A Stochastic Equilibrium Economy with Irreversible Investment *

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Abstract. A stochastic continuous-time economy on a finite time interval consists of (i) a commodity producing firm which must decide on cash holdings, employment levels and investment for capacity expansion, (ii) agents who maximize expected total utility of consumption, of money holding and of leisure, some of whom are employed by the firm, some facilitate capacity expansion ("construction") and some who are retired or on welfare. All agents participate in a financial market. The money supply (in real terms) is determined exogenously by the monetary authorities. We show how to construct an equilibrium where prices (and other parameters) are set so that both the agents and the firm can achieve their optimal choices and the markets clear.

Key words. general equilibrium, capacity expansion, capital asset pricing, consumption decisions, irreversible investment, convex analysis, stochastic analysis, stochastic control, local time.

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Miscellaneous Notation

$\mathcal{A}^j(x)$ is the set of admissible $(\mathbb{C}^j, \Phi^j)$ at x, cf. Definition 4.3
$\bar{\mathcal{A}}^j(x^j))$ is the set of admissible $(\mathbb{C}^j, \bar{x})$ at x^j, cf. Definition 4.5
$\hat{C}(t) := C(t+; \hat{\nu})$ is the optimal capacity process, cf. (6.2)
$\mathbb{C}^j := (c^j, m^j, l^j)$ is the consumption process of agent j, cf. Definition 4.1
$e^j(t) := x^j + g^j(t) + dw^j(t)$ is the production agent's real endowment, cf. (4.5)
$e(t) = \sum_j e^j(t)$ is the aggregate endowment process of the representative agent, cf. Section 5
$g(t) := \sum_j g^j(t) = M(t) - M(0) - \int_{[0,t)} M(s)\, \frac{dq(s)}{q(s)}$, cf.(6.14)
$g^j(t) := \int_{[0,t)} q(t)\, d\tilde{g}^j(t) + [q, \tilde{g}^j](t)$ is the real welfare received by agent j, cf. (4.4)
$G(C)$ is the scrap value of firm at capacity C, cf. Subsection 3.0.2
$\mathcal{H}^j$, $\bar{\mathcal{H}}^j$ are subsets of the admissible sets, cf. $(\mathcal{A}^j)$ and $(\bar{\mathcal{A}}^j)$ in Subsection 4.1.2
$I^{u_A}(y) := \arg\min_{x \in \mathbb{R}^n} \{x^\top y - u_A(x)\} = \{x : y \in \partial u_A(x)\}$, an extension of $(\nabla u_A)^{-1}$, cf. Section 4.2)
$\mathcal{I}_U^j$ and $\mathcal{I}^R$ are defined in (4.11) and (6.23)$(iv)'$ respectively
$(\hat{K}(t), \hat{L}(t)) := (K^{C(t;\hat{\nu})}, L^{C(t;\hat{\nu})}) = I^{R^{Q(M)}(C(t;\hat{\nu}),\cdot\,,\cdot\,)}(\tilde{r}(t), w'(t))$, cf. (3.14)
$(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))$ is the unique solution of (P_Λ), cf. Lemma 6.4
$\mathcal{L}$ is the generator of $(z(t), M(t))$, cf. (6.25)
ℓ is a local time, cf. (6.37) and following
$\mathcal{M}$ is generic notation for a martingale, with superscript identifying a linked process, e.g. $\mathcal{M}^q$, cf. (2.24)
$N\!\!S(t)$ is the market capitalization of the firm at time t, cf. Subsection 3.0.2
$N^{\mathrm{syn}}(t) := \sum_j \hat{\phi}_S^j(t) - N(t)$ is the number of synthetic shares, cf. preceeding Definition 6.1
$\wp^j(t) = (t, \int_0^t \tilde{r}(s)\, ds, w^j(t))^\top$, cf.(4.7)
$Q(M) = [0, M] \times [0, J^p]$
$\tilde{R}(C; \tilde{r}, w) := \max_{(K,L) \in [0,\infty) \times [0,1]} [R(C, K, L) - \tilde{r}K - wL]$ is the maximal profit rate at capacity C, cf. (3.10)
$\mathcal{T} := (t, C, M)$, $\hat{\mathcal{T}}(t) := (t, \hat{C}(t), M(t))$, cf. Definition 6.3
$\bar{\mathcal{T}} := (t, z, M)$, $\bar{\mathcal{T}}(t) := (t, z(t), M(t))$, cf. Definition 6.6
$U^j(t, c, m, l)$, $u^j(t, l)$ are consumption utilities of agent j, cf. Subsection 4.1.1
$U(t, c, m, l; \Lambda)$, $u(t, l; \Lambda)$ are consumption utilities of the representative agent, cf. Section 5
$\hat{U}(\mathcal{T}, K, L; \Lambda) = U(t, R(C, K, L), M - K, J^p - L; \Lambda)$ is the utility of operating capital and labor, cf. Def. 6.3
$\bar{U}(\bar{\mathcal{T}}, K, L; \Lambda) = U(t, z + R^2(K, L), M - K, J^p - L; \Lambda)$, cf. Definition 6.6
$U_c^{\Pi^\Lambda}(t, z, M)$ is $\bar{U}_1(\bar{\mathcal{T}}, \Pi^\Lambda(\bar{\mathcal{T}}); \Lambda)$ as a function of $\bar{\mathcal{T}} = (t, z, M)$
$u_A(x) := u(x) - \chi_A(x) = \begin{cases} u(x) & \text{if } x \in A, \\ -\infty & \text{if } x \notin A. \end{cases}$, cf. (3.11)
$V^j(x)$ is the utility of terminal wealth of agent j, cf. Subsection 4.1.1
$V(x; \Lambda)$ is the utility of terminal wealth of the representative agent, cf. Section 5
w' is the wage rate of production agents
w^c is the cumulative wage of agents in the construction sector

$X^j = X^{x^j, \mathcal{C}^j, \Phi^j}$ is the real wealth of agent j, cf. (4.3) and (4.7)
$z = R^1(C)$, cf. Definition 6.6 and preceeding
η^j is a solution of
$E\left\{\int_0^T \zeta(t)[\pi(t)^\top I^{U^j}(t, \eta^j\zeta(t)\pi(t))\,dt + u^j(t, \eta^j\zeta(t))\,dw^c(t) + \zeta(T)I^{V^j}(\eta^j\zeta(T))\right\} = \xi^j$, cf. (4.17)
η^Λ is a solution of
$E\left\{\int_0^T \zeta(t)\pi(t)^\top I^U(t, \eta^\Lambda\zeta(t)\pi(t);\Lambda)\,dt + u^j(t, \eta^j\zeta(t))\,dw^c(t) + \zeta(T)I^V(\eta^\Lambda\zeta(T);\Lambda)\right\} = \xi$, cf. (5.9)
$\hat{\nu}^+(t) := \hat{\nu}(t+)$, $z(t) := R^1(\hat{C}(t))$, cf. Definition 6.6
$\xi^j := x^j + E\ \int_0^T \zeta(t)\,de^j(t)$ is the expected "deflated value" of the endowment e^j, cf. (4.6)
$\xi = E\left\{X(0) + \int_0^T \zeta(t)\,de(t)\right\}$ is the expected "deflated value" of the aggregate endowment e, cf. Section 5
$\pi := (1, \tilde{r}, w)^\top$, cf. Section 4.2
$\Pi = (K, L)$,
$\Pi^\Lambda(\bar{\mathcal{T}}) := (K^\Lambda(\bar{\mathcal{T}}), L^\Lambda(\bar{\mathcal{T}}))$ is the unique solution of $\sup_{(K,L)\in[0,M]\times[0,1]} \bar{U}(\bar{\mathcal{T}}, K, L; \Lambda)$, cf. Definition 6.6
$\Sigma(t) := (\sigma_q(t), \sigma_D(t), \sigma_S(t))^\top$, $\Sigma'(t) := (\sigma_q(t), \sigma_S(t))^\top$, cf. (2.15)
ς^j_Λ is defined in (7.6)
$\Phi^j := (\phi^j_{\mathcal{B}}, \phi^j_B, \phi^j_D, \phi^j_S)$ is the portfolio process of agent j, cf. Definition 4.2
ψ^q is the integrand in the representation of the martingale $\mathcal{M}^q$ and similarly for other superscripts

1 Introduction

Loosely speaking, an economy is in (intertemporal) equilibrium if prices (of goods, of labour, of money and assets) vary over time such that the agents (who consume the goods and provide labour and investment funds) can act to maximize their individual utilities, the manager of the production facility can act so as to maximize profits and the "value" of the firm, and still the markets clear, i.e. for example the amount of labour desired by the manager will be provided by the agents acting in their own best interest. Included in the setting are the questions of equilibrium asset pricing, of irreversible investment decisions, and of fiscal and monetary policy.

The finance literature is quite rich in equilibrium asset pricing models like the classic [10], [13], [16], [18], [25]. In the economics literature there also exists a multitude of (usually discrete time) models that study various aspects of both open and closed economies in equilibrium, cf. [23], [26], [14] and the references therein. In particular, in some of these models the agents hold money for transaction purposes, e.g. in [3] Basak and Gallmeyer use a continuous time model based on partial equilibrium results in the literature to determine exchange rates as well as asset prices in equilibrium, thus linking the CAPM ideas to international economics. Furthermore, in some of these models, the producers of the goods, the firms, make decisions regarding labour levels and capital investment strategies. For example, Abel and Eberly, [1], provide an explicit solution of an irreversible investment problem in a continuous time Markovian setting. Employment rates are included in the model as a decision variable. In [20] a continuum of firms produces goods for a continuum of agents in a deterministic, discrete time economy. Each firm has an endowment which it can increase by borrowing money to invest. Its aim is to maximize profit. The agents buy goods to gain utility and hold money to facilitate these purchases (cash in advance). Their aim is to maximize utility. In [6] a one period, deterministic economy is considered. Agents maximize utility gained from consuming goods and holding resources. They have an initial endowment of the latter and must decide how much to sell in order to finance the purchase of the goods. The single producer of the goods purchases resources to produce the goods with the aim of maximizing profit. That problem can be solved using General Equilibrium Theory, but our approach there contains technical results used here.

Irreversible investment problems have been studied widely, cf. [12] and the references therein. Of particular interest is the work of Baldursson and Karatzas, [2] where it is shown that the solution of the myopic investor's problem, which can be solved in terms of stopping rules involving the Snell envelope, leads to a solution of the "social planner's" problem which is equivalent to the optimal irreversible investment problem facing a firm. These results motivated us to provide in [5] a solution for an irreversible investment problem that is closely linked to one studied in the present work. Paulsen [27] also considers an equilibrium model with irreversible investment to study money market returns. His model is rather simpler than ours but it leads to some interesting economic conclusions.

Our aim in this paper is to obtain an existence result for a general equilibrium model of an economy that includes irreversible investment in firms in addition to heterogeneous agents' decision problems, leading to market parameters, e.g. prices, interest rates and employment rates, that provide clearing of the markets. No government sector is included beyond an exogenous money supply process except for a "welfare" process paid to the agents; the latter is a device to pass changes in the money supply into the economy. We leave introduction of a government sector to the future. The model contains a CAP model and will give further insight into the dependence of interest rates and excess returns of risky assets on, among other factors, the money supply and industrial capacity. The point is that the endowment process of the partial equilibrium formulation in, e.g. [18], becomes endogenous in our setting.

In the recent paper [4] we made a first attempt to build a macroeconomic model based on the actions of both the individual agents and the firm. The firm employed the agents to produce the good; a production function $R(t, L(t))$ converted labour at level $L(t)$ units into goods. We endogenized the dividend and earning processes, the latter as salary of the agents and the former as the profits of the firm. The agents derived utility from consumption of both goods and leisure, hence there were two "prices", one for the good and one for leisure.

In the current work we push the previous results further by including money in the economy for the purpose of facilitating the transactions of both the agent and the firm, and by allowing capital investment in the production process. This new setting raises the complexity considerably. In fact, the optimization problem of the firm's manager changes from a static calculus of variations problem to a dynamic control problem. Fortunately we can treat it using the theory of [2] mentioned above in conjunction with [7].

The presence of money is handled by thinking of it as another good, and hence introducing the price of money. Therefore, when looking for an equilibrium, we work with real quantities, not nominal ones, so we adopt the setting of the "monetary equilibrium" model of Basak and Gallmeyer, [3], which itself is built on the framework of [18], rather than the nominal model of [4]. That is, we take an exogenous money supply (determined by the monetary authorities) and an endogenous financial market consisting of four instruments, a real bond to hedge against inflation, a nominal bond to finance production, shares in the productive asset, and a further auxiliary contract used to hedge remaining risk. The market may not be complete, unlike in

[18]. In [18] the instruments of the complete market are not related directly to any underlying economic fundamentals and the productive asset plays only an auxiliary role, whereas we take it as a fundamental part of the market and minimize the role of any contract introduced solely to hedge risk. Furthermore, contrary to what happens in [3], our productive asset pays an endogenous dividend stream $\delta(t)$ to the shareholders. Such $\delta(t)$ is determined by solving a stochastic capacity expansion control problem in which the firm's manager maximizes the profits by selecting not only the employment rate, as in [4], but also the money (operating capital) to be borrowed and the new capital to be invested to expand capacity. This investment capital is raised from the sale of new shares. The problem is set up as in [1] - first the employment rate and operating capital are optimized leaving an optimization with investment as the sole control entering the dynamics of the firm's capacity which, in turn, impacts production, cf. [8]. The investment funds are turned into expanded production capacity by the construction sector.

In fact we have a three sectors economy: the production sector is composed of agents who work to produce the consumption good in return for a wage at rate w', the construction sector is composed of agents who "construct" new capacity in return for a wage w^c which may be singular with respect to time. The last sector consists of agents without a wage income, i.e. retirees or welfare recipients; this is the welfare sector. We assume that production capacity and the labour pool of the second sector are completely elastic - the firm's capacity expansion demand is always met.

In the model several processes may be singular. Besides the firm's optimal investment process, the money supply may also have a singular input; in addition the "hard" constraints on the decision variables may yield singular processes of the "local time" sort. In fact, since each agent's consumption rate is constrained to be non-negative, since the firm's money holdings are non-negative and bounded above by the money supply, and since labour employed by the firm and operating capital borrowed by the firm are constrained to be non-negative with hard upper bounds, the optimal capital and labour turn out to contain singular terms which have an effect on the price of the real bond. These singular terms are identified as the respective local times at some boundaries. As a consequence, to achieve equilibrium the real bond will have to have a singular component. Such is also the case in the model of [19].

The Negishi method used in [18] works for utility functions of one variable, but ours have three variables. This raises some technical questions which we have studied in [7]. In particular, we provide there a formula for an extension of the inverse of the gradient map of a multivariable utility function, and we establish regularity properties of the utility function of a representative agent.

Another novel device is the auxiliary utility of capital and labour, whose maximization is a central pillar of our method. In fact establishing existence of an equilibrium usually reduces to solving a complicated fixed point problem due to the linkage of the firm's optimization problem and the optimization problem of the representative agent (aggregating the agents). We avoid this fixed point problem by decoupling the two problems: we assume that the production function is the sum of a term depending on the productive capacity (also called "installed capital") and a term depending on labour and operating capital levels. This allows us to find the optimal capacity independently of the other parameters. Then we solve for the optimal operating capital and labour maximizing the above-mentioned auxiliary utility of capital and labour, again avoiding reference to any parameters other than the optimal capacity which is known at this point. This was done in [8]. Of course the solution of the irreversible investment problem affects the equilibrium.

The model is specified in Section 2 and the financial market is set up there. The firm's capacity expansion problem is detailed and solved in Section 3. In Section 4 agents are introduced and the agent's problem is solved. The representative agent's utility function is defined in Section 5 and his maximization problem is solved. In Section 6 we define "equilibrium" and we derive some conclusions by exploiting a link between the firm's control problem and the representative agent's optimal expected total utility. The real interest rate, the nominal interest rate, the wage rate, the employment rate, the money balances, the consumption rate, the price of money, hence the price of the good and the price of the productive asset, are all determined from equilibrium considerations. We see in Remark 6.10 that some of the usual implications of the CAPM hold when suitably interpreted. Existence of an equilibrium is proved in Section 7. The Appendix contains a technical proof.

We stress that the *only* exogenous parameters in the model, aside from some initial conditions, are those of the money supply (chosen by the monetary authorities), and those of the capacity (or technology) process.

2 The Model

We build an economy with finite horizon T on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t : t \in [0,T]\}$ which is the usual augmentation of the filtration generated by an exogenous two-dimensional Brownian motion $\{W(t) : t \in [0,T]\}$. The financial market consists of a nominal bond with price process $\tilde{B}(t)$, a real bond (i.e. valued in real terms) with price process $\mathcal{B}(t)$, another type of contract, called "derivative" (without necessarily an underlying) with price process $\tilde{D}(t)$, and shares of one firm with price $\tilde{S}(t)$. Initially there are $N(0)$ shares outstanding. The owner of each share receives dividends paid at a rate $\tilde{\delta}(t)$ units of currency per unit of time. This firm produces a single kind of perishable consumption good which sells at price $p(t)$ units of local currency at time t. To do so it employs labour, borrows capital to facilitate its daily business, and sells shares to raise capital for capacity expansion. The division of roles for the two kinds of capital is arbitrary but useful in our context. Capacity expansion encompasses many factors: physical expansion of plant through construction or mergers, aquisition of new technologies, etc. Regarding notation, nominal prices stated in units of currency and denoted with a superscript $\tilde{}$ and real prices stated in units of the good are related by $\tilde{B}(t) = p(t)B(t)$. More on this shortly.

There are three kinds of agents, the first kind provides labour to the firm (production sector), the second turns investment cash into capacity expansion (construction sector) and the third kind who provides no labour (the welfare and retired sector). The agents consume the good, own the shares of the firm and hold the bonds. They will also hold money, supplied by the government, to facilitate purchases of the good and other financial transactions. The country's real money supply, M, is an *exogenous* (determined by the monetary authorities) continuous semimartingale. The price processes are *endogenous* and will be determined from equilibrium considerations.

We continue with details of the financial market; the details of the firm and behaviour of the agents will follow in the next sections. Without further comment we shall in future denote imposed constant lower bounds on a variable x by k_x and constant upper bounds by κ_x.

