, a compact set, then the supremum is attained uniquely at $z^\Lambda \in A^Z$.

As $\mathrm{ri}(\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda))) = \mathrm{int}(\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda)))$, then $A^Z \cap \mathrm{ri}(\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda))) = A^Z \cap \mathrm{int}(\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda))) = A^Z \cap \mathrm{int}(v^{-1}(A_0)) \cap \mathrm{int}(Z - A^\star) \supset \mathrm{int}(v^{-1}(A_0)) \cap \mathrm{int}(A^Z) = \mathrm{int}(A^Z \cap v^{-1}(A_0)) = \mathrm{int}(A^Z) \neq \emptyset$, (cf. (3.7) and (2.1)(iv), so Rockafellar 1970 [14], Theorems 23.5, 23.8, imply that z^Λ is characterized by

$$0 \in \partial[\hat{u}(z^\Lambda;\Lambda) - \chi_{A^Z}(z^\Lambda)] = \partial\hat{u}(z^\Lambda;\Lambda) - \mathcal{N}_{A^Z}(z^\Lambda).$$

Hence $z^\Lambda \in \mathrm{dom}(\partial(\hat{u}(\,\cdot\,;\Lambda)))$, i.e. $x^\Lambda \in \mathrm{dom}(\partial(u(\,\cdot\,;\Lambda))) = \tilde{A}$.

Since $\nabla u(\cdot;\Lambda)$ is defined on $\tilde{A}$, cf. Theorem 2.6(*iii*), then $\nabla\hat{u}(z^\Lambda;\Lambda) = u_c(x^\Lambda;\Lambda)\nabla v(z^\Lambda) - \nabla_r u(x^\Lambda;\Lambda)$ exists and is an element of $\partial\hat{u}(z^\Lambda;\Lambda)$. Now Rockafellar 1970 [14], Theorem 25.6, implies that

$$\partial\hat{u}(z^\Lambda;\Lambda) = \nabla\hat{u}(z^\Lambda;\Lambda) - \mathcal{N}_{\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda))}(z^\Lambda).$$

It follows that $\nabla\hat{u}(z^\Lambda;\Lambda) \in \mathcal{N}_{\mathrm{dom}(\hat{u}(\,\cdot\,;\Lambda))}(z^\Lambda) + \mathcal{N}_{A^Z}(z^\Lambda) = \mathcal{N}_{A^Z}(z^\Lambda)$ since the normal cone is $\{0\}$ at an interior point. The result follows. □

Corollary 3.2 *z^Λ is the unique solution of (3.5).*

Proof: Since $\nabla\hat{u}(z;\Lambda) = u_c(v(z), Z - z;\Lambda)\nabla v(z) - \nabla_r u(v(z), Z - z;\Lambda)$, then z^Λ satisfies uniquely (cf. (3.6) for $\hat{u}_i$)

$$\hat{u}_1(z^\Lambda;\Lambda)\nabla v(z^\Lambda) = \hat{u}_2(z^\Lambda;\Lambda) + \vec{n}(z^\Lambda) \tag{3.8}$$

where $\vec{n}(z^\Lambda)$ is an outward normal to A^Z at z^Λ.

With the projection $\mathcal{P}_{v_{A^Z}}$ defined as in Theorem 2.4(b), it follows from (3.8) that

$$\nabla v(z^\Lambda) = \mathcal{P}_{v_{A^Z}}\Big(\frac{\hat{u}_2(z^\Lambda;\Lambda)}{\hat{u}_1(z^\Lambda;\Lambda)}\Big) \quad \text{or} \quad z^\Lambda = I^{v_{A^Z}}\Big(\frac{\hat{u}_2(z^\Lambda;\Lambda)}{\hat{u}_1(z^\Lambda;\Lambda)}\Big). \tag{3.9}$$

As these steps can be reversed, the last equality characterizes z^Λ uniquely. The conclusion follows. $\square$

Let us now find a condition sufficient for existence of an equilibrium and then show that it holds. Fix $\Lambda \in \Re^J_{++}$. Find $z^\Lambda \in \arg\max_{z \in A^Z} \hat{u}(z;\Lambda)$, cf. (P_Λ) in Lemma 3.1. Set

$$\eta^\Lambda := u_c(v(z^\Lambda), Z - z^\Lambda; \Lambda), \qquad y^\Lambda := (\eta^\Lambda)^{-1} \nabla_r u(v(z^\Lambda), Z - z^\Lambda; \Lambda), \qquad \bar{y}^\Lambda = (1, (y^\Lambda)^\top)^\top. \tag{3.10}$$

Note that Lemma 3.1 implies that $x^\Lambda \in \tilde{A}$. Since $u(\cdot\,;\Lambda)$ is strictly increasing on $\tilde{A}$, cf. Theorem 2.6(iii), then $\nabla u(x^\Lambda;\Lambda) \in \Re^n_{++}$, i.e. the price vector $y^\Lambda \in \Re^N_{++}$.

Theorem 3.3 Let $\eta_j(\Lambda)$ be a solution of $(\bar{y}^\Lambda)^\top I^{u^j_{A^j}}(\eta_j \bar{y}^\Lambda) = \epsilon^j(y^\Lambda, z^\Lambda)$.
(a) If $\lambda_j \eta_j(\Lambda) = \eta^\Lambda,\ j = 1, \cdots, J$, then an equilibrium exists with $z^\Lambda = \hat{z}(y^\Lambda)$.
(b) An equilibrium exists.

Proof: (a) According to Corollary 3.2, z^Λ solves (1.2). Set

$$(\hat{c}^j, (\hat{r}^j)^\top)^\top := I^{u^j_{A^j}}(\eta_j(\Lambda)\bar{y}^\Lambda), \tag{3.11}$$

so it solves (1.4). It remains only to show that the markets clear.

If $\lambda_j \eta_j(\Lambda) = \eta^\Lambda$, then $\sum_j (\hat{c}^j, (\hat{r}^j)^\top)^\top = \sum_j I^{u^j_{A^j}}(\eta_j(\Lambda)\bar{y}^\Lambda) = \sum_j I^{u^j_{A^j}}(\frac{\eta^\Lambda}{\lambda_j}\bar{y}^\Lambda) = I^u(\eta^\Lambda \bar{y}^\Lambda;\Lambda)$. But $\eta^\Lambda \bar{y}^\Lambda = \nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)$, cf. (3.10), hence $I^u(\eta^\Lambda \bar{y}^\Lambda;\Lambda) = (v(z^\Lambda), (Z - z^\Lambda)^\top)^\top$. It follows that (1.5) holds.

(b) To establish existence of an equilibrium, we must show that there exists Λ such that $\eta_j(\Lambda) = \eta^\Lambda/\lambda_j$ satisfies (3.3) for all j. But (3.10) implies that $\nabla u(x^\Lambda;\Lambda) = \eta^\Lambda \bar{y}^\Lambda$ so Λ must be found to satisfy

$$\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)^\top I^{u^j_{A^j}}(\tfrac{1}{\lambda_j}\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)) = \eta^\Lambda \epsilon^j(y^\Lambda, z^\Lambda).$$

With $\vec{e}_0 := (1, 0, \cdots, 0)^\top \in \Re^{1+N}$, this reduces to

$$F_j(\Lambda) := \nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)^\top \Big(I^{u^j_{A^j}}(\tfrac{1}{\lambda_j}\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)) - \epsilon^j(y^\Lambda, z^\Lambda)\vec{e}_0 \Big) = 0, \tag{3.12}$$

for $j = 1, \cdots, J$. Lemma 4.4 establishes continuity of F_j on $\Re^J_{++}$. We will use Karatzas, Lehoczky and Shreve 1990 [11], Lemma 12.1, to show that the required Λ exists, i.e. that the markets clear.