2.1 The Financial Market

The market is *endogenous* in this model and consists of a real bond, a nominal bond, a productive asset, and a "derivative". To some extent we now follow [21], Chapter 1. The price of the real bond is a continuous, strictly positive, $\mathcal{F}_t$-adapted, finite variation process. It then decomposes into the sum of an absolutely continuous part, $\mathcal{B}^{ac}$ and a singular continuous part $\mathcal{B}^{sc}$. We take $\mathcal{B}(0) = 1$, i.e. initially the unit of account, in real terms, is worth one unit of good. Set

$$r(t) := \frac{\frac{d}{dt}\mathcal{B}^{ac}(t)}{\mathcal{B}(t)}, \quad \beta(t) := \int_0^t \frac{d\mathcal{B}^{sc}(t)}{\mathcal{B}(t)}$$

so that

$$d\mathcal{B}(t) = \mathcal{B}(t)[r(t)\,dt + d\beta(t)], \qquad t \in (0, T], \tag{2.1}$$

and $\mathcal{B}(t) = \exp\{\int_0^t [r(t)\,dt + d\beta(t)]\}$, $\beta(0) = 0$. If $\beta \equiv 0$, then r is the *real interest rate.* Although r is not the growth rate of $\mathcal{B}$ when $\beta \neq 0$ we will still call it the real interest rate. By writing $\mathcal{B}$ in nominal terms (that is, after multiplication by p) one obtains an inflation-indexed bond (or bank account) that allows hedging against inflation. With the variation on $[0, T]$ of the path $\beta(\cdot)$ denoted by $\|\beta\|_T$, it is assumed (to be verified later) that

$$\int_0^T |r(t)|\,dt < \infty \text{ a.s.}, \quad \|\beta\|_T \leq \infty \text{ a.s.} \tag{2.2}$$

The nominal bond (bank account) is used to finance production and its price is given by

$$\tilde{B}(t) = \exp \int_0^t \tilde{r}(s)\,ds, \quad t \in [0, T], \tag{2.3}$$

where $\tilde{r}$, the *nominal interest rate*, is continuous on $(0, T]$, adapted and satisfies

$$0 < k_{\tilde{r}} \leq \tilde{r}(t) \leq \kappa_{\tilde{r}} \tag{2.4}$$

for some constants $k_{\tilde{r}}, \kappa_{\tilde{r}}$. Again we have normalized $\tilde{B}$ so that at time 0 the unit of account is worth one unit of currency.

The nominal price of one share of the productive asset is the continuous $\mathcal{F}_t$-semimartingale $\tilde{S}(\cdot)$, satisfying (superscript $^\top$ stands for transpose)

$$d\tilde{S}(t) + \tilde{\delta}(t)\,dt = \tilde{S}(t)[\mu_{\tilde{S}}(t)\,dt + \sigma_{\tilde{S}}^\top(t) dW(t)], \qquad t \in [0, T], \tag{2.5}$$

where $\tilde{\delta}$ is the *nominal dividend process* and

$$\tilde{S}(T) < +\infty, \quad \tilde{\delta}(t) \geq 0, \quad \int_0^T \{\tilde{\delta}(t) + |\tilde{S}(t)\mu_{\tilde{S}}(t)| + \|\tilde{S}(t)\sigma_{\tilde{S}}(t)\|^2\}\, dt < \infty \quad \text{a.s.} \tag{2.6}$$

We do not give an initial condition because $\tilde{S}$ will be specified as the ratio of market capitalization to number of outstanding shares. The former will be determined as the discounted present value of future cash flows with a terminal condition supplied by a "rational market" assumption, and the latter will be determined by the investment policy of the firm and the initial number of outstanding shares, $N(0)$. Only if $N(0) = 0$ will we specify $\tilde{S}(0)$, and it will be the arbitrary share price specified in the Initial Public Offering (IPO) made when N assumes a positive value.

The derivative is a contract in zero net supply which the agents trade to offset risk. Its nominal price is

$$\tilde{D}(t) = \exp\Big(\int_0^t \sigma_{\tilde{D}}^\top(s)\, dW(s) + \int_0^t [\mu_{\tilde{D}}(s) - \tfrac{1}{2}\|\sigma_{\tilde{D}}(s)\|^2]\, ds\Big), \qquad t \in [0,T], \tag{2.7}$$

with

$$\int_0^T \{|\mu_{\tilde{D}}(t)| + \|\sigma_{\tilde{D}}(t)\|^2\}\, dt < \infty \quad \text{a.s.} \tag{2.8}$$

In fact, we shall do our accounting in real terms, so we shall work with *the price of money* (in units of the commodity), $q(t) = 1/p(t)$, rather than with $p(t)$. Then multiplication by q changes nominal quantities into real quantities with the former distinguished from real quantites by using a tilde $\tilde{}$. In general $x = q\tilde{x}$. We assume that the process q satisfies

$$\begin{cases} dq(t) = q(t)[\mu_q(t)dt + \sigma_q^\top(t)dW(t) + d\beta(t)], & t \in [0,T), \\ q(T) = q_T \end{cases} \tag{2.9}$$

where the parameters μ_q and σ_q are endogenous and must satisfy

$$\int_0^T \Big\{|\mu_q(t)| + \|\sigma_q(t)\|^2\Big\}\, dt < \infty, \quad \text{a.s.} \tag{2.10}$$

The term $d\beta$ must appear in the dynamics for q to avoid arbitrage, cf. [21]. The random variable q_T is an exogenous quantity and its inverse is the monetary authorities' inflation target. We assume $0 < q_T(\omega) \leq \kappa_q$. Previous work in the literature assumes $q_T = 0$, i.e. money has no value when "the world ends", but we do not have this scenario because we have terminal utility and scrap value of the firm.

The real prices $B = q\tilde{B}$ of the nominal bond, $D = q\tilde{D}$ of the derivative, and $S = q\tilde{S}$ of one share of the productive asset satisfy

$$\begin{aligned} dB(t) &= B(t)[(\tilde{r}(t) + \mu_q(t))dt + \sigma_q^\top(t)dW(t) + d\beta(t)], \quad t \in [0,T], \quad B(0) = q(0), \\ dD(t) &= D(t)[\mu_D(t)dt + \sigma_D^\top(t)dW(t) + d\beta(t)], \quad t \in [0,T], \quad D(0) = q(0), \\ dS(t) + \delta(t)\, dt &= S(t)[\mu_S(t)\, dt + \sigma_S^\top(t)\, dW(t) + d\beta(t)], \quad t \in [0,T], \end{aligned} \tag{2.11}$$

where

$$\mu_D = \mu_{\tilde{D}} + \mu_q + \sigma_{\tilde{D}}^\top \sigma_q,\ \sigma_D = \sigma_q + \sigma_{\tilde{D}},\ \mu_S = \mu_{\tilde{S}} + \mu_q + \sigma_{\tilde{S}}^\top \sigma_q,\ \sigma_S = \sigma_{\tilde{S}} + \sigma_q. \tag{2.12}$$

The measurable, $\mathcal{F}_t$-adapted, real-valued drift processes $r, \tilde{r}, \mu_q, \mu_{\tilde{D}}, \mu_{\tilde{S}}$ and vector-valued volatility processes $\sigma_q, \sigma_{\tilde{D}}$ and $\sigma_{\tilde{S}}$ as well as β will be determined endogenously by equilibrium arguments.

The money supply also enters the model. The monetary authorities can increase this supply with the increased money passing to the agents as welfare payments (or possibly taxes if these payments are negative). Indeed, if capacity is to increase, so must the real money supply. On the other hand, monetary authorities aim to maintain rates of inflation in a certain band; this is what the terminal value q_T (more precisely $1/q_T$) represents. In this model we shall take the real money supply $M(t)$ as exogenous, $M(0) \in [k_M, \kappa_M]$ with $0 < k_M < \kappa_M$ given constants, and

$$dM(t) = M(t)[\mu_M(t)\, dt + \sigma_M^\top(t)\, dW(t) + d\beta_M(t)]. \tag{2.13}$$

Here μ_M, σ_M, β_M are adapted processes, the first two being bounded and β_M continuous with $E\|\beta_M\|_T < \infty$, such that $k_M \le M(t) \le \kappa_M$. The nominal supply then satisfies

$$d\tilde{M}(t) = \tilde{M}(t)[\mu_{\tilde{M}}(t)\, dt + \sigma_{\tilde{M}}^\top(t)\, dW(t) + d\beta_{\tilde{M}}(t)]$$

with

$$\sigma_M = \sigma_{\tilde{M}} + \sigma_q, \qquad \mu_M = \mu_{\tilde{M}} + \mu_q + \sigma_{\tilde{M}}^\top \sigma_q, \qquad \beta_M = \beta_{\tilde{M}} + \beta. \tag{2.14}$$

Monetary authorities have not controlled the money supply per se since about 1980; they set the nominal interest rate to keep inflation within bounds while maintaining a reasonable rate of GDP growth. Nevertheless in this model the real money supply is exogenous to allow us to solve for an equilibrium.

2.2 The Risk-Neutral Probability Measure

Set

$$\Sigma(t) := \begin{pmatrix} \sigma_q^\top(t) \\ \sigma_D^\top(t) \\ \sigma_S^\top(t) \end{pmatrix}, \qquad \Sigma'(t) := \begin{pmatrix} \sigma_q^\top(t) \\ \sigma_S^\top(t) \end{pmatrix}. \tag{2.15}$$

We **assume** that a.e. a.s. $\Sigma'(t,\omega) \neq 0$, i.e. the market is in fact always random. Then we may assume that $\Sigma(t,\omega)$ has rank two a.e. a.s. by taking $\sigma_D(t) \in \ker(\Sigma'(t))$ bounded, non-zero unless $\ker(\Sigma'(t)) = \{0\}$. $\Re^n$ denotes Euclidean n-space. We assume (shown to hold in equilibrium) that there exists an $\Re^2$-valued process $\theta(\cdot)$ such that

$$\Sigma(t)\theta(t) = \begin{pmatrix} \mu_q(t) + \tilde{r}(t) - r(t) \\ \mu_D(t) - r(t) \\ \mu_S(t) - r(t) \end{pmatrix} \tag{2.16}$$

with

$$\begin{cases} (i) & \int_0^T \|\theta(t)\|^2\, dt < \infty \text{ a.s.} \\ (ii) & \mathcal{E}(t) := \exp\Big[-\int_0^t \theta^\top(s)\, dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds \Big] \text{ is a martingale.} \end{cases} \tag{2.17}$$

Remark 2.1 θ is the *market price of risk*. We introduce the new contract, the derivative, because we cannot be certain that agents can hedge risk without it. The market in the two bonds, the stock and the derivative is arbitrage free. □

It follows that $P^\circ(A) := E\{\mathcal{E}(T)\mathbb{1}_A\}, \quad A \in \mathcal{F}_T$, is a probability measure ($\mathbb{1}_A$ denotes the indicator function of A); in fact it is the *risk-neutral* probability measure equivalent to P with Radon-Nikodym derivative $\frac{dP^\circ}{dP}\Big|_{\mathcal{F}_t} = \mathcal{E}(t), \quad t \in [0,T]$. Set

$$\epsilon(t) := \exp\left[-\int_0^t r(s)\,ds - \beta(t)\right].$$

The process $\zeta(t) := \epsilon(t)\mathcal{E}(t)$ is the *real state-price density* (or *deflator*) and according to the foregoing satisfies

$$\begin{cases} d\zeta(t) = \zeta(t)[-r(t)dt - \theta^\top(t)dW(t) - d\beta(t)], & t \in (0,T], \\ \zeta(0) = 1. \end{cases} \tag{2.18}$$

We assume (shown to hold in equilibrium) that there exist finite constants k_ζ, κ_ζ such that a.e. a.s.

$$0 < k_\zeta \le \zeta(t,\omega) \le \kappa_\zeta. \tag{2.19}$$

Finally, $W^\circ(t) := W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under P° and

$$\begin{aligned} dB(t) &= B(t)[r(t)\,dt + \sigma_q^\top(t)dW^\circ(t) + d\beta(t)], \\ dD(t) &= D(t)[r(t)\,dt + \sigma_D^\top(t)dW^\circ(t) + d\beta(t)], \\ dS(t) + \delta(t)\,dt &= S(t)[r(t)\,dt + \sigma_S^\top(t)dW^\circ(t) + d\beta(t)], \\ dq(t) &= q(t)[(r(t) - \tilde{r}(t))\,dt + \sigma_q^\top(t)dW^\circ(t) + d\beta(t)], \\ dM(t) &= M(t)[(\mu_M(t) - \sigma_M^\top(t)\theta(t))\,dt + \sigma_M^\top(t)dW^\circ(t) + d\beta_M(t)]. \end{aligned} \tag{2.20}$$

We can now represent the price of money as the expected present value (under the risk-neutral measure) of the future nominal interest stream, but we give the representation in terms of the original measure.

Lemma 2.2 *If*

$$\operatorname{ess\,sup}_{t,\omega} q(t) < \infty \tag{2.21}$$

then

$$q(t) = \frac{1}{\zeta(t)} E\left\{e^{\int_t^T \tilde{r}(s)\,ds}\zeta(T)q_T\Big|\mathcal{F}_t\right\}. \tag{2.22}$$

Conversely, if q is defined by (2.22), then it is bounded and satisfies (2.9) with $\sigma_q = \theta + \psi^q$ and $\mu_q = r - \tilde{r} + \theta^\top\sigma_q$ for some process ψ^q such that $\int_0^T |\psi^q(t)|^2\,dt < \infty$ a.s. Moreover (2.10) holds.

Proof: Much as in [18], (8.5) and Theorem 8.2 using our (2.20), (2.4)and (2.19), it follows that

$$q(t) = \frac{1}{\zeta(t)} E\Big\{\zeta(T)q_T + \int_t^T \zeta(s)q(s)\tilde{r}(s)ds \Big| \mathcal{F}_t\Big\}. \tag{2.23}$$

Then $\zeta(t)q(t) + \int_0^t \zeta(s)q(s)\tilde{r}(s)\,ds$ is a continuous $\{\mathcal{F}_t\}$-martingale, hence so is

$$\mathcal{M}^q(t) := e^{\int_0^t \tilde{r}(s)\,ds}\zeta(t)q(t) = E\left\{e^{\int_0^T \tilde{r}(s)\,ds}\zeta(T)q(T)\Big|\mathcal{F}_t\right\} \tag{2.24}$$

and (2.22) follows.

Conversely with $\mathcal{M}^q(t)$ defined by the conditional expectation in (2.24) (with $q(T) := q_T > 0$), then $\mathcal{M}^q(t) > 0$ a.s. since $\zeta(T)q_T$ is, and the martingale representation theorem gives

$$\mathcal{M}^q(t) = \mathcal{M}^q(0) + \int_0^t \mathcal{M}^q(s)\psi^q(s)\,dW(s)$$

for some process ψ^q such that $\mathcal{M}^q\psi^q$ is a.s. square integrable, i.e. ψ^q is a.s. square integrable since $\inf_t \mathcal{M}^q(t) > 0$ a.s. Then (2.22) implies that $\mathcal{M}^q(t) = e^{\int_0^t \tilde{r}(s)\,ds}\zeta(t)q(t)$ and

$$dq(t) = q(t)\left\{(r(t) - \tilde{r}(t) + \theta(t)^\top[\theta(t) + \psi^q(t)])\,dt + [\theta(t) + \psi^q(t)]^\top\,dW(t) + d\beta(t)\right\}.$$

We can now identify μ_q and σ_q. Finally (2.10) follows from (2.17)(i), (2.2) and (2.4). □

Remark 2.3 Observe that the martingale $\mathcal{M}^q(t) = \zeta(t)B(t)$ is the deflated real value of the nominal bond. Moreover the nominal price of a unit of good is q^{-1}, where

$$\frac{dq^{-1}}{q^{-1}} = (\tilde{r} - r + \sigma_q^\top\psi^q)\,dt - \sigma_q^\top\,dW(t) - d\beta(t),$$

so the *inflation rate* (mean growth rate of q^{-1}) is defined (at least when $d\beta = 0$) to be $\tilde{r} - r + \sigma_q^\top(\sigma_q - \theta)$, i.e. the difference between the nominal and real interest rates adjusted by a risk premium equal to the product of the volatility of the price of a unit of good and the sum of this volatility plus the market price of risk (since $\sigma_q^\top(\sigma_q - \theta) = (-\sigma_q)^\top(\theta + (-\sigma_q))$. The term $d\beta$ accounts for abrupt (i.e. not proportional to dt) but still continuous changes in inflation.

When $q_T = 0$ we still have the representation (2.23), but now (2.22) implies $q \equiv 0$, the trivial solution; other solutions of (2.23) are possible when $q_T = 0$, cf. [3]. □

3 The Firm

We represent the productive sector of the economy by one firm. The firm produces the consumption good at rate $R(C, L, K)$ when it has capacity C, employs L units of labour and has *real* operating capital K. We assume this operating capital at time t is borrowed as nominal funds $\tilde{K}(t)$, with interest paid at rate $\tilde{r}(t)$. It is possible to invest in the sector (firm) to increase its productive capacity. This investment capital is realized through the sale of shares whose nominal price at time t is $\tilde{S}(t)$ and is passed to the construction sector which provides the increase in capacity. Initially there are $N(0)$ shares outstanding. If there are $N(t)$ shares outstanding at time t, where N is an almost surely finite, non-decreasing, lcrl (left-continuous with right hand limits) adapted process, then $\nu(t)$, the real capital raised through share offerings on the time interval $[0, t]$, satisfies

$$\nu(t) = \int_{[0,t)} S(s)\, dN(s), \quad \nu(0) = 0, \tag{3.1}$$

and is an almost surely finite, lcrl non-decreasing, adapted process if S has bounded paths. In nominal terms, the instantaneous investment is $(q(s))^{-1}\, d\nu(s)$. The irreversibility of investment is expressed in the non-decreasing nature of ν. Note that $N(0)$ is given (exogenous) but further increases in N will be decided by the firm's manager and will be endogenous in our problem. Also given is the initial holding of operating capital, $K(0)$. The optimal $\hat{K}(0)$ may be different from $K(0)$ so an instantaneous shift of money may take place at $t = 0$ to move it to equilibrium. $K(0)$ is required to account for the initial wealth in the economy at time $t = 0$.

Capacity is denoted by $C(t;\nu)$ and is **assumed** to satisfy

$$\begin{cases} dC(t;\nu) = C(t;\nu)[-\mu_C(t)dt + \sigma_C^\top(t)dW(t)] + f_C(t)d\nu(t), & t \in [0, T), \\ C(0;\nu) = C_0 \geq 0, \end{cases} \tag{3.2}$$

where $\mu_C \geq 0, \sigma_C$ and $f_C \geq k_f > 0$ are given bounded adapted processes, f_C being continuous. It is convenient to define

$$C^o(t) := C(t; 0)|_{C_0=1}, \qquad \overline{\nu}(t) := \int_{[0,t)} C^o(s)^{-1} f_C(s)\, d\nu(s). \tag{3.3}$$

Then $C(t;\nu) = C^o(t)[C_0 + \overline{\nu}(t)]$. C^o represents the decay of a unit of initial capital in the absence of investment.

3.0.1 The Production Function

The *production function* $R(C, K, L)$ translates capacity, money and labour into goods. For convenience it is defined on all of $\Re^3$ but may take the value $-\infty$. It is finite on $\text{dom}(R) := \{(C, K, L)^\top : R(C, K, L) > -\infty\}$. We define $\text{dom}(R(C)) := \{(K, L)^\top : R(C, K, L) > -\infty\}$, i.e. the C-section of $\text{dom}(R)$. Set $\nabla_{K,L} R(C, K, L) := (R_K(C, K, L), R_L(C, K, L))^\top$. Here R_K denotes the partial derivative with respect to K, etc. Let $\Re^n_{++}$ denote the positive orthant in $\Re^n$ and $\Re^n_+$ its closure, i.e. the non-negative orthant. Let $\text{bdy}(A)$ denote the boundary of the set A. Recall that κ_M is an upper bound on the money supply $M(t,\omega)$. Let J^p denote the number of agents in the production sector, i.e. the labour supply. We make the following

Assumption R

$$(3.4)\quad \begin{cases} \text{(i)} & R:\Re^3 \mapsto [-\infty,\infty) \text{ is upper semicontinuous, concave, non}-\text{decreasing;} \\ \text{(ii)} & \Re^3_+ \subset \mathrm{dom}(R),\ R(\Re^3_+) \subset [0,\infty),\ R \text{ is continuous on } \Re^3_+; \\ \text{(iii)} & R \text{ is twice continuously differentiable on } \mathrm{int}(\mathrm{dom}(R)); \\ \text{(iv)} & R \text{ is strictly concave, strictly increasing on} \\ & \mathrm{int}(\mathrm{dom}(R)) \cap (\Re_+ \times [0,\kappa_M] \times [0,J^p]); \\ \text{(v)} & \lim_{C\to\infty} \inf_{(K,L)\in[0,\kappa_M]\times[0,J^p]} R_C(C,K,L) = 0. \end{cases}$$

3.0.2 Dividends, Scrap Value and Market Capitalization

Labour at time t costs $\tilde{w}'(t)$ units of currency per unit of time or $w'(t) = q(t)\tilde{w}'(t)$ in real terms, and is provided by some of the agents present in the economy. The wage process $\{w'(t) : t \in [0,T]\}$ is *endogenous* and will be determined from equilibrium considerations, but we require that it be continuous on $(0,T)$ and

$$(3.5)\qquad 0 < w'(t) \le \kappa_w.$$

Costs to the firm consist of wages and interest payments. Corporate profits are distributed as dividends to the shareholders at a rate per unit of time per share (in real terms)

$$(3.6)\qquad \delta(t) := \{R(C(t;\nu),K(t),L(t)) - \tilde{r}(t)K(t) - w'(t)L(t)\}/N(t), \qquad t\in[0,T].$$

In fact the true dividend rate is the positive part of the right hand side of (3.6), but we shall see shortly that δ as defined is non-negative.

There is a *scrap value* $G(C(T;\nu))$ associated with the firm at time T. We make the following

Assumption G

$$(3.7)\quad \begin{cases} \text{(i)} & G:\Re_+ \mapsto \Re_+ \text{ is concave, non}-\text{decreasing, continuously differentiable;} \\ \text{(ii)} & \lim_{C\to\infty} G'(C) = 0, \qquad G'(0) f_C(T) \le 1 \text{ a.s.;} \\ \text{(iii)} & G(C(T;\nu)) = N(T)S(T). \end{cases}$$

An alternative to the limit condition in (ii) above is: $G(C) \le a_o + b_o C,\ b_o\kappa_f < 1,\ a_o, b_o \ge 0$.

Observe that $N(t)S(t)$ is the market capitalization of the firm at time t; we write it as $N\!\!S(t)$. If the market is rational, we certainly expect (3.7)(iii) to hold.

We can then represent market capitalization in terms of the scrap value, the future dividend stream and future capital investment as follows. Using (2.20) and (3.1) we obtain

$$d(N\!\!S(t)) = d\nu(t) - N(t)\delta(t)\,dt + N\!\!S(t)[r(t)\,dt + d\beta(t) + \sigma_S^\top\, dW^\circ(t)],$$

so as in the proof of Lemma 2.2, cf. (2.23), we have

Lemma 3.1 *Assume*

$$(3.8)\qquad \operatorname{ess\,sup}_{t,\omega}\{N(t)\delta(t) + N\!\!S(t)\} < \infty.$$

Then

$$(3.9)\qquad N\!\!S(t) = \frac{1}{\zeta(t)} E\Big\{\zeta(T)G(C(T;\nu)) + \int_t^T \zeta(s)N(s)\delta(s)ds - \int_{[t,T)} \zeta(s)\,d\nu(s)\Big|\mathcal{F}_t\Big\}.$$

3.1 Capacity Expansion with Irreversible Investment

The manager of the firm chooses labour L and real capital K to maximize production profits for the current capacity, wages and interest rate, i.e. he chooses them to obtain the maximal profit rate at capacity C. Take $M \le \kappa_M$ to be the current money supply and set $Q(M) := [0, M] \times [0, J^p]$. Define

$$\tilde{R}(C, \tilde{r}, w') := \max_{(K,L) \in Q(M)} [R(C, K, L) - \tilde{r}K - w'L]. \tag{3.10}$$

Observe that for fixed $C \ge 0$, $-\tilde{R}(C, \cdot, \cdot)$ is the (concave) conjugate of $R^{Q(M)}(C, \cdot, \cdot) = R(C, \cdot, \cdot) - \chi_{Q(M)}(\cdot, \cdot)$ where

$$\chi_{Q(M)}(x) := \begin{cases} 0 & \text{if } x \in Q(M), \\ \infty & \text{if } x \notin Q(M); \end{cases} \tag{3.11}$$

it is the indicator function of $Q(M)$. As in [8] Section 2 (with L replaced by (K, L)) a unique solution exists, denoted by $(K^C(\tilde{r}, w'), L^C(\tilde{r}, w'))^\top := I^{R^{Q(M)}(C,\cdot,\cdot)}(\tilde{r}, w')$ where $I^{R^{Q(M)}(C,\cdot,\cdot)}$ is an extension of the inverse of $\nabla_{K,L} R^{Q(M)}(C, \cdot, \cdot)$, cf. [7] Proposition 3.2.

We can now show that $N\delta \ge 0$ (hence $\delta \ge 0$) as long as we set $(K, L) = (K^C, L^C)$, which is the case for the optimal dividend. Note that we take $C \ge 0$, so $R(C, 0, 0) \ge 0$ according to Assumption R. Then, cf. (3.6),

$$N\delta = R(C, K^C, L^C) - \tilde{r}K^C - w'L^C = \tilde{R}(C, \tilde{r}, w') \ge R(C, 0, 0) \ge 0.$$

The manager chooses real investment $\nu(t, \omega)$ so as to maximize the expected total discounted real profit plus scrap value net of investment, i.e. he maximizes

$$\mathcal{J}_{C_0}(\nu) := E\Big\{ \int_0^T e^{-\mu_F(t)} \tilde{R}(C(t; \nu), \tilde{r}(t), w'(t))\, dt + e^{-\mu_F(T)} G(C(T; \nu)) - \int_{[0,T)} e^{-\mu_F(t)}\, d\nu(t) \Big\} \tag{3.12}$$

over the convex set $\mathcal{S} := \{\nu : \text{left continuous, non-decreasing, adapted process, } \nu(0) = 0\}$. Note that $\mu_F(t)$ is a discount factor; we assume that it is a bounded, continuous, adapted process. Note also that $(t, \omega) \mapsto \tilde{R}(C(t; \nu), \tilde{r}(t), w'(t))$ is measurable due to the continuity of $\tilde{R}(C, \tilde{r}, w')$.

We have the following estimates.

Proposition 3.2 *There exists a constant $K_{\mathcal{J}}$ depending on the bounds on μ_C, μ_F, f_C only such that*

(a) $\mathcal{J}_{C_0}(\nu) \le K_{\mathcal{J}}(1 + C_0)$ on $\mathcal{S}$,

(b) $E \int_{[0,T)} e^{-\mu_F(t)}\, d\nu(t) \le 2K_{\mathcal{J}}(1 + C_0)$ if $\mathcal{J}_{C_0}(\nu) \ge 0$,

(c) $\mathcal{J}_{C_0}$ is strictly concave on $\mathcal{S}$.

Proof: Parts (a) and (b) follow from [8] Proposition 2.1. Moreover [8] Proposition 5.1 implies that $\tilde{R}$ is strictly concave in C. Since G is concave and C is affine in ν, then part (c) also follows. □

Remark 3.3 We may look at the manager's situation slightly differently. He may think of the firm's net value as a claim to be sold. The no-arbitrage value of the claim is $\mathcal{J}_{C_0}$ above, with the discount factor $e^{-\mu_F(t)} := \zeta(t)$, i.e. the deflator, (2.18). In this version (3.9) and (3.6) imply

$$\mathcal{J}_{C_0}(\nu) = N(0)S(0) - E\left\{\zeta(T)[N(T)S(T) - G(C(T;\nu))]\right\} = N\!\!\!S(0). \tag{3.13}$$

Note that $N(0)$ is known, but $S(0)$ depends on future expected payments, so in fact *the manager would seek to maximize the present share value.* Although this interpretation is pleasing, in the end we will not be able to use it since we will assume that μ_F is exogenous unlike ζ. □

In any case the manager's optimal capacity expansion problem is

$$\max_{\nu\in\mathcal{S}} \mathcal{J}_{C_0}(\nu), \tag{$\mathcal{C}$}$$

and then N is found from $N(t) = N(0) + \int_{[0,t)} S(s)^{-1}\, d\nu(s)$.

3.2 Solution of the Optimal Capacity Problem

It follows from the strict concavity of $\mathcal{J}_{C_0}(\nu)$ that the solution of the manager's optimal capacity expansion problem ($\mathcal{C}$) is unique. We denote it by $\hat{\nu}$ and we write

$$(\hat{K}(t), \hat{L}(t)) := (K^{C(t;\hat{\nu})}(\tilde{r}(t), w'(t)), L^{C(t;\hat{\nu})}(\tilde{r}(t), w'(t))) = I^{R^{Q(M)}(C(t;\hat{\nu}),\cdot,\cdot)}(\tilde{r}(t), w'(t)). \tag{3.14}$$

The problem ($\mathcal{C}$) is a slight modification of the "social planning problem" discussed by Baldursson and Karatzas, [2]. Its solution is found as in [8] Section 3.

Define

$$\begin{aligned}\mathbf{R}^{C_0,T}(t) \quad := \quad & \int_0^t e^{-\mu_F(s)} C^o(s) \tilde{R}_C(C_0 C^o(s), \tilde{r}(s), w'(s))\, ds \\ & + e^{-\mu_F(t)} \frac{C^o(t)}{f_C(t)} \mathbb{1}_{\{t<T\}} + e^{-\mu_F(T)} C^o(T) G'(C_0 C^o(T)) \mathbb{1}_{\{t=T\}}.\end{aligned}$$

and

$$Z^{C_0,T}(t) := \operatorname{ess\,inf}_{\tau\in\Upsilon[t,T]} E\{\mathbf{R}^{C_0,T}(\tau)|\mathcal{F}_t\}$$

where $\Upsilon[t,T]$ denotes the stopping times with values in $[t,T]$. Let $\mathcal{Z}^{C_0,T}(\cdot)$ be a modification of $Z^{C_0,T}(\cdot)$ with rcll (right continuous with left limits) paths. Let

$$\tau^*(0, C_0) := \inf\{s \in [0,T) : \mathcal{Z}^{C_0,T}(s) = \mathbf{R}^{C_0,T}(s)\} \wedge T,$$

$$\overline{\nu}^{C_0}(t) := [\sup\{z \geq C_0 : \tau^*(0, z+) < t\} - C_0]^+ \quad \text{if } t > 0, \quad \overline{\nu}^{C_0}(0) = 0.$$

Thus $\overline{\nu}^{C_0}$ (modulo a shift) is $\tau^*(0,\cdot)$'s left-continuous inverse. Notice that $\tau^*(0,C_0)$ is non-decreasing in C_0 a.s., cf. [2], Lemma 1. Also, if $\hat{y}(0) := \sup\{z \geq 0 : \tau^*(0,z) = 0\}$ then $\overline{\nu}^{C_0}(0+) = \max\{\hat{y}(0) - C_0, 0\} := [\hat{y}(0) - C_0]^+$, i.e. expanding capacity up to level $\hat{y}(0)$ is the optimal strategy at time 0+.

Theorem 3.4 *For C_0 fixed, set*

$$\hat{\nu}(t) := \int_{[0,t)} \frac{C^o(s)}{f_C(s)}\, d\overline{\nu}^{C_0}(s). \tag{3.15}$$

Then (i) $\hat{\nu}$ is the unique solution of $\max_{\nu\in\mathcal{S}} \mathcal{J}_{C_0}(\nu)$.

(ii) $E\|\hat{\nu}\|_T \leq 2K_{\mathcal{J}}(1+C_0)\max_{t,\omega} e^{\mu_F(t)}$.

(iii) If $C(T;\hat{\nu}) = 0$ a.s. then $C_0 = 0$, $\hat{\nu} \equiv 0$ and a.e. a.s.

$$e^{\mu_F(t)} f_C(t) E\left\{ \int_t^T e^{-\mu_F(s)} \tilde{R}_C(0,\tilde{r}(s),w'(s))\frac{C^o(s)}{C^o(t)}\, ds + e^{-\mu_F(T)} G'(0)\frac{C^o(T)}{C^o(t)} \middle| \mathcal{F}_t \right\} \leq 1 \text{ a.e. a.s.} \tag{3.16}$$

The proof is the same as for [8] Theorem 3.1.

Remark 3.5 If R is of the Cobb-Douglas type with zero shift, i.e. $R(C,K,L) = \frac{1}{\alpha\beta\gamma} C^\alpha K^\beta L^\gamma$ with $\alpha,\beta,\gamma > 0$ and $\alpha+\beta+\gamma < 1$, then

$$\tilde{R}_C(C,\tilde{r}(t),w'(t)) = (1-\beta-\gamma)\left[\left(\frac{\beta}{\tilde{r}(t)}\right)^\beta \left(\frac{\gamma}{w'(t)}\right)^\gamma\right]^{\frac{1}{1-\beta-\gamma}} C^a,$$

where $a = \alpha/(1-\beta-\gamma) \in (0,1)$. Hence $\tilde{R}_C(0,\tilde{r}(t),w'(t)) = \infty$ for any t, cf. [8] Remark 3.2, so (3.16) fails, and in fact $C(t;\hat{\nu}) > 0$ for all $t > 0$.

On the other hand for general R if $\mu_F(t)$ is linear, i.e. $\mu_F(t) := \mu_F t$ and μ_C is constant, and if $\tilde{R}_C(0,\tilde{r}(s),w'(s))$ is constant (written as $\tilde{R}_C(0)$), then (3.16) reduces to

$$\tilde{R}_C(0)\left(\frac{1-e^{-(\mu_F+\mu_C)(T-t)}}{\mu_F+\mu_C}\right) + G'(0)e^{-(\mu_F+\mu_C)(T-t)} \leq \frac{1}{f_C(t)} \text{ a.e. a.s.} \tag{3.17}$$

If $C_0 = 0$ then (3.17) implies that the Gateau derivative of $\mathcal{J}_0(0)$ in any direction is non-positive so 0 is a local maximum and due to strict concavity it is then the global maximum, i.e. $\hat{\nu} \equiv 0$. This means that *the production facility will not be built if, for all times t, the marginal returns at zero capacity (the left side of (3.17)) are less than the marginal cost of a unit of new capacity.* □

As β is supposed to be continuous (although singular), we will **assume**

$$\hat{\nu}(t) \text{ continuous on } 0 < t \leq T \tag{3.18}$$

with possibly an initial jump. This is a reasonable expectation, but we can establish it only under some "Markovian" restrictions, cf. [8] Proposition 3.3.

Lemma 3.6 *Assume that* $R(C,K,L) = R^1(C) + R^2(K,L)$, $\mu_F(t) = t\,\mu_F$ *and* $\mu_F, \mu_C, \sigma_C, f_C$ *are constant. Further assume that* G *is thrice differentiable, that* $|G''(C)| \le \kappa_G(1+|C|^{k_G})$ *for some (possibly negative) constant* k_G *and that*

$$R^1_C(C) - (\mu_C + \mu_F)G'(C) + (\|\sigma_C\|^2 - \mu_C)CG''(C) + \|\sigma_C\|^2 C^2 G'''(C)/2 \ge 0. \tag{3.19}$$

Then $\hat{\nu}(t)$ *is continuous on* $0 < t \le T$.