Let $\Xi_+ := \{\sum_{j=1}^J \lambda_j \vec{e}_j : \lambda_j > 0,\ \sum_j \lambda_j = 1\}$ and $\Xi := \mathrm{cl}(\Xi_+)$. $F = (F_1, \cdots, F_J)^\top$ is defined on Ξ_+, but we can extend it continuously to Ξ by setting $I^{u^j_{A^j}}(\frac{y}{\lambda_j}) := 0$ for $\lambda_j = 0$ and still defining $u(x;\Lambda)$, $I^u(y;\Lambda)$ by (2.7) and (2.8). To establish the continuity we first show that if $\Lambda^m \to \tilde{\Lambda}$ with $\tilde{\lambda}_k > 0$ for some k, as is the case when $\tilde{\Lambda} \in \Xi$, then $x^{\Lambda^m} = (\nabla v(z^{\Lambda^m}), Z - z^{\Lambda^m})$

lies in a compact subset of $\tilde{A}$. This follows from a slight modification of the proof of Lemma 4.3: for L take $\{\Lambda \in \{\Lambda^m\}_{m=1}^{\infty} : \lambda_k \geq \tilde{\lambda}_k/2\}$ and replace (4.3) by

$$u^k_{x_i}(\tilde{g}^k(z,\Lambda)) > \frac{2\kappa_c\kappa_v}{\tilde{\lambda}_k} + 1.$$

The required compact subset of $\tilde{A}$ is now constructed as in the proof of Lemma 4.3.

Since ∇u is continuous and strictly increasing on $\tilde{A}$ then the $\nabla u(x^{\Lambda^m};\Lambda^m)$ lie in a compact subset of $\Re^n_{++}$. It follows from Lemma 4.1 that $\lim_{m\to\infty} I^{u^j}{}_{A^j}(\frac{\nabla u(x^{\Lambda^m},\Lambda^m)}{\lambda_j^m}) \to 0$ if $\lambda_j^m \to 0$. Now the continuity of $(x,\Lambda) \mapsto I^{u^j}{}_{A^j}(\frac{\nabla u(x^\Lambda,\Lambda)}{\lambda_j})$ established in Lemma 4.4 extends to $\tilde{A} \times \Xi$. This establishes the continuity of F on Ξ.

Observe now that if $\lambda_j = 0$, then from (3.12) and (2.13)

$$F_j(\Lambda) = -\epsilon^j(y^\Lambda, z^\Lambda)u_c(v(z^\Lambda), Z - z^\Lambda;\Lambda) = -\epsilon^j(y^\Lambda, z^\Lambda)\lambda_{j(1)}u_c^{j(1)}(\hat{x}^{j(1)}) < 0 \tag{3.13}$$

with $j(1)$ defined in Theorem 2.6(iv), since we can establish Theorem 2.6 after deleting the agents corresponding to the zero components of Λ.

Furthermore using (2.8) and (1.3) we obtain, for $\Lambda \in \Xi$,

$$\begin{aligned}
\sum_{j=1}^{J} F_j(\Lambda) &= \sum_j \nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)^\top \Big(I^{u^j}{}_{A^j}(\tfrac{1}{\lambda_j}\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)) - \epsilon^j(y^\Lambda, z^\Lambda)\vec{e}_0\Big) \\
&= \nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda)^\top \Big(I^u(\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda);\Lambda) - \sum_j \epsilon^j(y^\Lambda, z^\Lambda)\vec{e}_0\Big) \\
&= \begin{pmatrix} \eta^\Lambda \\ \eta^\Lambda y^\Lambda \end{pmatrix}^\top \left[\begin{pmatrix} v(z^\Lambda) \\ Z - z^\Lambda \end{pmatrix} - (v(z^\Lambda) + (y^\Lambda)^\top(Z - z^\Lambda))\vec{e}_0\right] \;=\; 0
\end{aligned} \tag{3.14}$$

since $\nabla u(v(z^\Lambda), Z - z^\Lambda;\Lambda) = (\eta^\Lambda, \eta^\Lambda\,(y^\Lambda)^\top)^\top$.

Equations (3.13) and (3.14) are (12.3) and (12.4) of Karatzas, Lehoczky and Shreve 1990 [11] suffice, as in their Lemma 12.1, to establish that there exists $\Lambda^* \in \Xi_+$ such that $F_j(\Lambda^*) = 0$ for all j. □

Example 3.4 We continue with Example 2.5, Remark 2.9. The constraints on r_1^j are compatible with the idea that resource one is the labor supplied by the agents. Resource two is thought of as raw material used in the production of the good. Take $\Lambda = (\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda})$ and compute

$$u(c, r_1, r_2;\Lambda) = \sup_{\substack{0 \leq c^1 \leq c \\ (r_1 - 1)\vee 0 \leq r^1 \leq r_1 \wedge 1}} [(c^1)^{\frac{1}{3}}(r^1)^{\frac{1}{3}}(r_2)^{\frac{1}{6}} + \lambda(c - c^1)^{\frac{1}{3}}(r_1 - r^1)^{\frac{1}{3}}](1+\lambda)^{-1}.$$

Write $f(c^1, r^1) := [(c^1)^{\frac{1}{3}}(r^1)^{\frac{1}{3}}(r_2)^{\frac{1}{6}} + \lambda(c - c^1)^{\frac{1}{3}}(r_1 - r^1)^{\frac{1}{3}}](1+\lambda)^{-1}$. Then f has one critical point at $(\hat{c}^1, \hat{r}^1) = \left(\frac{c\sqrt{r_2}}{\sqrt{r_2}+\lambda^3}, \frac{r_1\sqrt{r_2}}{\sqrt{r_2}+\lambda^3}\right)$. This point lies in the constraint set *provided* $0 \leq \hat{c}^1 \leq c$,

$0 \le \hat{r}^1 \le \min\{r_1, 1\}$ and $0 \le r_1 - \hat{r}^1 \le \min\{r_1, 1\}$. The first is always satisfied, but the other two depend on the values of λ, r_1, r_2. Set $\bar{a} := \frac{\sqrt{r_2}}{\lambda^3}$ and $a = \min\{\bar{a}, \bar{a}^{-1}\}$. The last two constraints reduce to $\hat{r}^1 \le 1 + \bar{a}^{-1}$ and $\hat{r}^1 \le 1 + \bar{a}$. There are two cases.
CASE 1: $r_1 \in [0, 1 + a]$. Now the critical point lies in the constraint set; it is the maximizing point and gives

$$u(c, r_1, r_2; \Lambda) = c^{\frac{1}{3}}\, r_1^{\frac{1}{3}}\, (\sqrt{r_2} + \lambda^3)^{\frac{1}{3}} (1 + \lambda)^{-1} = f(\hat{c}^1, \hat{r}^1)$$

CASE 2: $r_1 \in (1 + a, 2]$. The constraint is $r_1 - 1 \le r^1 \le 1$ and is not met by $r^1 = \hat{r}^1$; hence we look for the max on the boundary of the constraint region. An easy calculation shows that

$$\begin{aligned} u(c, r_1, r_2; \Lambda) &= \max\{f(c^{1,1}, r_1 - 1), f(c^{1,2}, 1)\} \\ &= c^{\frac{1}{3}} \max\Big\{[(r_1 - 1)^{\frac{1}{2}} r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}]^{\frac{2}{3}},\ [r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}} (r_1 - 1)^{\frac{1}{2}}]^{\frac{2}{3}}\Big\}/(1 + \lambda), \end{aligned}$$

where

$$c^{1,1} = c\,\frac{(r_1 - 1)^{\frac{1}{2}} r_2^{\frac{1}{4}}}{(r_1 - 1)^{\frac{1}{2}} r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}}, \qquad c^{1,2} = c\,\frac{r_2^{\frac{1}{4}}}{r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}} (r_1 - 1)^{\frac{1}{2}}}.$$

Hence case 2 breaks down into
CASE 2a: $\bar{a} \le 1,\ 1 + \bar{a} < r_1 \le 2$, where

$$u(c, r_1, r_2; \Lambda) = c^{\frac{1}{3}}\, [(r_1 - 1)^{\frac{1}{2}} r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}}]^{\frac{2}{3}}/(1 + \lambda),$$

and
CASE 2b: $\bar{a} > 1,\ 1 + \bar{a}^{-1} < r_1 \le 2$, where

$$u(c, r_1, r_2; \Lambda) = c^{\frac{1}{3}}\, [r_2^{\frac{1}{4}} + \lambda^{\frac{3}{2}} (r_1 - 1)^{\frac{1}{2}}]^{\frac{2}{3}}/(1 + \lambda).$$

Now assume that $v(z_1, z_2) = (z_1 + \varepsilon_1)^{\alpha_1} (z_2 + \varepsilon_2)^{\alpha_2} - \varepsilon_1^{\alpha_1} \varepsilon_2^{\alpha_2}$ with $0 < \alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2 < 1$, $\varepsilon_i \ge 0$. If both $\varepsilon_i > 0$ then the standard assumptions hold; otherwise the alternate assumptions hold. If we do not subtract the term $\varepsilon_1^{\alpha_1} \varepsilon_2^{\alpha_2}$, then $v(0) > 0$, and we can consider this amount of goods to be *imported*.