4 The Agents

We shall assume that the economy divides into three sectors, a production sector of J^p agents who supply labour to the firm, a construction sector of J^c agents who provide the capacity expansion, and a welfare sector (includes retirees). Agents in the first sector, i.e. $j \in \mathcal{J}^p$, experience the wage rate w' so that the cumulative wage process (when not reduced by leisure) is $w^j(t) = \int_0^t w'(s)\,ds$. Agents in the second sector, i.e. $j \in \mathcal{J}^c$, experience a wage process $w^j(t) := w^c(t)$, some non-decreasing process with $w^c(0) = 0$ which is continuous (but not necessarily absolutely continuous - employment can be more erratic) for $t > 0$. For agents in the third sector, i.e. $j \in \mathcal{J}^w$, we have $w^j(t) \equiv 0$ as they do not work. We will show that in equilibrium

$$E\, w^c(T) < \infty \tag{4.1}$$

Each agent selects a personal consumption, labour (or as we prefer, leisure) and money holding strategy by optimizing her utility. Of course the "retired" agents do not choose a leisure strategy.

Definition 4.1 The *consumption process* of the jth agent is a quadruple $(\bar{\mathcal{C}}^j(\cdot), \bar{\ell}^j(\cdot)) := (c^j(\cdot), m^j(\cdot), \bar{l}^j(\cdot), \bar{\ell}^j(\cdot))^\top : [0,T] \times \Omega \mapsto \bar{A}^j \times \bar{a}^j$ which is a $\mathcal{F}_t$-progressively measurable $\Re^4_+$ valued process such that $\sup_{t\in[0,T]} \|\bar{\mathcal{C}}^j(t)\| < +\infty$ a.s. In fact $\bar{A}^j = [0,\infty) \times [0,\infty) \times [0,1]$ for $j \in \mathcal{J}^p$, $\bar{A}^j = [0,\infty) \times [0,\infty) \times \{0\}$ otherwise and $\bar{a}^j = [0,1]$ for $j \in \mathcal{J}^c$, $\bar{a}^j = \{0\}$ otherwise.

Here $c^j(t)$, measured in units of the commodity per unit time, represents her rate of consumption of the good at time t, $m^j(t)$, measured in units of commodity, represents the money she holds for transaction purposes at time t, and $\bar{l}^j(t)$ (for $j \in \mathcal{J}^p$) or $\bar{\ell}^j(t)$ (for $j \in \mathcal{J}^c$) represents the amount of leisure she chooses at time t and by implication, the amount of time she spends working. It is convenient to write $\mathcal{C}^j(t) = (c^j(t), m^j(t), l^j(t))^\top$ where $l^j(t) := \bar{l}^j(t) + \bar{\ell}^j(t)$ as l^j then gives the leisure choice of the agent. The form of the sets $\bar{A}^j$ and $\bar{a}^j$ always allows us to disaggregate since $l^j = \bar{l}^j$ if $j \in \mathcal{J}^p$, $l^j = \bar{\ell}^j$ if $j \in \mathcal{J}^c$ and $l^j = 0$ if $j \in \mathcal{J}^w$. We distinguish leisure in the two sectors because for technical reasons utility of leisure will be accumulated over time using the Lebesgue-Stiltjes measure induced by w^c rather than Lebesgue measure. Note that $1 - l^j(t)$ denotes the intensity with which the agent works - the more intensely she works, the greater the stress and the more she will earn. We will also use $\mathcal{C}$ without argument to denote the triple $(c,m,l)^\top \in \Re^3$. It will be clear from the context whether we mean the process or the point in $\Re^3$.

The agent also holds a portfolio of financial instruments; ϕ_S^j represents the number of equity shares in the portfolio, and $\phi_{\mathcal{B}}^j, \phi_B^j, \phi_D^j$ represent the real bond, nominal bond and derivative holdings respectively.

Definition 4.2 The *portfolio process* of the agent consists of a progressively measurable quadruple $\Phi^j := (\phi_{\mathcal{B}}^j, \phi_B^j, \phi_D^j, \phi_S^j)^\top$ such that

$$(4.2)\quad \begin{array}{ll} (i) & \int_0^T |\mathcal{B}(t)\phi_{\mathcal{B}}^j(t) + B(t)\phi_B^j(t) + D(t)\phi_D^j(t) + S(t)\phi_S^j(t)|\,[|r(t)|\,dt + d\|\beta\|_t] < \infty \text{ a.s.} \\ (ii) & \int_0^T \left\|\Big(\phi_B^j(t)B(t),\, \phi_D^j(t)D(t),\, \phi_S^j(t)S(t)\Big)\Sigma(t)\right\|^2 dt < \infty \text{ a.s.} \end{array}$$

The components of the portfolio processes are measured in numbers of shares and may be fractions, either positive or negative, i.e. short selling and borrowing are allowed. The agent's strategy consists of a consumption process and a portfolio process.

The real wealth of the agent at time t is given by

$$(4.3)\qquad X^j(t) := \phi_{\mathcal{B}}^j(t)\mathcal{B}(t) + \phi_B^j(t)B(t) + \phi_D^j(t)D(t) + \phi_S^j(t)S(t) + m^j(t),$$

so the wealth lies in the holding of the financial assets, including the shares of the productive asset, and of cash. To examine how it changes over time it is useful to consider money as part of the portfolio, so (Φ^j, m^j) represents the *moneyed portfolio*. We assume that the initial moneyed portfolio, i.e. $(\Phi^j(0), m^j(0))$, is given (exogenously) so the initial wealth of the agent, $X^j(0)$ generically written as x^j, is fixed if the prices are determined at time 0. Again we point out that later equilibrium may impose different values for (Φ^j, m^j) at $t = 0$, so an instantaneous rebalancing of the moneyed portfolio may be required at time 0. The *gain* from the moneyed portfolio is the appreciation of the portfolio due to price movement and dividends, i.e.

$$\begin{aligned} \mathcal{G}^{\Phi^j,m^j}(t) \;:=\; & \int_0^t \phi_{\mathcal{B}}^j(s)\,d\mathcal{B}(s) + \int_0^t \phi_B^j(s)\,dB(s) + \int_0^t \phi_D^j(s)\,dD(s) + \int_0^t \phi_S^j(s)\,[dS(s) + \delta(t)\,dt] + \\ & + \int_0^t \tilde{m}^j(s)\,dq(s) \\ =\; & \int_0^t X^j(s)[r(s)\,ds + d\beta(s)] - \int_0^t m^j(s)\tilde{r}(s)\,ds \\ & + \int_0^t \Big(\phi_B^j(s)B(s) + m^j(s), \phi_D^j(s)D(s), \phi_S^j(s)S(s)\Big)\Sigma(s)\,dW^\circ(s). \end{aligned}$$

The conditions in (4.2) ensure that all the integrals are well defined. Note that $\mathcal{G}^{\Phi^j,m^j}(0+) = \mathcal{G}^{\Phi^j,m^j}(0) = 0$ due to the continuity of β, so an initial rebalancing of the portfolio cannot change the initial wealth.

Aside from the portfolio gains, the agent has at her disposal initial wealth and an income stream. Her *potential cumulative wage process* is given by the measurable, $\mathcal{F}_t$-adapted, nonnegative, bounded non-decreasing, left continuous process $\tilde{w}^j(t)$, $t \in [0, T]$, and it is measured in units of local currency. It will be reduced by the amount of leisure she chooses. In addition she receives "welfare" from the government, the cumulative amount to time t being $\tilde{g}^j(t)$, a continuous semi-martingale. These payments are funded through expansion of the money supply or, in case of a decrease in $\tilde{g}^j$, by taxation. Note that an increment $\Delta\tilde{g}^j(s)$ is worth, in real terms, $q(s+)\Delta\tilde{g}^j(s)$, and in the limit this yields $q(s)\,d\tilde{g}^j(s) + d[q,\tilde{g}^j](s)$. We define

$$\begin{aligned} g^j(t) &:= \textstyle\int_0^t q(s)\,d\tilde{g}^j(s) + [q,\tilde{g}^j](t), \\ w^j(t) &:= \textstyle\int_{[0,t)} q(s)\,d\tilde{w}^j(s) + [q,\tilde{w}^j](t). \end{aligned} \tag{4.4}$$

Let $\Gamma^{x^j,c^j,l^j}(t)$ denote the cumulative real income net of consumption and leisure to time t with initial wealth $X^j(0) = x^j \geq 0$, consumption rate c^j and leisure rate l^j. Then

$$\Gamma^{x^j,c^j,l^j}(t) := x^j + g^j(t) + w^j(t) - \int_0^t l^j(s)\,dw^j(s) - \int_0^t c^j(s)\,ds.$$

We insist that the portfolio be Γ-financed, i.e. the following budget equation holds:

$$X^j(t) = \mathcal{G}^{\Phi^j,m^j}(t) + \Gamma^{x^j,c^j,l^j}(t).$$

If we define the *real cumulative endowment* of agent j as

$$e^j(t) := x^j + g^j(t) + w^j(t), \tag{4.5}$$

then it represents initial wealth plus welfare income plus wage income to time t assuming no leisure, i.e. $l = 0$. We shall take this point of view - that the agents works at the maximum rate 1 but then consumes leisure! We assume (again shown to hold in equilibrium) that ξ^j, *the expected deflated value of total endowment* is positive and finite, i.e.

$$0 < \xi^j := x^j + E\int_0^T \zeta(t)\,de^j(t) < \infty. \tag{4.6}$$

The agent would not participate in the economy if $\xi^j \leq 0$.

Let us set $\wp^j(t) = (t, \int_0^t \tilde{r}(s)\,ds, w^j(t))^\top$; it is the same for all agents in a given sector. Define $X^{x^j,\mathcal{C}^j,\Phi^j}$ as the solution of

$$\begin{aligned} X(t) &= \int_0^t X(s)[r(s)\,ds + d\beta(s)] + e^j(t) - \int_0^t \mathcal{C}^j(s)^\top d\wp^j(s) \\ &\quad + \int_0^t \Big(\phi_B^j(s)B(s) + m^j(s),\, \phi_D^j(s)D(s)),\, \phi_S(s)^j S(s)\Big)\Sigma(s)\,dW^\circ(s). \end{aligned} \tag{4.7}$$

Note that the definition of $X^{x^j,\mathcal{C}^j,\Phi^j}$ does not require $\phi_{\mathcal{B}}^j$! Then it follows that for X^j defined in (4.3), $X^j = X^{x^j,\mathcal{C}^j,\Phi^j}$. We note that

$$\begin{aligned} \epsilon(t)X^{x^j,\mathcal{C}^j,\Phi^j}(t) &= x^j + \int_0^t \epsilon(s)[de^j(s) - \mathcal{C}^j(s)^\top d\wp^j(s)] \\ &\quad + \int_0^t \epsilon(s)\Big(\phi_B^j(s)B(s) + m^j(s),\, \phi_D^j(s)D(s),\, \phi_S^j(s)S(s)\Big)\Sigma(s)\,dW^\circ(s). \end{aligned} \tag{4.8}$$

At this point it is useful to introduce further restrictions which model economic reality.

Definition 4.3 Given $x \geq 0$, we say $(\mathcal{C}^j, \Phi^j)$ is *admissible* at x, i.e. $(\mathcal{C}^j, \Phi^j) \in \mathcal{A}^j(x)$, if $\mathcal{C}^j$ is a consumption process, Φ^j is a portfolio process and

$$X^{x,\mathcal{C}^j,\Phi^j}(t) + \frac{1}{\zeta(t)} E\left\{ \int_t^T \zeta(s) de^j(s) \Big| \mathcal{F}_t \right\} \geq 0 \ a.s. \ \forall t.$$

So admissible at x means that present wealth plus expected future deflated endowment is non-negative, i.e. the agent can only borrow against current holdings and expected future income.

4.1 The Agent's Problem

4.1.1 Utility Functions

Clearly an agent derives utility from consumption and, in the production and construction sectors, from leisure. Also, the agent holds money to facilitate her financial transactions, hence it is reasonable to assign utility to holding money. However, as prices inflate the utility of a unit of currency depreciates; it is for this reason that we use real money holdings in the utility function.

The jth agent has three personal *utility functions*, U^j, u^j and V^j. The measurable function $U^j(t, \mathcal{C}) : [0,T] \times \Re^3 \mapsto [-\infty, \infty)$ represents the agent's utility of consumption at rate $c \geq 0$, of holding cash $m \geq 0$ in real terms, and of leisure at rate l. As mentioned above for technical reasons agents in the construction sector accumulate utility of leisure over time not relative to Lebesgue measure but rather relative to the measure induced by w^c. Such utility is given by $u^j(t, \bar{\ell})$.

The function $V^j(x) : [0, +\infty) \mapsto [-\infty, \infty)$ represents the agent's utility of real wealth at the terminal time. It is strictly concave, strictly increasing, differentiable on $\mathrm{int}(\mathrm{dom}(V^j)) \supset (0, \infty)$, and such that

$$\lim_{x\to\infty} V^j_x(x) = 0. \tag{4.9}$$

It is convenient to have $U^j(t, c, m, l)$ defined for all c, m, l, but recall that $\mathrm{dom}(U^j(t,\cdot)) := \{\mathcal{C} : U^j(t, \mathcal{C}) > -\infty\}$ and $\mathrm{dom}(U^j) := \{(t, \mathcal{C}) : U^j(t, \mathcal{C}) > -\infty\}$. Also $\mathrm{dom}(\partial U^j(t,\cdot)) := \{\mathcal{C} : \partial U^j(t, \mathcal{C}) \neq \emptyset\}$ where $\partial U^j(t, \mathcal{C})$ is the supergradient of $U^j(t, \mathcal{C})$ with respect to $\mathcal{C}$ at $(t, \mathcal{C})$. Let ∇ denote gradient with respect to $\mathcal{C}$ and write U^j_s, $s \in \{t, c, m, l\}$, for the partial derivative with respect to s. We set

$$A^j := \bar{A}^j \cap \mathrm{dom}(U^j(t,\cdot)).$$

Note that A^j then depends on t, but we shall in fact assume that it is constant, cf. (4.10)(iv). The boundary of A^j decomposes into a finite number of relatively open sets, $\mathcal{C}^j_k$, i.e. corners, edges, faces, etc. We shall assume, cf. (4.10)(v), that if $\nabla U^j(t,\cdot)$ exists somewhere in $\mathcal{C}^j_k$ then it exists everywhere in $\mathcal{C}^j_k$. Recall that χ_A is the indicator function of A, as in (3.11). Define $U^j_{A^j}(t, \mathcal{C}) := U^j(t, \mathcal{C}) - \chi_{A^j}(\mathcal{C})$. This is the utility function restricted to the set A^j of interest.

Assumption U:

$$(4.10)\begin{cases} \text{(i)} & U^j(t,\cdot) \text{ is upper semicontinuous, concave, nondecreasing;} \\ \text{(ii)} & U^j \text{ is continuous on } \mathrm{dom}(U^j) \text{ and } [0,T]\times\Re^3_{++}\subset \mathrm{dom}(U^j); \\ \text{(iii)} & U^j\in C^{0,1}(\mathrm{int}(\mathrm{dom}(U^j)) \text{ and } \nabla U^j\in C^{1,2}(\mathrm{int}(\mathrm{dom}(U^j)) \text{ componentwise,} \\ & \frac{\partial^2 U^j}{\partial a\partial b}\geq 0 \text{ and } a,b\in\{c,m,l\},\ a\neq b; \\ \text{(iv)} & \Re^3_+\cap \mathrm{dom}(U^j(t,\cdot)) \text{ and } \Re^3_+\cap\mathrm{dom}(\partial U^j(t,\cdot)) \text{ are time independent} \\ & \text{and } \Re^3_{++}\subset\mathrm{dom}(\partial U^j(t,\cdot)); \\ \text{(v)} & \mathcal{C}^j_k\cap\mathrm{dom}(\partial U^j(t,\cdot))\neq\emptyset\Rightarrow \mathcal{C}^j_k\subset\mathrm{dom}(\partial U^j(t,\cdot)); \\ \text{(vi)} & \mathrm{dom}(\partial U^j_{A^j}(t,\cdot))\subset\mathrm{int}(\mathrm{dom}(U^j(t,\cdot)))\cap A^j \text{ (hence they are equal, cf. [7]);} \\ \text{(vii)} & U^j(t,\cdot) \text{ is strictly increasing, strictly concave on } \mathrm{int}(\mathrm{dom}(U^j(t,\cdot)))\cap A^j; \\ \text{(viii)} & \inf_{\mathcal{C}\in\mathrm{dom}(\partial U^j_{A^j}(t,\cdot))} U^j_s(t,\mathcal{C})=0 \text{ for } s\in\{c,m\}. \end{cases}$$

We will add parts (ix) and (x) shortly.

Part (viii) above gives the behaviour of the partial derivatives of U^j near ∞ in A^j. We also need the bahaviour as the coordinate planes are approached in A^j. When $j\notin\mathcal{J}^p$ we work on $\Re^2$ so set $U^j_0(t,c,m):=U^j(t,c,m,0)$.

Define

$$(4.11)\ \mathcal{I}^j_U:=\begin{cases}\{s\in\{c,m,l\}:\{(c,m,l)\in\bar{A}^j: s=0\}\cap\mathrm{dom}(\partial U^j(t,\cdot))=\emptyset\} & \text{if } j\in\mathcal{J}^p, \\ \{s\in\{c,m\}:\{(c,m:(c,m,0)\in\bar{A}^j, s=0\}\cap\mathrm{dom}(\partial U^j_0(t,\cdot))=\emptyset\} & \text{if } j\notin\mathcal{J}^p.\end{cases}$$

The set $\mathcal{I}^j_U$ gives the inaccessible parts of the boundary of A^j, i.e. if $s\in\mathcal{I}^j_U$ then $\max_{\mathcal{C}\in A^j} U^j(t,\mathcal{C})$ cannot occur at $s=0$ since $U^j_s=\infty$ there. The continuity of ∇U^j implies that for any convergent sequence $\{\mathcal{C}^k\}\subset A^j$, $\lim_k\|\nabla U^j(t,\mathcal{C}^k)\|=\infty$ only if $\mathcal{I}^j_U\neq\emptyset$.

We now add parts (ix) and (x) to **Assumption U**, i.e. to (4.10).

$$\begin{cases} \text{(ix)} & \text{there exist sets } G^j_s,\ s=c,m,l, \text{ closed in } A^j, \text{ with disjoint relative interiors} \\ & \text{such that } A^j=\bigcup_s G^j_s, \text{ and for } s\in\mathcal{I}^j_U,\ \lim_{\delta\to 0}\inf_{t\in[0,T]}\inf_{\mathcal{C}\in G^j_s(\delta)} U^j_s(t,\mathcal{C})=\infty, \\ & \text{where } G^j_s(\delta):=G^j_s\cap\{s\leq\delta\}. \\ & \text{Conversely, for any bounded sequence } \{\mathcal{C}^k=(c^k,m^k,l^k)\}\subset A^j \\ & \text{such that } U^j_s(t,\mathcal{C}^k)\to\infty \text{ for some } s\in\{c,m,l\}, \text{ we have } s^k\to 0; \\ \text{(x)} & \{m,l\}\subset\cup_j\mathcal{I}^j_U. \end{cases}$$

Suppose $\mathcal{C}^k\to\mathcal{C}$, $\mathcal{C}^k\in A^j$. The first part of (ix) implies that if $\mathcal{C}\in\mathrm{bdy}(A^j)$ but $\mathcal{C}\notin\mathrm{dom}(\partial U^j)$, then $\lim_{\mathcal{C}^k\to\mathcal{C}} U^j_s(\mathcal{C}^k)=\infty$ uniformly in G^j_s; it is void if ∇U^j is finite on $A^j\cap\mathrm{bdy}(\Re^3_{++})$. This part is used to establish the existence of an equilibrium, cf. Theorem 7.2 and [6] Lemma 4.3. The second part implies that if $U^j_s(\mathcal{C}^k)\to\infty$ then $s\in\mathcal{I}^j_U$ and is used to obtain existence of the Lagrange multipliers, cf. Lemma 4.6, [6] Lemma 4.1, and regularity of the representative agent's utility function, cf. Theorem 5.1. Part (x), e.g. $m\in\cup_j\mathcal{I}^j_U$, implies that there is at least one agent who when maximizing his utility will not do without cash (cf. the sentence after (4.11), and similarly for leisure in the production sector.

We point out that in the case $n = 1$ the Inada conditions imply (viii), (ix)and (x), although the latter are weaker.

A typical example for $j \in \mathcal{J}^p$ would be $U^j(t,c,m,l) = h^j(t)(c+\varepsilon_c^j)^{\alpha^j}(m+\varepsilon_m^j)^{\beta^j}(l+\varepsilon_l^j)^{\gamma^j}$ with h continuous, $\alpha^j, \beta^j, \gamma^j > 0$, $\alpha^j + \beta^j + \gamma^j < 1$, $\varepsilon_s^j \geq 0$ $(s = c, m, l)$ with $\varepsilon_m^j = 0$ for at least one j and $\varepsilon_l^j = 0$ for another j.

We now add utility of leisure for the construction sector via the *utility function* $u^j(t,l) : [0,T] \times \Re \mapsto [-\infty, \infty)$. We take $u^j = 0$ for $j \notin \mathcal{J}^c$. Again $a^j := \bar{a}^j \cap \mathrm{dom}(u^j(t,\cdot))$ which is assumed to be time independent.

Assumption u:

$$(4.12) \quad \begin{cases} \text{(i)} & u^j(t,\cdot) \text{ is upper semicontinuous, concave, nondecreasing;} \\ \text{(ii)} & u^j \text{ is continuous on } \mathrm{dom}(u^j); \\ \text{(iii)} & u^j \in C^{0,1}(\mathrm{int}(\mathrm{dom}(u^j)) \text{ and } \nabla u^j \in C^{1,2}(\mathrm{int}(\mathrm{dom}(u^j)) \text{ componentwise;} \\ \text{(iv)} & \Re^1_+ \cap \mathrm{dom}(u^j(t,\cdot)) \text{ is time independent and } \Re^1_{++} \subset \mathrm{dom}(\partial u^j(t,\cdot)); \\ \text{(v)} & \mathrm{dom}(\partial u^j_{a^j}(t,\cdot)) \subset \mathrm{int}(\mathrm{dom}(u^j(t,\cdot))) \cap a^j \text{ (hence they are equal, cf. [7]);} \\ \text{(vi)} & u^j(t,\cdot) \text{ is strictly increasing, strictly concave on } \mathrm{int}(\mathrm{dom}(u^j(t,\cdot)) \cap a^j). \\ \text{(vii)} & 0 \notin \cap_{j \in \mathcal{J}^c} \mathrm{dom}(\partial u^j(t,\cdot)) \\ \text{(viii)} & \text{for each } t \in [0,T],\ \min_{j \in \mathcal{J}^c} u^j_l(t,1) = 0 \\ \text{(ix)} & \text{if } 0 \notin \mathrm{dom}(\partial u^j(t,\cdot)) \text{ for some } t \text{ then } \lim_{l \downarrow 0} \inf_{0 \leq t \leq T} u^j_l(t,l) = +\infty. \end{cases}$$

Note that part (vii) is equivalent to (4.10)(x) and part (ix) to (4.10)(ix).

4.1.2 The Problem

The agent has initial wealth x^j and aims to maximize her expected total utility; that is, to solve

$$(\mathcal{A}^j) \qquad \max_{\mathcal{H}^j} E\Big\{ \int_0^T U^j(t, \bar{\mathcal{C}}^j(t))dt + \int_0^T u^j(t, \bar{\ell}^j(t))\, dw^c(t) + V^j(X^{x^j, \mathcal{C}^j, \Phi^j}(T)) \Big\}$$

where

$$\mathcal{H}^j := \{(\mathcal{C}^j, \Phi^j) \in \mathcal{A}^j(x^j) : E\{\textstyle\int_0^T (U^j)^-(t, \bar{\mathcal{C}}^j(t))dt + \int_0^T (u^j)^-(t, \bar{\ell}^j(t))\, dw^c(t) + (V^j)^-(X^{x^j, \mathcal{C}^j, \Phi^j}(T))\} < +\infty\}.$$

Here $u^- := -\min\{u, 0\}$ is the negative part of u. Recall that $\mathcal{C}^j$ separates uniquely into $(\bar{\mathcal{C}}^j, \bar{\ell}^j)$. For $j \notin \mathcal{J}^c$ the dw^c integral is zero. It may appear counterintuitive that a construction agent obtains increased benefit from leisure when the daily wage, dw^c, is increased; in fact the (real) wage increases when more work is demanded, i.e. more money is invested so immediately more capacity expansion is required. These demands exert greater stress on the agent, so the opportunity benefit of being "on vacation" is increased. The integral constraints of $\mathcal{H}^j$ imply that $\mathcal{C}^j(t) \in \mathrm{dom}(U^j(t,\cdot)$, $\bar{\ell}^j(t) \in \mathrm{dom}(u^j(t,\cdot)$ a.e. a.s. where the a.e. for u^j is with respect to the w^c measure. Hence U^j and u^j can be replaced by $U^j_{A^j}$ and $u^j_{a^j}$ in $(\mathcal{A}^j)$.

We will now present an equivalent problem which is independent of Φ^j. From (4.8) it follows that for all admissible strategies

$$\mathcal{M}_0(t) \quad := \quad x^j + E^\circ\Big\{ \int_0^T \epsilon(s) de^j(s) \Big| \mathcal{F}_t \Big\} +$$

$$+ \int_0^t \epsilon(s)\Big(\phi_B^j(s)B(s) + m^j(s),\, \phi_D^j(s)D(s),\, \phi_S^j(s)S(s)\Big)\,\Sigma(s)\,dW^\circ(s)$$

$$\geq \quad \int_0^t \epsilon(s)\,\mathcal{C}^j(s)^\top\, d\wp^j(s) \geq 0.$$

Thus $\mathcal{M}_0$ is a non-negative P°-local martingale, hence a supermartingale.

Then

$$E\,\zeta(T)X^{x^j,\mathcal{C}^j,\Phi^j}(T) + E\int_0^T \zeta(s)\mathcal{C}^j(s)^\top\, d\wp^j(s) = E^\circ\mathcal{M}_0(T) \leq E^\circ\mathcal{M}_0(0) = \xi^j. \tag{4.13}$$

In other words, expected "present value" of terminal wealth and "consumption" cannot exceed expected "present value" of endowment. We can now show that any consumption process and terminal wealth satisfying the budget constraint (4.13) corresponds to an admissible portfolio process.

Theorem 4.4 *Assume (4.6). Given $x^j \geq 0$ and a consumption process $\mathcal{C}^j$, let $\bar{x} \geq 0$ be $\mathcal{F}_T$-measurable such that*

$$E\Big\{\int_0^T \zeta(t)\mathcal{C}^j(t)^\top\, d\wp^j(t) + \zeta(T)\bar{x}\Big\} \leq \xi^j. \tag{4.14}$$

Then there exists a portfolio process Φ^j such that $(\mathcal{C}^j, \Phi^j) \in \mathcal{A}(x^j)$ and $X^{x^j,\mathcal{C}^j,\Phi^j}(T) \geq \bar{x}$. Moreover if (4.14) is an equality, then

$$X^{x^j,\mathcal{C}^j,\Phi^j}(t) = \zeta(t)^{-1}E\Big\{\int_t^T \zeta(s)\mathcal{C}^j(s)^\top\, d\wp^j(s) - \int_t^T \zeta(s)\,de^j(s) + \zeta(T)\bar{x}\Big|\mathcal{F}_t\Big\}. \tag{4.15}$$

Proof: We use some ideas from [21]. Define

$$\begin{cases} \mathcal{N}^j(t) := \int_0^t \epsilon(s)\mathcal{C}^j(s)^\top\, d\wp^j(s) - \int_0^t \epsilon(s)\,de^j(s), \\ \mathcal{M}^j(t) := E^\circ\{\mathcal{N}^j(T) + \epsilon(T)\bar{x}\,|\,\mathcal{F}_t\} = \mathcal{M}^j(0) + \int_0^t \psi^j(s)^\top dW^\circ(s), \end{cases}$$

with $\int_0^T \|\psi^j\|^2 ds < \infty$, where ψ^j is given by the martingale representation theorem. Note that (4.6) and (4.14) imply that $\mathcal{M}^j$ is a P°-martingale.

Define the multifunction F^j on $[0,T]\times\Omega$ as $F^j(t,\omega) := \{\varphi \in \Re^3 : \Sigma^\top(t,\omega)\varphi = \psi^j(t,\omega)\}$. Then $F^j(t,\omega) \neq \emptyset$ since $\Sigma(t)$ has rank two, and $F^j(t,\omega)$ is a closed set. As $\{(t,\omega) : \psi^j(t,\omega) \in \text{range}(\Sigma^\top(t,\omega)) \cap \mathcal{O}\}$ for any open set $\mathcal{O} \subset \Re^2$ is progressively measurable, then F^j is weakly measurable, cf. [31]. Theorem 4.1 of [31] implies that there exists a $\Re^3$-valued, progressively measurable process $\bar{\varphi}^j$ such that $\psi^j = \Sigma^\top\bar{\varphi}^j$.

If $\bar{X}(t) := \epsilon(t)^{-1}[\mathcal{M}^j(t) - \mathcal{N}^j(t)]$, then $\bar{X}(0) = \mathcal{M}^j(0)$, $\bar{X}(T) = \bar{x}$ and

$$d[\epsilon(t)\bar{X}(t)] = \epsilon(t)\,de^j(t) - \epsilon(t)\mathcal{C}^j(t)^\top\, d\wp^j(t) + \epsilon(t)\varphi^j(t)^\top\Sigma(t)\,dW^\circ(t)$$

where $\varphi^j := \epsilon^{-1}\bar{\varphi}^j$. Define $(\phi_B^j, \phi_D^j, \phi_S^j)$ such that $\varphi^j = (\phi_B^j B + m, \phi_D^j D, \phi_S^j S)^\top$. Define $X^{x^j,\mathcal{C}^j,\Phi^j}$ from (4.7) (recall that $\phi_{\mathcal{B}}^j$ is not needed for this construction), and then define $\phi_{\mathcal{B}}^j := \mathcal{B}^{-1}[X^{x^j,\mathcal{C}^j,\Phi^j} - \phi_B^j B - \phi_D^j D - \phi_S^j S - m^j]$.

Let us prove (4.2)(i),(ii) for $\Phi^j = (\phi^j_{\mathcal{B}}, \phi^j_B, \phi^j_D, \phi^j_S)^\top$. Part (ii) follows from the definition of $(\phi^j_B, \phi^j_D, \phi^j_S)$, the almost sure square integrability of ψ^j and σ_q, and the pathwise boundedness of m^j and ϵ^{-1}. Moreover the left side of (4.2)(i) is almost surely finite since $X^{x^j,\mathcal{C}^j,\Phi^j}$ has continuous, hence bounded, paths, m^j has bounded paths and r is integrable, $\|\beta\|_T < \infty$ a.s. So Φ^j is a portfolio process.

From (4.6) and (4.14) it follows readily that $\bar{X}(0) = \mathcal{M}^j(0) \le x^j = X^{x^j,\mathcal{C}^j,\Phi^j}(0)$ and from (4.8) that $d[\epsilon(X^{x^j,\mathcal{C}^j,\Phi^j} - \bar{X})] = 0$, so $X^{x^j,\mathcal{C}^j,\Phi^j}(t) \ge \bar{X}(t)$ a.s. Hence

$$\begin{aligned}\epsilon(t)X^{x^j,\mathcal{C}^j,\Phi^j}(t) + E^\circ\Big\{\int_t^T \epsilon(s)de^j(s)\Big|\mathcal{F}_t\Big\} &\ge \epsilon(t)\bar{X}(t) + E^\circ\Big\{\int_t^T \epsilon(s)de^j(s)\Big|\mathcal{F}_t\Big\}\\ &= E^\circ\Big\{\int_t^T \epsilon(s)\mathcal{C}^j(s)^\top d\wp^j(s) + \epsilon(T)\bar{x}\Big|\mathcal{F}_t\Big\} \ge 0.\end{aligned}$$

Then $(\mathcal{C}^j, \Phi^j) \in \mathcal{A}^j(x^j)$; moreover (4.15) follows readily since under equality in (4.14) $\bar{X}(0) = x^j = X^{x^j,\mathcal{C}^j,\Phi^j}(0)$, i.e. $X^{x^j,\mathcal{C}^j,\Phi^j} = \bar{X}$. □

Under equality in (4.14) it follows from (4.15) that $\bar{x} = X^{x^j,\mathcal{C}^j,\Phi^j}(T)$. We can think of $\mathcal{N}^j(t)$ as the endowment deficit to time t, i.e. excess of consumption over endowment. Adding an (anticipated) terminal wealth $\bar{x}$ produces the total deficit, $\mathcal{M}^j(T)$. The Theorem simply states that the initial wealth $X^j(0)$ is sufficient to finance the claim $\mathcal{M}^j(T)$, and Φ^j is the appropriate hedge.

Rather than looking at admissible pairs $(\mathcal{C}^j, \Phi^j) \in \mathcal{A}^j(x^j)$ we can now consider pairs $(\mathcal{C}^j, \bar{x})$.

Definition 4.5 *Given $x^j \ge 0$, a consumption process $\mathcal{C}^j$ and a $\mathcal{F}_T$-measurable random variable $\bar{x} \ge 0$, we say $(\mathcal{C}^j, \bar{x})$ is admissible at x^j, i.e. $(\mathcal{C}^j, \bar{x}) \in \bar{\mathcal{A}}^j(x^j)$, if $(\mathcal{C}^j, \bar{x})$ satisfies (4.14).*

Then problem $(\mathcal{A}^j)$ is equivalent to the static problem

$$(\bar{\mathcal{A}}^j) \qquad \max_{\bar{\mathcal{H}}^j} E\Big\{\int_0^T U^j(t, \bar{\mathcal{C}}^j(t))dt + \int_0^T u^j(t, \bar{\ell}^j(t))\, dw^c(t) + V^j(\bar{x})\Big\}$$

where
$\bar{\mathcal{H}}^j := \Big\{(\mathcal{C}^j, \bar{x}) \in \bar{\mathcal{A}}^j(x^j) : E\left\{\int_0^T (U^j)^-(t, \bar{\mathcal{C}}^j(t))dt + \int_0^T (u^j)^-(t, \bar{\ell}^j(t))\, dw^c(t) + (V^j)^-(\bar{x})\right\} < \infty\Big\}$.
Since U^j is strictly increasing in c, then we may take equality in (4.14), so $(\bar{\mathcal{A}}^j)$ is an isoperimetric problem.

4.2 Solution of the Agent Problem

Problem $(\bar{\mathcal{A}}^j)$ is solved using convex analysis. This was done for a utility function depending on two variables in [4]; in fact the main difficulty concerns inverting $\nabla U^j_{A^j}$. In [7], Propositions 3.2, 3.3, 3.14 and Remark 3.7 it is shown that for each $t \in [0, T]$ there exists a continuous, monotone function $I^{U^j_{A^j}}(t, \cdot)$, strictly monotone on $\mathcal{R}^j(t) := \nabla U^j(t, A^j \cap \mathrm{int}(\mathrm{dom}(U^j(t, \cdot))))$, which extends $\nabla U^j_{A^j}(t, \cdot)^{-1}$. Moreover $\mathrm{dom}(I^{U^j_{A^j}}) = [0, T] \times \Re^2_{++} \times \Re$, $I^{U^j_{A^j}} \in C^{1,2}(\mathrm{int}(\cup_{t\in[0,T]}\{t\} \times \mathcal{R}^j(t)))$ and is piecewise in $C^{1,2}$ on its domain. Recall that $I^{U^j_{A^j}}$ as the superdifferential of a concave function is monotone, i.e. is a set valued function such that $(x-x')^\top(y-y') \le 0$ for all $y \in I^{U^j_{A^j}}(x)$ and $y' \in I^{U^j_{A^j}}(x')$ with $x, x' \in A^j$, cf. [7] for more details.

It is easy to define $I^{u^j_{a^j}}(t)$ on $\Re$ as a continuous extension of $[\frac{\partial}{\partial l}u^j_{a^j}(t,\cdot)]^{-1}$, constant off $\frac{\partial}{\partial l}u^j_{a^j}(t,a_j)$. I^{V^j} is defined similarly on $(0,\infty)$.