$\hat{u}(z; \Lambda)$ is defined on A^Z with $Z = (2, Z_2)$ for $Z_2 > 0$ (since (2.5) implies that $Z_1 = 2$). With $(c, r_1, r_2) = (v(z), 2 - z_1, Z_2 - z_2)$ we have $a = a(z_2) := \min\Big\{\frac{\sqrt{Z_2 - z_2}}{\lambda^3}, \frac{\lambda^3}{\sqrt{Z_2 - z_2}}\Big\}$ and CASE 1 is $z_1 \in [1 - a(z_2), 2]$ whereas CASE 2 is $z_1 \in [0, 1 - a(z_2))$.

Now z^Λ is found as the solution of (P_Λ), cf. Lemma 3.1. Once we have obtained an expression for z^Λ we can solve (3.12) for Λ, hence obtain a numeric value for z^Λ and y^Λ, cf. (3.10). Note that y_1^Λ is the price of labor and y_2^Λ is the price of the resource, all in terms of the value of the good produced.

We used Matlab to solve (3.12) for λ. Since $F_1 + F_2 = 0$, we only need to solve $F_1 = 0$ to obtain Λ. We used the following parameter values: $\alpha_1 = 0.5,\ \alpha_2 = 0.3,\ \rho = \begin{pmatrix} 1 & 1 \\ 1 & Z_2 - 1 \end{pmatrix},\ \varphi = (0.6, 0.4)$ and varied Z_2 and ε_i.

Table 1

Z_2	ε_1	ε_2	λ	y_1^Λ	y_2^Λ	z_1^Λ	z_2^Λ	$v(z^\Lambda)$
3.0000	1.0000	1.0000	1.1254	0.5548	0.1720	0.6009	2.0978	0.7762
5.0000	1.0000	1.0000	1.2702	0.6426	0.1264	0.5187	3.6320	0.9519
9.0000	1.0000	1.0000	1.4838	0.7615	0.0873	0.4615	6.6508	1.2259
3.0000	0	1.0000	1.1959	0.7892	0.1355	0.6667	1.3291	1.0522
3.0000	1.0000	1.0000	1.1254	0.5548	0.1720	0.6009	2.0978	0.7762
3.0000	2.0000	1.0000	1.0993	0.4646	0.2015	0.3479	2.2478	0.7676
3.0000	1.0000	0	1.2872	0.5664	0.2097	0.1058	1.7922	1.2527
3.0000	1.0000	1.0000	1.1254	0.5548	0.1720	0.6009	2.0978	0.7762
3.0000	1.0000	2.0000	1.1427	0.5793	0.1489	0.7084	1.9882	0.7482
3.0000	1.0000	1.0000	1.1254	0.5548	0.1720	0.6009	2.0978	0.7762
3.0000	2.0000	2.0000	1.1121	0.4832	0.1748	0.5345	2.2029	0.7080
3.0000	3.0000	3.0000	1.1046	0.4405	0.1753	0.4891	2.2606	0.6655

In our calculations the constraints on z^Λ are never active. Increasing Z_2, the amount of raw material present, produces an increase in the number of goods produced, v, as expected, cf. rows 1 to 3 of Table 1. As either ε_i increases, $v(z^\Lambda)$ decreases for fixed Z_2, i.e. production becomes less efficient. When this effect is due to ε_1 (rows 4 to 6), then the amount of labour used decreases as does its price, and the amount of material used and price thereof increases. When this effect is due to ε_2 (rows 7 to 9), the corresponding implication is not quite true. There is no trend in the amount of raw material used, only in its price (decreasing) and for labour the amount used is increasing but the price of labour has no trend. When both ε_i increase (rows 10 to 12), the pattern of rows 4 to 6 is reestablished; reversing the values of α_1 and α_2 does not change this. There are similar anomalies when we use $v(z_1, z_2) = (z_1 + \varepsilon_1)^{0.5}(z_2 + \varepsilon_2)^{0.3}$, i.e. $v(0) > 0$, although now $v(z^\Lambda)$ increases with ε_i. The example shows that comparative statics results will not be obvious in these models.

Remark 3.5 As usual in General Equilibrium Theory uniqueness of the equilibrium does not follow. However in cases such as the above example, where (3.12) is solved numerically for Λ, one may check whether the algorithm yields the same Λ (up to scaling) for a random selection of starting points. This would indicate uniqueness of the equilibrium, but may require added hypotheses, cf. Karatzas, Lehoczky and Shreve 1990 [11], Theorem 11.1. In this example varying the starting point did not produce different values for Λ, so we suspect that the equilibrium is unique.

4 Appendix

We establish the existence of the Lagrange multiplier η_j. Let $\mathcal{K}^j := \{k : \mathcal{C}_k^j = \{a_k^j\},\ \text{a singleton}\}$, so a_k^j are the vertices of A^j. Write $\tilde{m}^j$ for card$(\mathcal{M}^j \cup \tilde{\mathcal{M}}^j)$ and decompose $\Re^n = \Re^{\tilde{m}^j} \oplus \Re^{n-\tilde{m}^j}$ where $\Re^{\tilde{m}^j} = \mathrm{aff}(A^j)$. Any $\bar{y} \in \Re^n$ decomposes as $\bar{y} = \bar{y}_+ \oplus y_\perp$. Consider a (price) vector y such that $\bar y = (1, y^\top)^\top = \bar y_+ \oplus y_\perp \in \mathrm{dom}(I^{u^j_{A^j}})$ with $\bar y_+ \in \Re^{\tilde m^j}_{++}$. Define $\bar\eta^j_y := \sup\{\eta : \bar y^\top I^{u^j_{A^j}}(\eta \bar y) > 0\}$ and $\underline{\eta}^j_y := \inf\{\eta : \eta\, \bar y \in \mathrm{dom}(I^{u^j_{A^j}})\}$. Then $0 \le \underline{\eta}^j_y \le \max_{i \in \mathcal{M}^j}[\mu^j_i/\bar y_i]$, cf. Theorem 2.4.

Lemma 4.1 *For $\bar y \in \mathrm{dom}(I^{u^j_{A^j}})$ with $\bar y_+ \in \Re^{\tilde m^j}_{++}$, $\eta \to \bar y^\top I^{u^j_{A^j}}(\eta \bar y)$ is continuous, non-increasing on $(\underline{\eta}^j_y, \infty)$ and strictly decreasing on $(\underline{\eta}^j_y, \bar\eta^j_y) \backslash \bigcup_{k \in \mathcal{K}^j} \{\eta : \bar y^\top I^{u^j_{A^j}}(\eta \bar y) = \bar y^\top a^j_k\}$. Moreover $\lim_{\eta \downarrow \underline{\eta}^j_y} \bar y^\top I^{u^j_{A^j}}(\eta \bar y) = \infty$, $\lim_{\eta \uparrow \bar\eta^j_y} \bar y^\top I^{u^j_{A^j}}(\eta \bar y) = 0$ and $\lim_{\eta \to \infty} \sup_{\bar y \in \mathcal{Y}} \|I^{u^j_{A^j}}(\eta \bar y)\| = 0$ for any set $\mathcal{Y}$ such that $\mathcal{Y} \subset \mathrm{dom}(I^{u^j_{A^j}}) \cap [\Re^{\tilde m^j}_{++} \oplus \Re^{n - \tilde m^j}]$ with $\mathcal{Y} = \mathcal{Y}_+ \oplus \mathcal{Y}_\perp$ and $\mathcal{Y}_+ \subset \Re^{\tilde m^j}_{++}$ compact.*

Proof: We write $h(\eta) := \bar y^\top I^{u^j_{A^j}}(\eta \bar y)$ for $\eta \in (\underline{\eta}^j_y, \infty)$; then $h(\eta) \ge 0$. Continuity of h follows from that of $I^{u^j_{A^j}}$. h is non-increasing since $I^{u^j_{A^j}}$ is.