Now as usual to solve $(\bar{\mathcal{A}}^j)$ we first maximize over $\bar{\mathcal{A}}^j(x^j)$ and then check that the solution is in $\bar{\mathcal{H}}^j$. We write $\pi := (1, \tilde{r}, w')^\top$. For the maximization we will need to solve the following equation for the Lagrange multiplier:

$$\mathcal{X}^j(\eta) := E\Big\{\int_0^T \zeta(t)\Big[\pi(t)^\top I^{U^j_{A^j}}(t,\eta\zeta(t)\pi(t))\,dt + I^{u^j_{a^j}}(t,\eta\zeta(t))\,dw^c(t)\Big] + \zeta(T)I^{V^j}(\eta\zeta(T))\Big\} = \xi^j. \tag{4.16}$$

Note that for $j \notin \mathcal{J}^p$, $(0,0,1)I^{U^j_{A^j}}(t,\eta\zeta(t)\pi(t)) = 0$ and for $j \notin \mathcal{J}^c$, $I^{u^j_{a^j}}(t,\eta\zeta(t)) = 0$. Define

$$\mathring{A}(t,\omega) := \{\eta : I^{U^j_{A^j}}(t,\eta\zeta(t)\pi(t)) = (0,0,1)^\top,\ \eta\zeta(t) > \lim_{l\downarrow 0} u^j_l(t,l),\ \eta\zeta(T) > \lim_{x\downarrow 0} V^j_x(x)\}.$$

Lemma 4.6 *$\mathcal{X}^j(\eta)$ maps $(0,\infty)$ into itself, is continuous and non-increasing. Let $\bar{\eta} := \sup\{\eta > 0 : \mathcal{X}^j(\eta) > 0\}$. Then $\mathcal{X}^j$ is strictly decreasing on $(0,\bar{\eta})\setminus \cap_{\mathrm{ess}(t,\omega)} \mathring{A}(t,\omega)$, and*

$$\lim_{\eta\downarrow 0}\mathcal{X}^j(\eta) = \infty, \qquad \lim_{\eta\uparrow\bar{\eta}}\mathcal{X}^j(\eta) = 0.$$

Here $\cap_{\mathrm{ess}(t,\omega)}$ denotes the essential intersection, i.e. largest intersecting over almost all (t,ω).

Proof: Set

$$y(t) = \zeta(t)\pi(t), \quad y' = (1/2)k_\zeta(1, k_{\tilde{r}}, 0)^\top \le y(t)/2$$

where the inequality is due to (2.4), (2.19) and (3.5). Since $I^{U^j_{A^j}}(t,\cdot)$ is monotone and $y(t) \ge y'$ then $\eta(y(t)-y')^\top(I^{U^j_{A^j}}(t,\eta y(t)) - I^{U^j_{A^j}}(t,\eta y')) \le 0$. Now $y(t) - y' \ge y(t)/2$ so

$$\begin{aligned} y(t)\cdot I^{U^j_{A^j}}(t,\eta y(t)) &\le 2(y(t)-y')^\top I^{U^j_{A^j}}(t,\eta y(t)) \le 2(y(t)-y')^\top I^{U^j_{A^j}}(t,\eta y') \\ &\le 3\kappa_\zeta\|\pi(t)\|\|I^{U^j_{A^j}}(t,\eta y')\|, \end{aligned}$$

an integrable expression if $I^{U^j_{A^j}}(t,\eta y')$ is bounded. Since $I^{U^j_{A^j}}(t,\eta y')$ is continuous, it is bounded.

Also $\zeta(t)I^{u^j_{a^j}}(t,\eta\zeta(t)) \le \kappa_\zeta I^{u^j_{a^j}}(t,\eta k_\zeta)$ is integrable. Similarly for $\zeta(T)I^{V^j}(\eta\zeta(T)$, so $\mathcal{X}^j(\eta) < \infty$ and hence maps $(0,\infty)$ into itself.

Set $h_t(\eta) := y(t)^\top I^{U^j_{A^j}}(t,\eta y(t))$. Since $\mathcal{X}^j(\eta) = E\{\int_0^T h_t(\eta)\,dt + \int_0^T I^{u^j_{a^j}}(t,\eta\zeta(t))\,dw^c(t) + \zeta(T)I^{V^j}(\eta\zeta(T))\}$ then Lemma 4.1 of [6] and the non-increasing (i.e. monotone) nature of I^{V^j} imply that $\mathcal{X}^j$ is non-increasing. Monotone convergence gives right continuity of $\mathcal{X}^j$ and $\lim_{\eta\downarrow 0}\mathcal{X}^j(\eta) = \infty$ (n.b. $I^{V^j}(0+) = \infty$ if $V^j \not\equiv 0$, cf. (4.9)). Left continuity follows from dominated convergence since $h_t(\cdot)$ is non-increasing.

Let us show that $\lim_{\eta\uparrow\bar{\eta}} \mathcal{X}^j(\eta) = 0$. If $\bar{\eta}_1(t,\omega) := \sup\{\eta : h_t(\eta) > 0\}$, then Lemma 4.1 of [6] implies that $h_t(\eta) \downarrow 0$ when $\eta \uparrow \bar{\eta}_1(t,\omega)$. Set $\eta_1 := \operatorname{ess\,sup}_{t,\omega} \bar{\eta}_1(t,\omega) = \sup\{\eta : E\int_0^T h_t(\eta)\,dt > 0\}$. Similarly for $\bar{\eta}_2(t,\omega) := \sup\{\eta : I^{u^j_{a^j}}(t,\eta\zeta(t)) > 0\}$ and $\eta_2 := \operatorname{ess\,sup}_{t,\omega} \bar{\eta}_2(t,\omega) = \sup\{\eta : E\int_0^T I^{u^j_{a^j}}(t,\eta\zeta(t))\,dw^c(t) > 0\}$, where the t-measure in ess sup is not Lebesgue, but given by w^c. Set $\eta_3 := \sup\{\eta : E\,\zeta(T)I^{V^j}(\eta\zeta(T)) > 0\}$. If $\bar{\eta}_3(\omega) := \sup\{\eta : \zeta(T)I^{V^j}(\eta\zeta(T)) > 0\}$ then $\eta_3 = \operatorname{ess\,sup}_\omega \bar{\eta}_3(\omega)$ and $\lim_{\eta\uparrow\eta_3} E\zeta(T)I^{V^j}(\eta\zeta(T)) = E\lim_{\eta\uparrow\eta_3}\zeta(T)I^{V^j}(\eta\zeta(T)) \leq$
$\leq E\lim_{\eta\uparrow\bar{\eta}_3(\omega)}\zeta(T)I^{V^j}(\eta\zeta(T)) = 0$. Then $\bar{\eta} = \max\{\eta_1,\eta_2,\eta_3\}$ and $\lim_{\eta\uparrow\bar{\eta}}\mathcal{X}^j(\eta) = 0$ follows.

If $\mathcal{X}^j(\eta) > 0$, i.e. $\eta < \bar{\eta}$, then at least one of the following holds. $I^{V^j}(\eta\zeta(T)) > 0$ on a set of positive probability, so $\eta\zeta(T) < V^j_x(0)$ there, hence $I^{V^j}(\eta\zeta(T))$ is strictly decreasing near η on a set of positive probability. Alternatively, $h_t(\eta) > 0$ on a (t,ω) set of positive measure, hence $h_t(\cdot)$ is strictly decreasing as long as $\eta \notin \mathring{A}(t,\omega)$, cf. Lemma 4.1 of [6]. Or else $I^{u^j_{a^j}}(t,\eta\zeta(t)) > 0$ on a set of positive $P\times dw^c$ measure, hence $I^{u^j_{a^j}}(t,\eta\zeta(t))$ is strictly decreasing as long as it is less than 1. In any case $\mathcal{X}^j$ is strictly decreasing. The result follows. □

Then (4.16) has a solution η^j (not necessarily unique), i.e.

$$\mathcal{X}^j(\eta^j) = \xi^j. \tag{4.17}$$

Note however that $I^{U^j_{A^j}}(t,\eta^j\zeta(t)\pi(t))$ is unique, cf. [6], Corollary 4.2, as are $I^{u^j_{a^j}}(t,\eta^j\zeta(t))$ and $I^{V^j}(\eta^j\zeta(T))$. We have

Proposition 4.7 *Given $x^j \geq 0$, there exists an optimal strategy $\hat{\mathcal{C}}^j$ for the agent given by*

$$\hat{\mathcal{C}}^j(t) := I^{U^j_{A^j}}(t,\eta^j\zeta(t)\pi(t)) + (0,0,I^{u^j_{a^j}}(t,\eta^j\zeta(t)))^\top, \qquad t\in[0,T], \tag{4.18}$$

i.e. $\hat{\bar{\mathcal{C}}}^j(t) = I^{U^j_{A^j}}(t,\eta^j\zeta(t)\pi(t))$, $\hat{\bar{\ell}}^j(t) = I^{u^j_{a^j}}(t,\eta^j\zeta(t))$. The corresponding portfolio process $\hat{\Phi}^j$ is given by Theorem 4.4 with $\bar{x} = \bar{x}^j := I^{V^j}(\eta^j\zeta(T))$ which is the optimal terminal real wealth $X^{x^j,\hat{\mathcal{C}}^j,\hat{\Phi}^j}(T)$. The strategy is unique.

Proof: We may proceed as in [18], Theorem 9.4, to show that $(\hat{\mathcal{C}}^j,\bar{x}^j)$ solves $(\bar{\mathcal{A}}^j)$ uniquely. Recall that the integral constraint in $\bar{\mathcal{H}}^j$ implies that maximization is only over admissible consumption processes $\mathcal{C}^j$ with $\bar{\mathcal{C}}^j(t)\in A^j$, $\bar{\ell}^j(t)\in a^j$. Since $I^{u^j_{a^j}} \neq 0$ only for $j\in\mathcal{J}^c$ then (4.18) and (4.17) imply equality in (4.14) for $(\hat{\mathcal{C}}^j,\bar{x}^j)$, so from (4.15) we have that $\bar{x}^j = X^{x^j,\hat{\mathcal{C}}^j,\hat{\Phi}^j}(T)$. Any $(\mathcal{C}^j,\bar{x})\in\bar{\mathcal{H}}^j$ satisfies (4.14), cf. Definition 4.5. For such $(\mathcal{C}^j,\bar{x})$ (4.17) gives

$$E\left\{\int_0^T \zeta(t)[I^{U^j_{A^j}}(\eta^j\zeta(t)\pi(t))^\top + (0,0,I^{u^j_{a^j}}(t,\eta^j\zeta(t))) - \mathcal{C}^j(t)^\top]\,d\wp^j(t) + \zeta(T)[I^{V^j}(\eta^j\zeta(T)) - \bar{x}]\right\} \geq 0.$$

As $I^{U^j_{A^j}}(t,y) = \arg\min_{x\in A^j}\{x^\top y - U^j(t,x)\}$, cf. (3.4) of [7], and similarly for $I^{u^j_{a^j}}(t,y)$ and $I^{V^j}(y)$, then using $y = \eta^j\zeta(t)\pi(t))^\top$ for U^j and $y = \eta^j\zeta(t)$ for u^j and $y = \eta^j\zeta(T)$ for V^j it follows from the previous inequality that

$$E\Big\{\int_0^T U^j(t,\bar{\mathcal{C}}^j(t))\,dt \;+\; \int_0^T u^j(t,\bar{\ell}^j(t))\,dw^c + V^j(\bar{x})\Big\}$$

$$\leq \;\; E\Big\{\int_0^T U^j(t,\hat{\bar{\mathcal{C}}}^j(t))\,dt + \int_0^T u^j(t,\hat{\bar{\ell}}^j(t))\,dw^c + V^j(\bar{x}^j)\Big\}. \tag{4.19}$$

Then $(\hat{\mathcal{C}}^j, \bar{x}^j)$ solves $(\bar{\mathcal{A}}^j)$ provided it lies in $\bar{\mathcal{H}}^j$. However for $j \notin \mathcal{J}^w$, $\mathcal{C}^\top = (c,m,l) := \varepsilon\,(1,1,1)$ with $\varepsilon \in (0,1]$ is a consumption process since $l = \varepsilon \leq 1$. For $\bar{x} = \varepsilon$

$$E\int_0^T \zeta(t)\mathcal{C}(t)^\top\, d\wp^j(t) + \zeta(T)\bar{x} = \varepsilon E\Big\{\int_0^T \zeta(t)[(1+\tilde{r}(t))\,dt + \,dw^j(t)] + \zeta(T)\Big\} \leq \xi,$$

if ε is sufficiently small, cf. (4.6). Hence by Theorem 4.4, we have admissibility of the constant strategy $\mathcal{C}$. But $|E\{\int_0^T [U^j(t,\mathcal{C})\,dt + u^j(t,\varepsilon)\,dw^c] + V^j(\varepsilon)\}| < \infty$ by continuity of U^j and u^j, by (2.4), (3.5), (4.1), and because $\mathcal{C} \in \mathrm{dom}(U^j(t,\cdot\,)$ and $\varepsilon \in \mathrm{dom}(u^j(t,\cdot\,) \cap \mathrm{dom}(V^j)$. Now (4.19) implies that $(\hat{\mathcal{C}}^j, \bar{x}^j) \in \bar{\mathcal{H}}^j$. For $j \in \mathcal{J}^w$ we just use $\mathcal{C}^\top = \varepsilon(1,1,0)$. □

5 The Representative Agent

For equilibrium, we will have to solve the market clearing conditions, e.g. $\sum_j \phi_S^j(t) = N(t)$. This is much easier if only one agent (representative) is present, for then $\phi_S(t) = N(t)$. For this reason we aggregate the actions of the agents into the action of a single representative agent. His utility function must opportunely weight the utility functions of the individual agents in the economy; the factor Λ below will accomplish this. It will be arbitrary to begin with, but in the end it will be chosen so as to produce an equilibrium. Of course, we will also have to disaggregate in the end.

For $\Lambda = (\lambda_1,\ldots,\lambda_J) \in \Re^J_{++}$ let us define the functions $U(t,\bar{\mathcal{C}};\Lambda)$, $u(t,\ell;\Lambda)$ and $V(x;\Lambda)$ as follows. $J := J^p + J^c + J^w$.

$$U(t,\bar{\mathcal{C}};\Lambda) := \sup_{\sum_j \bar{\mathcal{C}}_j = \bar{\mathcal{C}}} \sum_{j=1}^J \lambda_j U^j_{A^j}(t,\bar{\mathcal{C}}_j), \quad u(t,\bar{\ell};\Lambda) := \sup_{\sum_j \bar{\ell}_j = \bar{\ell}} \sum_{j=1}^J \lambda_j u^j_{a^j}(t,\bar{\ell}_j), \tag{5.1}$$

$$V(x;\Lambda) := \sup_{z^j \geq 0,\ \sum_j z^j = x} \sum_{j=1}^J \lambda_j V^j(z^j),$$

$$I^U(t,y;\Lambda) := \sum_{j=1}^J I^{U^j_{A^j}}\Big(t,\frac{y}{\lambda_j}\Big),\ y \in \Re^2_{++}\times\Re,\ I^u(t,y;\Lambda) := \sum_{j\in\mathcal{J}^c} I^{u^j_{a^j}}\Big(t,\frac{y}{\lambda_j}\Big),\ y \in \Re \tag{5.2}$$

$$I^V(y;\Lambda) := \sum_{j=1}^J I^{V^j}\Big(\frac{y}{\lambda_j}\Big),\ y \in \Re_{++}.$$

From [7], Theorem 4.3, Corollary 4.4, Remark 3.7 and Proposition 3.14 we obtain

Lemma 5.1 *(i) For each* $\Lambda \in \Re^J_{++}$, $U(t,\cdot\,;\Lambda) : \Re^3 \mapsto [-\infty,\infty)$ *is a closed, proper, concave, increasing function on* $\Re^3$ *with* $\mathrm{dom}(U(t,\cdot;\Lambda)) = A := \sum_{j=1}^J A^j \subset [0,\infty)^2 \times [0, J^p]$. *For each* $\bar{\mathcal{C}} \in A$ *there exist* $\hat{\bar{\mathcal{C}}}_j(t) \in A^j$ *such that*

$$\bar{\mathcal{C}} = \sum_j \hat{\bar{\mathcal{C}}}_j(t), \quad U(t,\bar{\mathcal{C}};\Lambda) = \sum_j \lambda_j U^j_{A^j}(t,\hat{\bar{\mathcal{C}}}_j(t)). \tag{5.3}$$

Moreover $I^U(t,\cdot\,;\Lambda)$ *is the inverse of* $\partial U(t,\cdot\,;\Lambda)$ *and is monotone (in the operator sense of concave analysis).* $U(t,\cdot\,;\Lambda)$ *is strictly concave on* $\mathrm{dom}(\partial U(t,\cdot\,;\Lambda)) = \mathrm{im}(I^U(t,\cdot\,;\Lambda))$.

(ii) $\mathrm{im}(I^U(t,\cdot\,;\Lambda)) = \tilde{A} := \sum_j \mathrm{dom}(\partial U^j_{A^j}(t,\cdot))$ *is convex. For* $\bar{\mathcal{C}} \in \tilde{A}$, *there exists* $y(t) \in (I^U(t,\cdot\,;\Lambda))^{-1}(\bar{\mathcal{C}})$ *such that*

$$\hat{\bar{\mathcal{C}}}_j(t) = I^{U^j_{A^j}}(t,\tfrac{y(t)}{\lambda_j}). \tag{5.4}$$

(iii) $U(t,\cdot;\Lambda)$ *is continuously differentiable on* $\tilde{A}$. *This set is dense in* A. *Moreover* $(\nabla U(t,\cdot;\Lambda))^{-1} = I^U(t,\cdot;\Lambda)$ *on* $\nabla U(t,\tilde{A};\Lambda) := \tilde{\mathcal{R}}(t;\Lambda)$, *so* $I^U(t,y;\Lambda)$ *is a continuous, monotone extension of* $(\nabla U(t,\cdot;\Lambda))^{-1}$, *strictly monotone on* $\mathrm{cl}(\tilde{\mathcal{R}}(t,\Lambda)) \cap (\Re^2_{++} \times \Re)$. $U(t,\cdot\,;\Lambda)$ *is strictly increasing on* $\tilde{A}$. *For* $y \in \tilde{\mathcal{R}}(t,\Lambda)$, *we have*

$$\nabla_{c,m,l} U(t, I^U(t,y;\Lambda);\Lambda) = y. \tag{5.5}$$

(iv) For each $\bar{\mathcal{C}} \in \tilde{A}$,

$$\nabla_{\bar{\mathcal{C}}} U(t,\bar{\mathcal{C}};\Lambda) = \lambda_j \nabla_{\bar{\mathcal{C}}} U^j(t,\hat{\bar{\mathcal{C}}}_j(t)) - \vec{n}^j(\hat{\bar{\mathcal{C}}}_j(t)), \quad j = 1,\cdots,J, \tag{5.6}$$

where $\vec{n}^j(\hat{\bar{\mathcal{C}}}_j(t)) \in \mathcal{N}_{A^j}(\hat{\bar{\mathcal{C}}}_j(t))$, *the cone of outward normals to* A^j *at* $\hat{\bar{\mathcal{C}}}_j(t)$. *For each* $s \in \{c,m,\bar{l}\}$ *there exists* $j(s)$ *such that the component* $[\vec{n}^{j(s)}(\hat{\bar{\mathcal{C}}}_j(t))]_s = 0$.