As $(\eta_1 - \eta_2)(h(\eta_1) - h(\eta_2)) = (\eta_1 \bar y - \eta_2 \bar y)^\top \Big(I^{u^j_{A^j}}(\eta_1 \bar y) - I^{u^j_{A^j}}(\eta_2 \bar y)\Big)$ and $I^{u^j_{A^j}}$ is strictly monotone on $\mathcal{R}_{u^j_{A^j}}$, then h is strictly decreasing provided $\eta \bar y \in \mathcal{R}_{u^j_{A^j}}$. For $\eta_i \bar y \in \mathcal{S}^j_k$, $i = 1, 2$, $\mathcal{P}_{u^j_{A^j}}(\eta_i \bar y) - \eta_i \bar y \in \mathcal{N}_{A^j}(\mathcal{C}^j_k)$ and $I^{u^j_{A^j}}(\eta_i \bar y) \in \mathcal{C}^j_k$ so

$$\Big(\mathcal{P}_{u^j_{A^j}}(\eta_i \bar y) - \eta_i \bar y\Big)^\top \Big(I^{u^j_{A^j}}(\eta_1 \bar y) - I^{u^j_{A^j}}(\eta_2 \bar y)\Big) = 0.$$

It follows that

$$\begin{aligned}
(\eta_1 - \eta_2)(h(\eta_1) - h(\eta_2)) &= \Big(\eta_1 \bar y - \eta_2 \bar y\Big)^\top \Big(I^{u^j_{A^j}}(\eta_1 \bar y) - I^{u^j_{A^j}}(\eta_2 \bar y)\Big) \\
&= \Big(\mathcal{P}_{u^j_{A^j}}(\eta_1 \bar y) - \mathcal{P}_{u^j_{A^j}}(\eta_2 \bar y)\Big)^\top \Big(I^{u^j_{A^j}}(\eta_1 \bar y) - I^{u^j_{A^j}}(\eta_2 \bar y)\Big) \\
&= \Big(\mathcal{P}_{u^j_{A^j}}(\eta_1 \bar y) - \mathcal{P}_{u^j_{A^j}}(\eta_2 \bar y)\Big)^\top \Big(I^{u^j_{A^j}}(\mathcal{P}_{u^j_{A^j}}(\eta_1 \bar y)) - I^{u^j_{A^j}}(\mathcal{P}_{u^j_{A^j}}(\eta_2 \bar y))\Big) \\
&< 0
\end{aligned}$$

provided $\mathcal{P}_{u^j_{A^j}}(\eta_1 \bar y) - \mathcal{P}_{u^j_{A^j}}(\eta_2 \bar y) \ne 0$, since $\mathcal{P}_{u^j_{A^j}}(\eta_i \bar y) \in \mathcal{R}_{u^j_{A^j}}$ and we have strict monotonicity on $\mathcal{R}_{u^j_{A^j}}$. But $\mathcal{P}_{u^j_{A^j}}(\eta_1 \bar y) \ne \mathcal{P}_{u^j_{A^j}}(\eta_2 \bar y)$ unless $\bar y \in \mathrm{aff}(\mathcal{N}_{A^j}(\mathcal{C}^j_k))$. Moreover $\mathcal{N}_{A^j}(\mathcal{C}^j_k) = \tilde{\mathcal{N}}_{A^j} \oplus \Re^{n - \tilde m^j}$ where $\mathcal{N}_{A^j}(\mathcal{C}^j_k)$ is the outward normal cone of A^j at any point in $\mathcal{C}^j_k$ relative to $\mathrm{aff}(A^j) = \Re^{\tilde m^j}$. If $\dim(\mathrm{aff}(\tilde{\mathcal{N}}_{A^j}(\mathcal{C}^j_k))) < \tilde m^j$, then it is orthogonal to one of the coordinate axes of $\Re^{\tilde m^j}$, hence cannot contain $\bar y_+ \in \Re^{\tilde m^j}_{++}$. If $\dim(\mathrm{aff}(\tilde{\mathcal{N}}_{A^j}(\mathcal{C}^j_k))) = \tilde m^j$ then $\bar y_+ \in \mathrm{aff}(\tilde{\mathcal{N}}_{A^j}(\mathcal{C}^j_k)) = \Re^{\tilde m^j}$, but $k \in \mathcal{K}^j$, i.e. $\mathcal{C}^j_k$ is a vertex and $I^{u^j_{A^j}}(\eta \bar y) = a^j_k = (a^j_k)_+ \oplus 0$. The Proposition does not claim strict monotonicity there, hence we have established the claimed monotonicity properties of h.

For $\eta > \underline{\eta}^j_y$, $\eta \bar y \in \mathcal{S}^j_k$ for some $k \ge 0$. If $k = 0$, i.e. $\mathcal{S}^j_k = \mathrm{int}(\mathcal{R}_{u^j_{A^j}})$, then $\eta \bar y = \mathcal{P}_{u^j_{A^j}}(\eta \bar y)$; otherwise $\mathcal{P}_{u^j_{A^j}}(\eta \bar y) \in \mathrm{bdy}(\mathcal{R}_{u^j_{A^j}})$. As η increase, i.e. $\eta \bar y$ moves along the ray away from 0, then $\mathcal{P}_{u^j_{A^j}}(\eta \bar y)$ moves on $\mathrm{bdy}(\mathcal{R}_{u^j_{A^j}})$ except when and if $\eta \bar y \in \mathrm{int}(\mathcal{R}_{u^j_{A^j}})$ where $\mathcal{P}_{u^j_{A^j}}(\eta \bar y)$ coincides with $\eta\bar{y}$. If the ray emerges from $\mathrm{int}(\mathcal{R}_{u^j_{A^j}})$ the two separate again. If $0 \in \mathrm{dom}(\partial u^j)$, i.e. $\|\nabla u^j(0)\| < \infty$, then $\mathcal{P}_{u^j_{A^j}}(\eta\bar{y})$ eventually reaches $\nabla u^j(0)$. This last claim follows because $\mathcal{N}_{A^j}(0) = -\Re^{\tilde{m}^j}_+ \oplus \Re^{n-\tilde{m}^j}$ so for k_0 such that $\mathcal{C}^j_{k_0} = \{0\}$, $\mathcal{S}^j_{k_0} = \nabla u^j(0) + \Re^{\tilde{m}^j}_+ \oplus \Re^{n-\tilde{m}^j}$, hence contains $\eta\bar{y}$ for large η. If $0 \notin \mathrm{dom}(\partial u^j)$ then $\mathcal{P}_{u^j_{A^j}}(\eta\bar{y})$ wanders off to ∞ on $\mathrm{bdy}(\mathcal{R}_{u^j_{A^j}})$. We can now examine the limits.

We show that $\lim_{\eta\downarrow\underline{\eta}^j_y} \bar{y}^\top I^{u^j_{A^j}}(\eta\bar{y}) = \infty$. For η sufficiently close to $\underline{\eta}^j_y$, $\eta\bar{y} \in \mathcal{S}^j_{k_y}$ for some (fixed, depending on y) $k_y \geq 0$. Set $x^\eta := I^{u^j_{A^j}}(\eta\bar{y})$. We argue by contradiction.

If x^η remains bounded as $\eta \downarrow \underline{\eta}^j_y$, then a subsequence (again denoted by x^η converges to $x_o \in \mathrm{cl}(\mathrm{dom}(\partial u^j) \cap A^j)$. If $x_o \notin \mathrm{dom}(\partial u^j)$ then $\|\nabla u^j(x^\eta)\| = \|\mathcal{P}_{u^j_{A^j}}(\eta\bar{y})\| \to \infty$ so $\eta\bar{y} \to +\infty$, i.e. $\underline{\eta}^j_y = +\infty$, a contradiction. Hence $x_o \in \mathrm{dom}(\partial u^j)$ and in A^j since A^j is closed in $\mathrm{dom}(u^j)$.

If $x^\eta \in \mathrm{int}(A^j)$, i.e. $\{\eta\bar{y}\} \in \mathrm{int}(\mathcal{R}_{u^j_{A^j}})$, i.e. $k_y = 0$, cf. Definition 2.3, then

$$\underline{\eta}^j_y\bar{y} = \lim_{\eta\downarrow\underline{\eta}^j_y} \eta\bar{y} = \lim_{\eta\downarrow\underline{\eta}^j_y} \nabla u^j(x^\eta) = \nabla u^j(x^o) \in \mathcal{R}_{u^j_{A^j}} \subset \mathrm{dom}(I^{u^j_{A^j}}) \tag{4.1}$$

contradicting the definition of $\underline{\eta}^j_y$ since $\mathrm{dom}(I^{u^j_{A^j}})$ is open. Hence we may assume that $k_y > 0$, i.e. $x^\eta \in \mathcal{C}^j_{k_y} \subset \mathrm{bdy}(A^j)$. Then $\eta\bar{y} = \mathcal{P}_{u^j_{A^j}}(\eta\bar{y}) - \vec{n}(\eta\bar{y}) = \nabla u^j(x^\eta) - \vec{n}(\eta\bar{y})$ with $\vec{n}(\eta\bar{y}) \in \mathcal{N}_{A^j}(\mathcal{C}^j_{k_y})$. Taking limits as in (4.1) implies that $\underline{\eta}^j_y\bar{y} \in \nabla u^j(x^o) - \mathcal{N}_{A^j}(\mathcal{C}^j_{k_y}) \subset \mathrm{dom}(I^{u^j_{A^j}})$, again a contradiction. So $x^\eta \in A^j \subset \Re^{\tilde{m}^j} \oplus \{0\}$ is unbounded and $h(\eta) = \bar{y}^\top x^\eta \to \infty$ since $\bar{y}_+ \in \Re^{\tilde{m}^j}_{++}$.

Next we must show that $\lim_{\eta\uparrow\bar{\eta}^j_y} \bar{y}^\top I^{u^j_{A^j}}(\eta\bar{y}) = 0$. At the same time we shall establish the last result, i.e. $\lim_{\eta\to\infty} \sup_{\bar{y}\in\mathcal{Y}} \|I^{u^j_{A^j}}(\eta\bar{y})\| = 0$. Above we saw that if $0 \in \mathrm{dom}(\partial u^j)$ then $\mathcal{P}_{u^j_{A^j}}(\eta\bar{y}) = \nabla u^j(0)$ eventually, hence $I^{u^j_{A^j}}(\eta\bar{y}) = 0$ for large η, so $\bar{\eta}^j_y < \infty$. In fact $\sup_{y\in\mathcal{Y}} \bar{\eta}^j_y$ is attained since $y \mapsto \bar{\eta}^j_y$ is continuous, cf. the geometry of the intersection of the cone subtended by $\mathcal{Y}_+$ with $(\nabla u^j(0))_+ + \Re^{\tilde{m}^j}_+$. The result follows from continuity of h. This is the only case where $\bar{\eta}^j_y < \infty$ since if $h(\eta) = 0$ and $\bar{y} = \bar{y}_+ \oplus y_\perp$ with $\bar{y}_+ \in \mathcal{R}^{\tilde{m}^j}_{++}$ then $I^{u^j_{A^j}}(\eta\bar{y}) = x^\eta_+ \oplus 0$ (with $x^\eta_+ \in \Re^{\tilde{m}^j}$ since A^j "lies" in this space), so $x^\eta_+ = 0$. Hence $x^\eta = 0$, i.e. $\mathcal{P}_{u^j_{A^j}}(\eta\bar{y}) = \nabla u^j(0)$, i.e. $0 \in \mathrm{dom}(\partial u^j)$.

If $0 \notin \mathrm{dom}(\partial u^j)$ then for η sufficiently large, $x^\eta = I^{u^j_{A^j}}(\eta\bar{y})$ is not at a vertex of A^j and $\eta\bar{y} \in \mathcal{S}^j_{k_o}$ for some (fixed, depending on y) $k_o \geq 0$. Then h is strictly decreasing (for η sufficiently large) and x^η_+ must lie in the compact set $\{x \in \Re^{\tilde{m}^j}_+ : \bar{y}_+^\top x \leq \sup_{\bar{y}\in\mathcal{Y}} h(1)\}$. Nb. $x^\eta = x^\eta_+ \oplus 0$ so $h(\eta) = \bar{y}^\top x^\eta$ is independent of $\bar{y}_\perp$ and $\bar{y}_+ \to h(1)$ is continuous. Hence $\bigcup_{\bar{y}\in\mathcal{Y}} I^{u^j_{A^j}}(\eta\bar{y})$ lies in a compact set $\mathcal{X} \subset \mathrm{cl}(A^j)$. Let x^∞ be an accumulation point as $\|\eta\bar{y}\| \to \infty$.

It follows from the converse part of (2.4) that for $\varepsilon > 0$ there exists $M^\varepsilon_i < \infty$, the same for all $\bar{y} \in \mathcal{Y}$, such that if $u^j_{x_i}(x^\eta) > M^\varepsilon_i$ then $x^\eta_i < \varepsilon$. Write M^ε for $(M^\varepsilon_0, \ldots, M^\varepsilon_N)^\top$. If

$\{\eta\bar{y}\} \subset \mathrm{ri}(\mathcal{R}_{u^j_{A^j}})$, i.e. $x^\eta \in \mathrm{ri}(A^j)$, and $\eta\bar{y}_+ \in M^\varepsilon_+ + \Re^{\tilde{m}^j}_+$ then $(x^\eta_+)_i < \varepsilon$ for all i and $x^\eta_\perp = 0$. Hence $\lim_{\|\eta\bar{y}\|\to\infty} x^\eta = 0$ uniformly for $\bar{y} \in \mathcal{Y}$. And $x^\infty = 0$.

Now consider the case when $\{x^\eta\} \subset \mathrm{bdy}(A^j)$ (for large η). Then $\eta\bar{y} = \mathcal{P}_{u^j_{A^j}}(\eta\bar{y}) - \vec{n}(\eta\bar{y}) = \nabla u^j(x^\eta) - \vec{n}(\eta\bar{y})$ with $\vec{n}(\eta\bar{y}) \in \mathcal{N}_{A^j}(\mathcal{C}^j_{k_j})$. If $\nabla u^j(x^\eta)_o$ is the component of $\nabla u^j(x^\eta) = \eta\bar{y} + \vec{n}(\eta\bar{y})$ orthogonal to $\mathcal{N}_{A^j}(\mathcal{C}^j_{k_j})$, then $\nabla u^j(x^\eta)_o \to \infty$ so the corresponding components of x^η converge to 0 uniformly for $\bar{y} \in \mathcal{Y}$ by the argument of the previous paragraph. The other coordinates of x^∞ are fixed by the requirement that $x^\infty \in \mathcal{C}^j_{k_j}$. Since $u^j_{x_i}$ is decreasing in x_i then for large η this boundary, $\mathcal{C}^j_{k_j}$ here, is defined by the intersection of surfaces of the form $x_i = 0$ rather than $x_i = 1$ (an alternative choice only for $i \in \tilde{\mathcal{M}}^j$) since the surface $y = \nabla u^j|_{x_i=1}$ is closer to the origin than the surface $y = \nabla u^j|_{x_i=0}$. It follows that the components other than x^η_o are 0, hence $x^\infty = 0$. □