(v) $\nabla_{\bar{\mathcal{C}}} U(\cdot\,,\cdot\,;\Lambda)$ *is continuous on* $[0,T] \times \tilde{A}$ *and is piecewise in* $C^{1,2}$.

Although the cited references only provide ∇U picewise continuously differentiable, the same proof provides twice piecewise differentiable since the ∇U^j are. Furthermore, it is evident that

$$\tilde{A} = \{(c,m,\bar{l}) \in A : c > 0 \text{ if } c \in \cup_j \mathcal{I}^j_U,\ m > 0,\ \bar{l} > 0\}, \tag{5.7}$$

cf. (4.11) and (4.10)(x).

The corresponding result for $u(\cdot\,;\Lambda)$ is easier since the argument ℓ is one-dimensional. Then $\tilde{a} = (0, J^c]$ since $0 \notin \cap_{j\in\mathcal{J}^c}\mathrm{dom}(\partial u^j(t,\cdot))$.

A comment on our notation: $(\hat{c}^j(t), \hat{m}^j(t), \hat{\bar{l}}^j(t), \hat{\bar{\ell}}^j(t))$ gives the optimal "consumption" choices made by agent j, whereas $(\hat{c}_j(t), \hat{m}_j(t), \hat{\bar{l}}_j(t), \hat{\bar{\ell}}_j(t))$ is the argument that provides the maxima in (5.1) for given $(c,m,\bar{l},\bar{\ell})$. We shall see that they agree in equilibrium.

We can think of the representative agent as an agent whose endowment process is given by the *aggregate endowment process*

$$\begin{aligned} e(t) &= \sum_j e^j(t) = \sum_j x^j + \sum_j g^j(t) + \sum_j w^j(t) \\ &= x + g(t) + w(t) \end{aligned}$$

with the obvious identifications so $w(t) = J^p \int_0^t w'(s)\, ds + J^c\, w^c(t)$. Our representative agent now has the same problem as the production agent with $\xi^o := E\left\{x + \int_0^T \zeta(t)\, de(t)\right\}$ except that the sets A and a do not allow the identification of $\bar{\mathcal{C}}$ and $\bar{\ell}$ from $\mathcal{C}$ so we work with $(\bar{\mathcal{C}}, \bar{\ell})$ and set $\mathcal{C} := \bar{\mathcal{C}} + (0,0,1)^\top \ell$. We replace $\mathcal{C}^\top\, d\wp^j$ by $\bar{\mathcal{C}}^\top \pi\, dt + \bar{\ell}\, dw^c$. The solution is then given by Proposition 4.7.

Proposition 5.2 *There exists an optimal strategy* $(\hat{\bar{\mathcal{C}}}^\Lambda, \hat{\bar{\ell}}^\Lambda)$ *for the agent given by*

$$(\hat{\bar{\mathcal{C}}}^\Lambda, \hat{\bar{\ell}}^\Lambda)(t) = (I^U(t, \eta^\Lambda \zeta(t)\pi(t); \Lambda), I^u(t, \eta^\Lambda \zeta(t); \Lambda)), \qquad t \in [0,T]. \tag{5.8}$$

Here η^Λ *is such that*

$$E\left\{\int_0^T \zeta(t)\Big[\pi(t)^\top I^U(t, \eta^\Lambda\zeta(t)\pi(t);\Lambda)\, dt + I^u(t, \eta^\Lambda\zeta(t);\Lambda)\, dw^c(t)\Big] + \zeta(T) I^V(\eta^\Lambda \zeta(T);\Lambda)\right\} = \xi^o. \tag{5.9}$$

The corresponding $\hat{\Phi}^\Lambda$ *is given by Theorem 4.4 with* $\bar{x} = I^V(\eta^\Lambda\zeta(T))$*, the optimal terminal wealth. The strategy is unique.*

We observe that if η^j is a solutions of (4.17) for all j, and if we set $\Lambda = ((\eta^1)^{-1}, \ldots, (\eta^J)^{-1})^\top$, then $\eta^\Lambda = 1$ is a solution of (5.9). It follows that $\hat{\mathcal{C}}^\Lambda = \sum_j \hat{\mathcal{C}}^j$, $\hat{\bar{l}}^\Lambda = \sum_{j\in\mathcal{J}^p} \hat{\bar{l}}^j$, $\hat{\bar{\ell}}^\Lambda = \sum_{j\in\mathcal{J}^c} \hat{\bar{\ell}}^j$ and the optimal wealth of the representative agent at time T is $X(T) := \bar{x} = \sum_j X^{x^j, \hat{C}^j, \Phi^j}(T)$. Moreover, an examination of how Φ^j is defined in Theorem 4.4, reveals that $\hat{\Phi}^\Lambda = \sum \hat{\Phi}^j$, hence $X(t) = \sum X^{x^j, \hat{C}^j, \Phi^j}(t)$ for all t. Note that η^Λ depends on ξ^o hence on q.

6 Equilibrium

The market parameters are $r, \beta, \tilde{r}, \mu_{\tilde{S}}, \sigma_{\tilde{S}}, \mu_{\tilde{D}}, \sigma_{\tilde{D}}, \mu_q, \sigma_q, \tilde{w}', \tilde{w}^c, \tilde{g}^j, j = 1, \ldots, J$. Recall that $w' = q\tilde{w}'$. The money supply and q_T, determined by the monetary authorities, and the technology parameters, μ_C, σ_C, f_C, C_0, $R(\cdot)$, $G(\cdot)$ and μ_F are exogenous. The agents and the firm's manager take the market parameters as given, but we want equilibrium to specify them. Equilibrium requires that the agents act optimally, that the manager of the firm chooses investment, labour and operating capital to maximize the expected total discounted profits net of investments, that profits are distributed as dividends, that investment capital be passed as wages to the non-production sector, that changes in the money supply be passed into the economy as "welfare", and that the markets (goods, money, labour, bonds, derivatives and equity) clear.

Note that the possible initial jump of the optimal investment policy moves the capacity from C_0 to its optimal value. This explains the definition and use of $\hat{C}$ below. At times when Σ' is not invertible, σ_S is a multiple of σ_q, i.e. $\sigma_S(t) = \alpha(t)\sigma_q(t)$ for some α. Then it turns out that shares of the stock can be replicated by holding a linear combination of nominal and real bond units, cf. the proof of Theorem 7.2. In particular at time t we can replicate one share of the stock by holding $\phi_B^{\text{syn}}(t) := \alpha(t)S(t)B^{-1}(t)$ units of the nominal bond and $\phi_{\mathcal{B}}^{\text{syn}}(t) := (1-\alpha(t))S(t)\mathcal{B}^{-1}(t)$ units of the real bond. We must account for such synthetic shares so we set $N^{\text{syn}}(t) := \sum_j \hat{\phi}_S^j(t) - N(t)$.

Definition 6.1 The market is in *equilibrium* (Arrow-Radner equilibrium, cf. [11]) if the market parameters satisfy (2.2), (2.4), (2.6), (2.10), (2.17), (2.19), (2.21), (3.5), (3.8), (4.1), (4.6), and are such that there exists a real dividend process δ, a real investment process $\hat{\nu}$, a labour process $\hat{L}$, a real capital process $\hat{K}$, and strategies $(\hat{\mathcal{C}}^j, \hat{\Phi}^j) := \left(\hat{c}^j, \hat{m}^j, \hat{l}^j, \hat{\phi}^j_{\mathcal{B}}, \hat{\phi}^j_B, \hat{\phi}^j_D, \hat{\phi}^j_S\right)$ (recall that $l^j = \bar{l}^j + \bar{\ell}^j$), such that if we define $N(t) = N(0) + \int_{[0,t)} S(s)^{-1}\, d\hat{\nu}(s)$, then

(6.1) $\left(\hat{c}^j, \hat{m}^j, \hat{l}^j, \hat{\phi}^j_{\mathcal{B}}, \hat{\phi}^j_B, \hat{\phi}^j_D, \hat{\phi}^j_S\right)$ is optimal for agent j, $j = 1, \dots, J$;

(6.2) $\hat{\nu}(\cdot) \in \arg\max\{\mathcal{J}_{C_0}(\nu) : \nu \in \mathcal{S}\}, \quad \hat{C}(t) := C(t+; \hat{\nu})$;

(6.3) $(\hat{K}(t), \hat{L}(t))^\top = I^{R^{Q(M)}(\hat{C}(t),\cdot,\cdot)}(\tilde{r}(t), w'(t))$, a.e. $t \in [0,T]$, a.s., cf. (3.10) ff. for $I^{R^{Q(M)}}$;

(6.4) $N(t)\delta(t) = R(\hat{C}(t), \hat{K}(t), \hat{L}(t)) - \tilde{r}(t)\hat{K}(t) - w'(t)\hat{L}(t)$, a.e. $t \in [0,T]$, a.s.;

(6.5) $\displaystyle\sum_{j=1}^J g^j(t) = \int_0^t q(s)\, d\tilde{M}(s) + [q, \tilde{M}](t)$, a.e. $t \in [0,T]$, a.s.;

(6.6) $\displaystyle\sum_{j=1}^J \hat{c}^j(t) = R(\hat{C}(t), \hat{K}(t), \hat{L}(t))$, a.e. $t \in [0,T]$, a.s.;

(6.7) $\displaystyle\sum_{j=1}^J \hat{m}^j(t) + \hat{K}(t) = M(t)$, a.e. $t \in [0,T]$, a.s.;

(6.8) $\displaystyle\sum_{j\in\mathcal{J}^p} \hat{l}^j(t) = J^p - \hat{L}(t)$, a.e. $t \in [0,T]$, a.s.;

(6.9) $\displaystyle\sum_{j\in\mathcal{J}^c} \hat{l}^j(t) = J^c - \frac{d\hat{\nu}}{dw^c}(t)$, a.e. $t \in [0,T]$, a.s.;

(6.10) $\displaystyle\sum_{j=1}^J \hat{\phi}^j_{\mathcal{B}}(t) + \phi^{\mathrm{syn}}_{\mathcal{B}}(t) N^{\mathrm{syn}}(t) \mathbb{1}_{\{\Sigma'(t) \text{ singular}\}} = 0$, a.e. $t \in [0,T]$, a.s.

(6.11) $\displaystyle\sum_{j=1}^J \hat{\phi}^j_B(t) + \phi^{\mathrm{syn}}_B(t) N^{\mathrm{syn}}(t) \mathbb{1}_{\{\Sigma'(t) \text{ singular}\}} = B(t)^{-1}\hat{K}(t)$, a.e. $t \in [0,T]$, a.s.

(6.12) $\displaystyle\sum_{j=1}^J \hat{\phi}^j_D(t) = 0$, a.e. $t \in [0,T]$, a.s.;

(6.13) $\displaystyle\sum_{j=1}^J \hat{\phi}^j_S(t) = N(t) + N^{\mathrm{syn}}(t) \mathbb{1}_{\{\Sigma'(t) \text{ singular}\}}$, a.e. $t \in [0,T]$, a.s.

When Σ' is invertible, i.e. $\mathbb{1}_{\{\Sigma'(t) \text{ singular}\}} = 0$, then the market clearing conditions (6.13), (6.10) and (6.11) assume the expected form. Note that (6.5) is also

$$g(t) := \sum_{j=1}^J g^j(t) = M(t) - M(0) - \int_{[0,t)} M(s)\, \frac{dq(s)}{q(s)}, \tag{6.14}$$

which is the change in real money supply net of an adjustment for the change in the price of money. Observe also that we may have $C(0;\hat{\nu}) \neq C(0+;\hat{\nu})$, i.e. there is an initial jump in capacity to the equilibrium value. All other variables are then determined relative to the capacity after this initial adjustment - that is why $\hat{C}$ is used.

Suppose we have an equilibrium, what can we say about it? The crucial market parameters are the nominal interest rate $\tilde{r}$ and the wage rate w'. With these we can find $\tilde{R}(\cdot\,,\tilde{r}(t),w'(t))$, cf. (3.10), then $\hat{\nu}$, cf. Theorem 3.4, and then $C(\cdot\,;\hat{\nu})$ and $\hat{K},\hat{L}$, cf. (3.14).

Take $\lambda_j = 1/\eta^j$ where the η^j are given by (4.17) so that $\eta^\Lambda = 1$, cf. Proposition 5.2 and following. Then

$$(R(\hat{C}(t),\hat{K}(t),\hat{L}(t)), M(t)-\hat{K}(t), J^p-\hat{L}(t)) = I^U(t,\zeta(t)\pi(t);\Lambda) \tag{6.15}$$

using first (6.6) - (6.8), then (4.18), and finally (5.2). If $\zeta(t)\pi(t) \in \tilde{\mathcal{R}}(t;\Lambda)$ then (5.5) implies

$$\nabla U(t,R(\hat{C}(t),\hat{K}(t),\hat{L}(t)), M(t)-\hat{K}(t), J^p-\hat{L}(t);\Lambda) = \zeta(t)\pi(t), \tag{6.16}$$

hence

$$\begin{cases} \tilde{r}(t) = \frac{U_m(t,R(\hat{C}(t),\hat{K}(t),\hat{L}(t)),M(t)-\hat{K}(t),J^p-\hat{L}(t);\Lambda)}{U_c(t,R(\hat{C}(t),\hat{K}(t),\hat{L}(t)),M(t)-\hat{K}(t),J^p-\hat{L}(t);\Lambda)}, \\ w'(t) = \frac{U_l(t,R(\hat{C}(t),\hat{K}(t),\hat{L}(t)),M(t)-\hat{K}(t),J^p-\hat{L}(t);\Lambda)}{U_c(t,R(\hat{C}(t),\hat{K}(t),\hat{L}(t)),M(t)-\hat{K}(t),J^p-\hat{L}(t);\Lambda)}. \end{cases} \tag{6.17}$$

Furthermore commencing with (6.9) we obtain similarly

$$J^c - \frac{d\hat{\nu}}{dw^c}(t) = I^u(t,\zeta(t);\Lambda), \tag{6.18}$$

hence

$$w^c(t) = \int_0^t \Big(J^c - I^u(s,\zeta(s);\Lambda)\Big)^{-1} d\hat{\nu}(s). \tag{6.19}$$

We point out that (6.9) states that $\sum_{j\in\mathcal{J}^c}(1-l^j)\,dw^c = d\hat{\nu}$, i.e. the capital raised for capacity expansion is paid as wages to the construction sector. Let us show that $\frac{d\hat{\nu}}{dw^c}(t) > 0$ so that (6.19) is valid. From (5.6) it follows that $u_l(t,l;\Lambda) \geq 0$ since this is true for all u^j, in particular for the one where the outward normal is 0. Now consider j_o such that $u_l^{j_o}(t,1) = 0$, cf. (4.12)(viii). If $\hat{\vec{\ell}}^{j_o}(t) = 1$, then (5.6) implies that $u_l(t,\hat{\vec{\ell}}^\Lambda(t);\Lambda) \leq 0$ hence $= 0$. If $\hat{\vec{\ell}}^j(t) < 1$ this and (5.6) imply that $\lambda_j u_l(t,\hat{\vec{\ell}}^\Lambda(t);\Lambda) = \vec{n}^j(\hat{\vec{\ell}}^\Lambda(t)) \leq 0$, a contradiction. Hence $\hat{\vec{\ell}}^j(t) = 1$ for all j, so that $\frac{d\hat{\nu}}{dw^c}(t) = 0$, cf. (6.9). But now $0 = u_l(t,J^c - \frac{d\hat{\nu}}{dw^c}(t);\Lambda) = \zeta(t) \geq k_z > 0$. This contradiction shows that the original assumption ($\hat{\vec{\ell}}^{j_o}(t) = 1$) is false, i.e. for at least one j, $\hat{\vec{\ell}}^j(t) < 1$, hence for all t, $\frac{d\hat{\nu}}{dw^c}(t) = J^c - \sum_{j\in\mathcal{J}^c}\hat{l}^j > 0$.