Corollary 4.2 *A solution of (3.3) exists; it is unique provided* $\epsilon^j(y, \hat{z}(y)) \notin \bigcup_{k\in\mathcal{K}^j} \bar{y}^\top a^j_k$. *In any case the solution of (1.4), i.e.* $(\hat{c}^j, (\hat{r}^j)^\top)^\top = I^{u^j_{A^j}}(\eta\bar{y})$, *is unique.*

Proof: Existence and uniqueness of η_j follows from the previous Lemma; only the uniqueness of $(\hat{c}^j, (\hat{r}^j)^\top)^\top$ requires comment. Non-uniqueness of η_j occurs if $\epsilon^j(y, \hat{z}(y)) = \bar{y}^\top a^j_k$ for some k, but then $(\hat{c}^j, (\hat{r}^j)^\top)^\top = a^j_k$ and hence is still unique. □

Lemma 3.1 shows that $x^\Lambda = (v(z^\Lambda), (Z - z^\Lambda)^\top)^\top \in \tilde{A}$ but we want to show that it lies in a compact subset of $\tilde{A}$, i.e. it stays away from boundary points of $\tilde{A}$ or A which are not in $\tilde{A}$. This means that the maximization in Lemma 3.1 can be carried out over a smaller compact set, $A^Z \backslash \tilde{\mathcal{G}}$, cf. the proof below.

Lemma 4.3 *For* $\Lambda \in L$, *a compact subset of* $\Re^J_{++}$, $(v(z^\Lambda), (Z - z^\Lambda)^\top)^\top$ *lies in* $\tilde{A}(L)$, *a compact subset of* $\tilde{A}$.

Proof: Fix $\Lambda \in L$. We first work under the standard assumptions. Recall from Remark 2.8 that $\tilde{A} = A \backslash \bigcup_{i\in\mathcal{I}}\{x \in A : x_i = 0\}$. Define $\tilde{g}(z) := (v(z), (Z - z)^\top)^\top$ on $\mathrm{dom}(v)$, $g^j(x, \Lambda) := I^{u^j_{A^j}}(\frac{\nabla u(x;\Lambda)}{\lambda_j})$ on $\tilde{A}$, $\tilde{g}^j(z, \Lambda) := g^j(\tilde{g}(z), \Lambda)$ on $\tilde{g}^{-1}(\tilde{A})$. Since $\nabla u(\cdot;\cdot)$ is continuous, cf. Lemma 4.4, then all three functions are continuous and $\tilde{g}(A^Z) \subset \mathrm{cl}(A)$, $\tilde{g}(z^\Lambda) \in \tilde{A}$, cf. Lemma 3.1, Theorem 2.6 and Theorem 2.4.

Recall $\mathcal{I}^j$ defined in (2.4). If $\mathcal{I} := \bigcup_j \mathcal{I}^j = \emptyset$ then $\tilde{A} = A$ and we take $\tilde{A}(L) = \tilde{g}(A^Z) \subset A$. So assume $\mathcal{I} \neq \emptyset$. We know that $x^\Lambda = \tilde{g}(z^\Lambda) \in \tilde{A}$, so $Z_i - z^\Lambda_i > 0$ for $i \in \mathcal{I}\backslash\{0\}$ and $v(z^\Lambda) > 0$ if $0 \in \mathcal{I}$. Take $i \in \mathcal{I}^j$. Then $\tilde{g}^j(z^\Lambda, \Lambda) \notin \{x \in A^j : x_i = 0\}$ and the latter set is contained in $\mathrm{bdy}(G^j_i(\delta))$ where $G^j_i(\delta) := \{x \in G^j_i : x_i \leq \delta\}$, cf.(2.4). In fact, we will show that $\cup_{\Lambda\in L}\tilde{g}^j(z^\Lambda, \Lambda) \cap G^j_i(\delta^j_i) = \emptyset$ near the "bad" boundary of A^Z (cf. the sets $\mathcal{B}$ below) for suitable $\delta^j_i > 0$ small enough.

Define $\mathcal{B}(\varepsilon) := \{z \in \tilde{g}^{-1}(\tilde{A}) : z_i < Z_i$ for all $i \in \mathcal{I}\backslash\{0\}$, $Z_i - z_i < \varepsilon$ for some $i \in \mathcal{I}\backslash\{0\}\}$. Set $\varepsilon_1 := \frac{1}{2}\min_i Z_i$. Since $0 \notin \mathrm{cl}(\mathcal{B}(\varepsilon_1))$ then for $z \in \mathrm{cl}(\mathcal{B}(\varepsilon_1))$ we have $v(z) > 0$, cf. (2.1)(iv), and by continuity of v, $\inf_{z\in\mathcal{B}(\varepsilon_1)} v(z) := v_o > 0$. Recall from (2.10) that $u(\tilde{g}(z); \Lambda) = \sum_j \lambda_j u^j_{A^j}(\tilde{g}^j(z, \Lambda))$

and $\tilde{g}^j(z,\Lambda) \in A^j$, $\sum_j \tilde{g}^j(z,\Lambda) = \tilde{g}(z)$. Write $c^j := \tilde{g}^j_0(z,\Lambda)$. Since $\sum_j c^j = v(z) \geq v_o > 0$ for $z \in \mathcal{B}(\varepsilon_1)$, then there exists j_o such that $c^{j_o} \geq v_o/J > 0$, and by (2.13) and (2.2)(iii), for $z \in \mathcal{B}(\varepsilon_1)$

$$u_c(\tilde{g}(z);\Lambda) = \lambda_{j_o} u_c^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) - \vec{n}_0^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) \leq \lambda_{j_o} u_c^{j_o}(\frac{v_o}{J}, Z) \leq \kappa_c \tag{4.2}$$

with $\kappa_c := \sup_{\Lambda \in L} \sup_j \lambda_j u_c^j(\frac{v_o}{J}, Z) < \infty$. Note that $\vec{n}_0^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) \geq 0$ since $c^{j_o} > 0$. Define $\kappa_v = \sup_{\mathcal{B}(\varepsilon_1)} \|\nabla v(z)\| < \infty$, cf. (2.1)(vi), so $\sup_{z \in \mathcal{B}(\varepsilon_1)} u_c(\tilde{g}(z);\Lambda) v_{z_i}(z) \leq \kappa_c \kappa_v$ for all i.

On the other hand we can choose δ_i^j so small that

$$u_{x_i}^j(\tilde{g}^j(z,\Lambda)) > \frac{\kappa_c \kappa_v}{\min_{\Lambda \in L} \min_j \lambda_j} + 1 \tag{4.3}$$

for $z \in \bigcup_{\Lambda \in L} (\tilde{g}^j(\cdot,\Lambda))^{-1}(G_i^j(\delta_i^j))$. It follows that (3.8) has no solution in $\mathcal{B}(\varepsilon_1) \cap \bigcup_j \bigcup_{i \in \mathcal{I}^j \setminus \{0\}} \bigcup_{\Lambda \in L} (\tilde{g}^j(\cdot,\Lambda))^{-1}(G_i^j(\delta_i^j))$. Note again that $\tilde{g}^j(z,\Lambda) \in G_i^j(\delta_i^j) \setminus \{x_i = 0\}$ implies that $z \in \mathrm{int}(A^Z)$, so $\vec{n}_i^j(z) = 0$ (we already know that $z_i^\Lambda < Z_i$ since $i \in \mathcal{I} \setminus \{0\}$).

We require an extra step if $i = 0 \in \mathcal{I}^j$. Set $\mathcal{B}_0(\varepsilon) := \{z \in \tilde{g}^{-1}(\tilde{A}) : v(z) \leq \varepsilon\}$ so for ε small, $z \in \mathcal{B}_0(\varepsilon)$ lies in a small (possibly empty) neighborhood of 0, i.e. $z_i < Z_i$ for all i. Set $Z_\varepsilon := \min_i \inf_{z \in \mathcal{B}_0(\varepsilon)} [Z_i - z_i] > 0$ and $|v| := \max_{A^Z} v(z)$. As above, for $i > 0$ there exists j_o such that $\tilde{g}_i^{j_o}(z) \geq \frac{Z_i - z_i}{J} > 0$. With $\tilde{z}_i := \frac{Z_\varepsilon}{J}$ and $\tilde{z}_k := Z_k,\ k \neq i$, we have

$$\begin{aligned} u_{x_i}(\tilde{g}(z);\Lambda) &= \lambda_{j_o} u_{x_i}^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) - \vec{n}_i^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) \leq \lambda_{j_o} u_{x_i}^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) \\ &\leq \lambda_{j_o} u_{x_i}^{j_o}(|v|, \tilde{z}) \leq \max_{\Lambda \in L} \max_{j,i} \lambda_j u_{x_i}^j(|v|, \tilde{z}) := \kappa_o. \end{aligned}$$

Furthermore $\min_{z \in \mathcal{B}_0(\varepsilon)} \|\nabla v(z)\| := v_{oo} > 0$. Now we can choose δ_0^j so small that

$$u_c^j(\tilde{g}^j(z)) > \frac{\kappa_o}{v_{oo} \min_{\Lambda \in L} \min_j \lambda_j} + 1$$

for $z \in \bigcup_{\Lambda \in L} (\tilde{g}^j(\cdot,\Lambda))^{-1}(G_0^j(\delta_0^j))$. Again it follows that (3.8) has no solution in $\mathcal{G} := [\mathcal{B}(\varepsilon_1) \cap \cup_j \cup_{i \in \mathcal{I}^j \setminus \{0\}} \cup_{\Lambda \in L} (\tilde{g}^j(\cdot,\Lambda))^{-1}(G_i^j(\delta_i^j))] \cup [\mathcal{B}_0(\varepsilon) \cap \cup_{\{j : 0 \in \mathcal{I}^j\}} \cup_{\Lambda \in L} (\tilde{g}^j(\cdot,\Lambda))^{-1}(G_0^j(\delta_0^j))]$, i.e. z^Λ must lie in the complement of this set, hence in the compact set $A^Z \setminus \mathrm{int}(\mathcal{G})$. Then $\tilde{A}(L) := \tilde{g}(A^Z \setminus \mathrm{int}(\mathcal{G}))$ is a compact subset of $\tilde{A}$. Moreover $\cup_{i \in \mathcal{I}} \{x \in A : x_i = 0\} \subset \tilde{g}(\mathcal{G})$ for all $\delta_i^j, \varepsilon, \varepsilon_1$, so $\cup_{i \in \mathcal{I}} \{x \in A : x_i = 0\} \subset \tilde{g}(\mathrm{int}(\mathcal{G}))$, i.e. $\tilde{A}(L) \subset \tilde{A}$.

Now let us establish the result under the alternate assumptions. For the above proof to work, we need to keep a positive distance from the "bad" boundary pieces of A^Z, i.e. those which are not in $\mathrm{dom}(\partial v)$. Set $\mathcal{B}_a(\varepsilon) := \{z \in \tilde{g}^{-1}(\tilde{A}) : z_i > 0 \text{ for all } i \in \mathcal{I}^v,\ z_i \leq \varepsilon \text{ for some } i \in \mathcal{I}^v\}$. Take $\varepsilon < \varepsilon_1$, $z \in \mathcal{B}_a$ with $i \in \mathcal{I}^v$ such that $z_i \leq \varepsilon$. As above there exists j_o such that $\tilde{g}_i^{j_o}(z,\Lambda) \geq \frac{Z_i - z_i}{J} \geq \frac{Z_i - \varepsilon}{J} > 0$. Recall (2.2)$'$(ix) and set $Z_\varepsilon := \min_i Z_i - \varepsilon$. Then

$$\begin{aligned} \frac{u_{r_i}(\tilde{g}(z);\Lambda)}{u_c(\tilde{g}(z);\Lambda)} &= \frac{u_{r_i}^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) - \vec{n}_i^{j_o}(\tilde{g}^{j_o}(z,\Lambda))}{u_c^{j_o}(\tilde{g}^{j_o}(z,\Lambda)) - \vec{n}_0^{j_o}(\tilde{g}^{j_o}(z,\Lambda))} \leq \frac{u_{r_i}^{j_o}(\tilde{g}^{j_o}(z,\Lambda))}{u_c^{j_o}(\tilde{g}^{j_o}(z,\Lambda))} \\ &\leq \frac{u_{r_i}^{j_o}(|v|, \frac{Z_\varepsilon}{J})}{u_c^{j_o}(|v|, \frac{Z_\varepsilon}{J})} \leq \max_j \frac{u_{r_i}^j(|v|, \frac{Z_\varepsilon}{J})}{u_c^j(|v|, \frac{Z_\varepsilon}{J})} := \kappa_i, \end{aligned}$$

since $\vec{n}^j_0(\tilde{g}^j(z,\Lambda)) \leq 0$ and $\vec{n}^j_i(\tilde{g}^j(z,\Lambda)) < 0$ only if $\tilde{g}^j_i(z,\Lambda) = 0$.

On the other hand, for $i \in \mathcal{I}^v$, we have $v_{z_i}(z) > \kappa_i$ for $z \in G^v_i(\delta^v_i)$ with δ^v_i sufficiently small. Then there is no solution of (3.8) on $\mathcal{B}_a(\varepsilon) \cap \bigcup_{i\in\mathcal{I}^v} G^v_i(\delta^v_i)$. We can now replace $\tilde{g}^{-1}(\tilde{A})\backslash\{0\}$ by $\tilde{g}^{-1}(\tilde{A})\backslash[\mathcal{B}_a(\varepsilon) \cap \bigcup_{i\in\mathcal{I}^v} G^v_i(\delta^v_i)]$ in the proof given for the standard case, i.e. in the definition of v_o and v_{oo}, to arrive at the same conclusion. $\square$

Recall from (3.12) that

$$F_j(\Lambda) := \nabla u(v(z^\Lambda), Z - z^\Lambda; \Lambda)^\top \Big(I^{u^j_{A^j}}(\tfrac{1}{\lambda_j}\nabla u(v(z^\Lambda), Z - z^\Lambda; \Lambda)) - \epsilon^j(y^\Lambda, z^\Lambda)\vec{e}_0 \Big),\ j = 1,\ldots,J.$$

Lemma 4.4 *∇u and $I^{u^j_{A^j}}(\frac{\nabla u}{\lambda_j})$ are continuous on $\tilde{A} \times \Re^J_{++}$. Moreover $F := (F_1, F_2, \ldots, F_J)^\top$ is continuous on $\Re^J_{++}$.*

Proof:

We will established that $\Lambda \mapsto z^\Lambda$ is continuous on $\Re^J_{++}$ and $\nabla u(\cdot\,;\cdot)$ is continuous on $\{((v(z^\Lambda), Z - z^\Lambda); \Lambda) : \Lambda \in \Re^J_{++}\}$ hence also $\Lambda \to y^\Lambda$.

First, note that $x \to u(x;\Lambda)$ is concave, closed on A, hence continuous on A, cf. Rockafellar 1970 [14], Theorem 10.2. Fix $(x_o, \Lambda_o) \in \tilde{A} \times \Re^J_{++}$, so there exists a unique $y_o := \nabla u(x_o; \Lambda_o) \in \operatorname{int}(\tilde{\mathcal{R}}(\Lambda_o))$ such that $x_o = I^u(y_o; \Lambda_o) = \sum_j I^{u^j_{A^j}}(\frac{y_o}{\lambda_{oj}})$. As $I^{u^j_{A^j}}$ is piecewise continuously differentiable, then $I^u(\cdot\,;\cdot)$ is locally Lipschitz on $\mathcal{D}(\Lambda_o) \times \Re^J_{++}$.

We want to apply the implicit function theorem for Lipschitz functions, cf. Clarke 1983 [6]. For $y \in \mathcal{S}^j_k$, cf. Definition 2.3, $\mathcal{N}_{A^j}(I^{u^j_{A^j}}(y)) = \mathcal{N}_{A^j}(\mathcal{C}^j_k)$, i.e. the cone does not depend on the particular y. Let P^j_k denote the orthogonal projection of $\Re^n$ onto $\operatorname{aff}(\mathcal{N}_{A^j}(\mathcal{C}^j_k))^\perp$. Then $\frac{\partial}{\partial y} I^{u^j_{A^j}}(y) = \left(P^j_k H_{u^j}(I^{u^j_{A^j}}(y)) P^j_k\right)^\dagger := \tilde{H}^j(y)$, cf. Chiarolla and Haussmann 2008 [4], (3.9), where † denotes the pseudo-inverse. This matrix is symmetric and negative semi-definite, but may be different on different sides of an interface between $\mathcal{S}^j_{k_1}$ and $\mathcal{S}^j_{k_2}$. For $\xi \in \Re^n$

$$\xi^\top \frac{\partial}{\partial y} I^u(y_o; \Lambda_o)\xi = \sum_j \lambda_j \xi^\top \tilde{H}^j(\frac{y_o}{\lambda_j})\xi \geq 0$$

and this is zero only if every term in the sum is zero, i.e. if $\xi \in \bigcap_j \operatorname{aff}(\mathcal{N}_{A^j}(I^{u^j_{A^j}}(\frac{y_o}{\lambda_j})))$, i.e. if $\xi = 0$ since $y_o \in \operatorname{int}(\tilde{\mathcal{R}}(\Lambda_o)) = \mathcal{R}_o(\Lambda_o)$, cf. (2.14). Hence the Jacobian of I^u is piecewise non-singular, and hence the y-component of the generalized Jacobian of $I^u(y_o; \Lambda_o)$ has maximal rank, cf. Clarke 1983 [6], 2.6.1, 2.6.4, 7.1. Now the implicit function theorem, cf. Clarke 1983 [6], Corollary, p.256, yields that there exists a Lipschitz continuous function ζ mapping a neighborhood $\mathcal{X}$ of (x_o, Λ_o) into a neighborhood $\mathcal{Y}$ of y_o such that $I^u(\zeta(x,\Lambda);\Lambda) = x$ for $x \in \mathcal{X}$. We may assume that $\mathcal{Y} \subset \operatorname{int}(\tilde{\mathcal{R}}(\Lambda_o))$.

For $(x,\Lambda) \in \mathcal{X} \cap \tilde{A} \times \Re^J_{++}$, $I^u(\zeta(x,\Lambda);\Lambda) = x$ by construction of ζ, and $x = I^u(\nabla u(x;\Lambda);\Lambda)$ since I^u is the inverse of ∇u on $\tilde{A}$. Set-continuity of $\operatorname{int}(\tilde{\mathcal{R}}) = \mathcal{R}_o$, cf. Chiarolla and Haussmann

2008 [4], Lemma 4.1, implies that $\zeta(x,\Lambda) \in \mathrm{int}(\tilde{\mathcal{R}}(\Lambda))$ for Λ close to Λ_o. Now strict monotonicity of I^u on $\mathrm{int}(\tilde{\mathcal{R}}(\Lambda))$, implies that $\nabla u = \zeta$ and hence ∇u is locally Lipschitz continuous on $\tilde{A} \times \Re^J_{++}$.

It follows that $(x,\Lambda) \to \hat{x}^j = I^{u^j_{A^j}}(\frac{\nabla u(x,\Lambda)}{\lambda_j})$ is also continuous. Since $u(x;\Lambda) = \sum_j \lambda_j u^j_{A^j}(\hat{x}^j)$, then $(x,\Lambda) \to u(x;\Lambda)$ is continuous on $\tilde{A} \times \Re^J_{++}$, hence on $\tilde{A}(L) \times L$ where L is a compact set in $\Re^J_{++}$, cf. Lemma 4.3; hence $\hat{u}(\cdot,\cdot) = u(\tilde{g}(\cdot),\cdot)$ is uniformly continuous on the compact set $[A^Z \backslash \mathrm{int}(\mathcal{G})] \times L$, cf. the proof of Lemma 4.3 for $\mathcal{G}$. It follows that $\Lambda \mapsto \max_{z \in A^Z \backslash \mathrm{int}(\mathcal{G})} \hat{u}(z;\Lambda)$ is continuous. Now suppose $\Lambda_k \to \Lambda_o$ with $\Lambda_k \in L$; then for a subsequence $z^{\Lambda_k} \to z_o \in A^Z \backslash \mathrm{int}(\mathcal{G})$ and for $z \in A^Z \backslash \mathrm{int}(\mathcal{G})$, $\hat{u}(z;\Lambda_o) = \lim_k \hat{u}(z;\Lambda_k) \le \lim_k \max_{z \in A^Z \backslash \mathrm{int}(\mathcal{G})} \hat{u}(z;\Lambda_k) = \lim_k \hat{u}(z^{\Lambda_k};\Lambda_k) = \hat{u}(z_o;\Lambda_o)$. Hence $z_o \in \arg\max_{z \in A^Z \backslash \mathrm{int}(\mathcal{G})} \hat{u}(z;\Lambda)$. As the last set is a singleton, then the limit is the same for all subsequences, hence $\Lambda \mapsto z^\Lambda$ is continuous on $\Re^J_{++}$. It follows that F is continuous. □

References

[1] Abel, A.B., Eberly, J.C.: An exact solution for the investment and value of a firm facing uncertainty, adjustment costs, and irreversibility. J Econ Dynamics Control **21**, 831-852 (1997)

[2] Bank, P., Riedel, F.: Optimal dynamic choice of durable and perishable goods. Discussion Paper 03-009, Dept of Econ, Stanford Univ., (2003)

[3] Chiarolla, M.B., Haussmann, U.G.: Equilibrium in a stochastic model with consumption, wages and investment. J Math Econ **35**, 1 - 31 (2001). A version without typos can be found at http://www.math.ubc.ca/ uhaus/wage.pdf

[4] Chiarolla, M.B., Haussmann, U.G.: Multivariable utility functions. SIAM J. Optim. **19**, 1511 - 1533 (2008).

[5] Chiarolla, M.B., Haussmann, U.G.: A stochastic equilibrium economy with irreversible investment. Preprint (2009). http://www.math.ubc.ca/˜uhaus/SEEII.pdf

[6] Clarke, F.H.: Optimization and Nonsmooth Analysis. New York: John Wiley & Sons (1983)

[7] Debreu, G.: Theory of Value - an Axiomatic Analysis of Economic Equilibrium. Cowles Foundation 17th Monograph. New York: John Wiley (1959)

[8] Deelstra, G., Pham, H., Touzi, N.: Dual formulation of the utility maximization problem under transaction costs. Ann. Applied Probab. **11**, 1353 - 1383 (2001)

[9] Florenzano, M.: General Equilibrium Analysis. Dordrecht, NL: Kluwer Academic (2003)

[10] Kannai, Y.: The ALEP definition of complementarity and least concave utility functions. J. Econ. Theory **22**, 115 - 117 (1980)

[11] Karatzas, I., Lehoczky, J.P., Shreve, S.E.: Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model. Math Oper Research **15**, 80 - 128 (1990)

[12] Lakner, P.: Optimal consumption and investment on a finite horizon with stochastic commodity prices. In: Byrnes, C.I., Martin, C.F., Saeks, R.E. (Eds): Linear Circuits, Systems and Signal Processing: theory and application, 457 - 464: Elsevier (1988)

[13] Negishi, T.: Welfare economics and existence of anequilibrium for a competitive economy. Metroeconomica **12**, 92 - 97 (1960)

[14] Rockafellar, R.T.: Convex Analysis. Princeton, NJ: Princeton University Press (1970)