Remark 6.2 The interpretation of (6.16) is that *the deflator ζ is the marginal utility of consumption; the nominal interest rate $\tilde{r}$ is the deflated marginal utility of money*, and *the real wage rate w' is the deflated marginal utility of leisure in the production sector*. Moreover, *the marginal utility of consumption is also equal to the marginal utility of leisure in the construction sector*, $u_l(t,J^c - \frac{d\hat{\nu}}{dw^c}(t);\Lambda)$. □

Our objective is to find the market parameters so if we define $\varrho(\tilde{r}(\cdot), w'(\cdot))$ to be the right side of (6.17) (it is a function of these parameters since $\hat{K}, \hat{L}$ are functions of $\tilde{r}, w'$, cf. (3.14)), then $(\tilde{r}, w')$ is a fixed point of ϱ. Finding such a point is a rather daunting task. On the other hand we can solve for $(\hat{K}, \hat{L})$ without resorting to $\tilde{r}$ and w'. To this end we make the

Definition 6.3 Set $\mathcal{T} := (t, C, M)$, $\hat{\mathcal{T}}(t) := (t, \hat{C}(t), M(t))$. The *utility of operating capital and labour* at time t, capacity C and money supply M is $\hat{U}(\mathcal{T}, K, L; \Lambda)$ where

$$\begin{aligned}
\hat{U}(\mathcal{T}, K, L; \Lambda) &:= U(t, R(C, K, L), M - K, J^p - L; \Lambda). \\
\hat{U}_1(\mathcal{T}, K, L; \Lambda) &:= U_c(t, R(C, K, L), M - K, J^p - L; \Lambda), \\
\hat{U}_2(\mathcal{T}, K, L; \Lambda) &:= U_m(t, R(C, K, L), M - K, J^p - L; \Lambda), \\
\hat{U}_3(\mathcal{T}, K, L; \Lambda) &:= U_l(t, R(C, K, L), M - K, J^p - L; \Lambda).
\end{aligned}$$

Since $\bar{A}^j = [0, \infty) \times [0, \infty) \times [0, 1]$ or $\times\{0\}$ then $A^j = \bar{A}^j \cap \mathrm{dom}(U^j) = D^j_1 \times D^j_2 \times D^j_3$. We decompose $A = A_o \times A^\star$ where $A_o := \sum_j D^j_1$, and where $A^\star := \sum_j D^j_2 \times D^j_3$, cf. Lemma 5.1. Note that A_o is the positive half-axis, open unless all the D^j_1 are closed. Then

$$\mathrm{dom}(\hat{U}(\mathcal{T}, \cdot, \cdot; \Lambda)) = \{K, L : (R(C, K, L), M - K, J^p - L) \in A\} = R(C, \cdot, \cdot)^{-1}(A_o) \cap ((M, J^p) - A^\star).$$

We use results from [6] with $Z = (M, J^p)$, $A^Z = Q(M) = [0, M] \times [0, J^p]$, $z = (K, L)$, $z^\Lambda = (K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))$. Then [6] Lemma 3.1 and Corollary 3.2 give the next Lemma and Corollary.

Lemma 6.4 *$\hat{U}(\mathcal{T}, \cdot, \cdot; \Lambda)$ is strictly concave on $Q(M) = [0, M] \times [0, J^p]$. Hence*

$$(P_\Lambda) \qquad \sup_{(K,L) \in Q(M)} \hat{U}(\mathcal{T}, K, L; \Lambda)$$

is attained uniquely at a point $(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))$. Moreover $\nabla_{K,L} \hat{U}(\mathcal{T}, \cdot, \cdot; \Lambda)$ is normal to $Q(M)$ at $(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))$.

Corollary 6.5 *$(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))^\top$ is the unique solution of*

$$(6.20) \qquad (K, L)^\top = I^{R_{Q(M)}(C, \cdot, \cdot)} \left(\frac{\hat{U}_2(\mathcal{T}, K, L; \Lambda)}{\hat{U}_1(\mathcal{T}, K, L; \Lambda)}, \frac{\hat{U}_3(\mathcal{T}, K, L; \Lambda)}{\hat{U}_1(\mathcal{T}, K, L; \Lambda)} \right).$$

Moreover $K^\Lambda(\mathcal{T}) < M$ and $L^\Lambda(\mathcal{T}) < J^p$.

We note that (4.10)(xi) and (5.4) imply that at least one $\hat{m}_j > 0$ and one $\hat{l}_j > 0, j \in \mathcal{J}^p$ so (6.7), (6.8) imply that $K^\Lambda < M$, $L^\Lambda < J^p$. Similarly

Observe that (6.3) and (6.17) imply that $(\hat{K}(t), \hat{L}(t))^\top$ satisfies (6.20) at $\mathcal{T} = \hat{\mathcal{T}}(t)$, $Q(M) = Q(M(t))$ and $C = \hat{C}(t)$, hence

$$(6.21) \qquad (\hat{K}(t), \hat{L}(t)) = (K^\Lambda(\hat{\mathcal{T}}(t)), L^\Lambda(\hat{\mathcal{T}}(t))),$$

but of course to obtain $\hat{\mathcal{T}}(t)$ we require $\hat{C}$ hence $\hat{\nu}$. In fact, given $\hat{\nu}$ we solve (3.2) for $C(\cdot;\hat{\nu})$, then (P_Λ) for $(\hat{K}(t),\hat{L}(t)) := (K^\Lambda(\hat{\mathcal{T}}(t)), L^\Lambda(\hat{\mathcal{T}}(t)))$. This provides $\tilde{r}$ and w' via (6.17), hence $\tilde{R}(C,\tilde{r}(t),w'(t)) = R(C,\hat{K}(t),\hat{L}(t)) - \tilde{r}(t)\hat{K}(t) - w'(t)\hat{L}(t)$. But $\hat{\nu}$ must satisfy (3.15), so again a fixed point problem must be solved. However, if we take R of an additive form, we can isolate the $\max_\nu \mathcal{J}_{C_0}$-problem and find $\hat{\nu}$ independently of the market parameters. Hence we make the

Assumption

$$R(C,K,L) = R^1(C) + R^2(K,L) \tag{6.22}$$

with $R^1 \geq 0$. This is a severe restriction. Now $\tilde{R}(C,\tilde{r},w') = R^1(C) + \tilde{R}^2(\tilde{r},w')$ so

$$\begin{aligned}\mathcal{J}_{C_0}(\nu) \ &:= \ E\Big\{\int_0^T e^{-\mu_F(t)} R^1(C(t;\nu))\,dt + e^{-\mu_F(T)} G(C(T;\nu)) - \int_{[0,T)} e^{-\mu_F(t)}\,d\nu(t)\Big\}\\ &\quad + E\Big\{\int_0^T e^{-\mu_F(t)} \tilde{R}^2(\tilde{r}(t),w'(t))\,dt\Big\}\end{aligned}$$

where the second expectation is independent of ν.

This gives an approach to existence. By replacing $\tilde{R}$ by R^1 in Theorem 3.4, we obtain $\hat{\nu}$ without knowledge of $\tilde{r}, w'$ (this assumes that μ_F is known, i.e. $e^{-\mu_F}$ cannot be the deflator ζ as intended in Remark 3.3), then $\hat{C}$ can be computed, cf.(6.2), so $\hat{\mathcal{T}}(t)$ is now known. Moreover $I^{R_{Q(M)}(\hat{C},\cdot\,,\cdot\,)}$ is $(I^{R^2})^{Q(M)} := I^{R^{2Q(M)}}$, independent of $\hat{C}$. This is all independent of Λ. Our candidate for optimal $(K(t),L(t))$ will be $(K^\Lambda(\hat{\mathcal{T}}(t)), L^\Lambda(\hat{\mathcal{T}}(t)))$, cf. (6.21). Finally we can find ζ from (6.16) with $(\hat{K}(t),\hat{L}(t))$ replaced by $(K^\Lambda(\hat{\mathcal{T}}(t)), L^\Lambda(\hat{\mathcal{T}}(t)))$ and q from (2.22). Of course each choice of Λ gives η^j from (4.17) and we required $\eta^j = 1/\lambda_j$, i.e. we have another fixed point problem to solve for Λ. This we do in the next section.

We add further conditions to ensure that we can solve for the remaining parameters; in examples where the solutions are exhibited, some of the conditions can be dispensed with. We make the

Assumption

$$\left\{\begin{array}{rl} (i) & \mu_F(t) \text{ is linear, i.e. } \mu_F(t) = t\mu_F, \text{ and } \mu_C, \sigma_C, f_C \text{ are constant;}\\ (ii) & G(0)=0,\ G(C) \leq \kappa_G,\\ & \text{either } C_0 > 0 \text{ or (3.17) fails (so that } \hat{\nu}\equiv 0 \text{ is not optimal),}\\ & G \text{ is thrice differentiable, } |G''(C)| \leq \kappa_G(1+|C|^{k_G}), \text{ and (3.19) holds;}\\ (iii) & R^1 \in C^2(\Re_+), R^1(0) \geq 0, \max\{R^1(C), R^1_C(0), -C^2 R^1_{CC}(C)\} \leq \kappa_{R^1} \text{ on } [0,\infty);\\ (iv) & R^2 \in C^3(\operatorname{int}(\operatorname{dom}(R^2))), \text{ and}\\ & ([0,\kappa_M]\times[0,J^p])\backslash\{0\} \subset \operatorname{dom}(\partial R^2),\ R^2(([0,\kappa_M]\times[0,J^p])\backslash\{0\}) \subset \Re^1_{++};\\ (v) & u^j_l(T,1) \geq V^j_x(k_M) \text{ for all } j, \textbf{ or}\\ (vi) & U^j_c(T,0,\kappa_M,1) \leq V^j_x(\kappa_G+\kappa_M) \text{ for all } j. \end{array}\right. \tag{6.23}$$

The last two conditions are only used to ensure that at time T the total wealth in the economy consists of the scrap value of the firm plus the money supply. In fact these conditions "explain" T: they imply that at time T under any scenario the money supply and scrap value provide less marginal utility than does labour in the construction sector in the case of (v), or more than consumption in the case of (vi).

The bound on G introduced in (ii) is used in (vi) and to establish (2.6). Further, as $\tilde{R}(C, \tilde{r}(t), w'(t))$ is replaced by $R^1(C)$ in the $\mathcal{J}_{C_0}$ problem, then (6.23)$(i)(ii)$ and Lemma 3.6 imply that $\hat{\nu}$ is continuous after time $t = 0$. In fact from Remark 3.5 it follows that $\hat{\nu}(0+) > 0$ if $C_0 = 0$, hence $C(0+; \hat{\nu}) > 0$. From (3.4) it follows that $R^1 > 0$ on $(0, \infty)$.

An alternative to (iv) which allows R^2 to have the Cobb-Douglas form $K^\alpha L^\gamma$ with $0 < \alpha, \gamma,\ \alpha + \gamma < 1$ is the following.

$(iv)'$ $R^2 \in C^3(\mathrm{int}(\mathrm{dom}(R^2)))$;
there exist closed sets $G_i^R,\ i = 1, 2,$ with disjoint interiors such that
$[0, \kappa_M] \times [0, J^p] = G_1^R \cup G_2^R,$ and for
$i \in \mathcal{I}^R := \Big\{ i \in \{1, 2\} : \{Z \in [0, \kappa_M] \times [0, J^p] : Z_i = 0\} \cup \mathrm{dom}(\partial R^2) = \emptyset \Big\},$
$\lim_{\delta \downarrow 0} \inf_{Z \in G_i^R(\delta)} R^2_{Z_i}(Z) = \infty,$ where $G_i^R(\delta) = G_i^R \cap \{Z_i \leq \delta\}$;
for all j and all $i \in \mathcal{I}^R$,
$U^j_m(c, m, l)/U^j_c(c, m, l)$ is independent of l and
$U^j_l(c, m, l)/U^j_c(c, m, l)$ is independent of m.

An example of a function satisfying (iii) is $R^1(C) := a_0 + a_1(1 - e^{-bC})$ with $a_0 \geq 0, a_1, b > 0$.

It turns out to be useful to work with $z := R^1(C)$ rather than C. We collect further notation in

Definition 6.6
$\hat{\nu}^+(t) := \hat{\nu}(t+),\ \hat{C}(t) := C(t+; \hat{\nu}),\ \hat{z}(t) := R^1(\hat{C}(t)),\ \bar{\mathcal{T}} := (t, z, M),\ \bar{\mathcal{T}}(t) := (t, \hat{z}(t), M(t)),$ $\Pi^\Lambda(\bar{\mathcal{T}}) := (K^\Lambda(\bar{\mathcal{T}}), L^\Lambda(\bar{\mathcal{T}})),\ \bar{U}(\bar{\mathcal{T}}, K, L; \Lambda) := U(t, z + R^2(K, L), M - K, J^p - L; \Lambda).$

Then $\Pi^\Lambda(\bar{\mathcal{T}})$ is the unique solution of the problem as defined in Lemma 6.4 but with $\mathcal{T}$ replaced by $\bar{\mathcal{T}}$, i.e.

$$(P_\Lambda) \qquad \sup_{(K,L) \in Q(M)} \bar{U}(\bar{\mathcal{T}}, K, L; \Lambda).$$

Observe that $\hat{C}(t) = C(t; \hat{\nu})$ except $\hat{C}(0) = C_0 + \hat{\nu}^+(0)$, and $\hat{C}$ is continuous, $\Re_{++}$-valued, cf. (3.18). Moreover $\hat{C}(t)$ is not bounded, but $z(t)$ is (cf. (6.23)(iii)); that is why we switched from C to $z \in \mathrm{im}(R^1)$. The dynamics of $\hat{z}(t)$ are given by

$$\begin{aligned} \hat{z}(t) &= R^1(C_0 + f_C \hat{\nu}^+(0)) \\ &\quad + \int_0^t \Big[\tfrac{1}{2} R^1_{C\,C}(\hat{C}(s)) \hat{C}(s)^2 \|\sigma_C\|^2 - R^1_C(\hat{C}(s)) \hat{C}(s) \mu_C \Big]\, ds \\ &\quad + \int_0^t R^1_C(\hat{C}(s)) \hat{C}(s) \sigma_C^\top \, dW(s) + \int_{[0,t)} R^1_C(\hat{C}(s)) f_C \, d\hat{\nu}^+(s). \end{aligned} \tag{6.24}$$

Defining μ_z, σ_z and ρ_z appropriately, we have

$$d\hat{z}(t) = \mu_z(t)\, dt + \sigma_z(t)^\top \, dW(t) + \rho_z(t)\, d\hat{\nu}^+(t).$$

Since $R^1_C(\hat{C}(t))\hat{C}(t) \leq R^1(\hat{C}(t)) - R^1(0) \leq \kappa_{R^1}$ by concavity, then (6.23)(iii) implies that $\mu_z,\ \sigma_z,\ \rho_z$ are bounded and $\rho_z \geq 0$.

To compute further parameters, we require the dynamics of Π^Λ. Lemma 6.4 gives us the existence of this function; here is its regularity.

Proposition 6.7 *The function* $(\bar{\mathcal{T}}, \Pi, \Lambda) \rightarrow \bar{U}(\bar{\mathcal{T}}, \Pi; \Lambda)$ *is continuous on its domain. The function* $(\bar{\mathcal{T}}, \Pi, \Lambda) \rightarrow \nabla_\Pi \bar{U}(\bar{\mathcal{T}}, \Pi; \Lambda)$ *is continuous on its domain and is piecewise continuously differentiable in* t *and twice piecewise continuously differentiable in* (z, M, Π). *Moreover* $(\bar{\mathcal{T}}, \Lambda) \rightarrow \Pi^\Lambda(\bar{\mathcal{T}})$ *is continuous on* $[0,T] \times [0, \kappa_{R^1}] \times [0, \kappa_M] \times \Re^J_{++}$, *piecewise continuously differentiable in* t *and twice piecewise continuously differentiable in* (z, M).

Proof: The results for $\bar{U}$ follow from our Lemma 5.1 and the proof of Lemma 4.4 in [6].

The results about Π^Λ follow from [7], Section 3 applied to $\bar{U}(\bar{\mathcal{T}}, \cdot\,; \Lambda)$ after possibly using the Whitney Extension Theorem to extend U onto an open set containing A (because we require $\mathrm{dom}(\partial_\Pi \bar{U}_{Q(M)}(\bar{\mathcal{T}}, \cdot\,; \Lambda)) \subset \mathrm{int}(\mathrm{dom}(\partial_\Pi \bar{U}(\bar{\mathcal{T}}, \cdot\,; \Lambda)) \cap Q(M)$, cf. [7] Assumption 2(ii)). The monotonicity of the utility function required in [7] is needed only to determine the domain of $I^{\bar{U}(\bar{\mathcal{T}}, \cdot\,;\Lambda)}$, the extension of the inverse of $\nabla \bar{U}$, but we only need $I^{\bar{U}(\bar{\mathcal{T}}, \cdot\,;\Lambda)}(0)$ $(= \Pi^\Lambda(\bar{\mathcal{T}}))$ and 0 is known to be in the domain since (P_Λ) has a solution. The results in [6] only consider the first derivatives of $I^{\bar{U}} = \Pi^\Lambda$ but the argument extends to second derivatives when these exist for $\nabla_\Pi \bar{U}$.

Observe that $c \in \cup_j \mathcal{I}^j_U$ means that $\mathrm{dom}(\partial U^j(t, \cdot))$ cannot intersect $(\{c = 0\} \cap A^j)$ whence $z + R^2(\Pi) > 0$ on $\mathrm{dom}(\nabla_\Pi \bar{U})$, cf. (5.6). This plus (4.10)(x) implies that $\mathrm{dom}(\nabla_\Pi \bar{U}(\bar{\mathcal{T}}, \cdot\,; \Lambda)) = ([0, M) \times [0, J^p)) \cap \{\Pi : z + R^2(\Pi) > 0 \text{ if } c \in \cup_j \mathcal{I}^j_U\}$. □

To find the dynamics of Π^Λ define the generator of $(z(t), M(t))$ as

$$\begin{aligned}(6.25)\quad \mathcal{L}V(z, M) :=\ & \frac{1}{2}\Big[\|\sigma_z\|^2 V_{zz}(z,M) + 2M\sigma_z^\top \sigma_M V_{zM}(z,M) + M^2\|\sigma_M\|^2 V_{MM}(z,M)\Big] \\ & + \mu_z V_z(z,M) + M\mu_M V_M(z,M).\end{aligned}$$

At this point we have to pay for not imposing the Inada conditions on all the utility functions, i.e. for not avoiding "hard" constraints. We want to apply a change of variable formula, i.e. Itô's formula, to the function $\Pi^\Lambda(\bar{\mathcal{T}})$ to obtain the dynamics of $\Pi^\Lambda(\bar{\mathcal{T}}(t))$. Π^Λ is non-differentiable on surfaces where the constraints on $\hat{\mathcal{C}}^j$ become binding. The boundary $\{l^j = 1\}$ of $A^j,\ j \in \mathcal{J}^p$, is always accessible, i.e. in $\mathrm{dom}(\partial U^I)$, so the constraint can be active there. N.b. inaccessible boundaries are never visited by $\hat{\mathcal{C}}(t)$ so those constraints will never be active. $\nabla U(t, \cdot\,; \Lambda)$ is not differentiable at $\mathcal{C} = \sum_j \hat{\mathcal{C}}^j$ for which at least one of the $\hat{l}^j$ in $\hat{\mathcal{C}}^j$ is equal to 1. This translates to

$$I_3^{U^j_{A^j}}\Big(t, \frac{\nabla_{\mathcal{C}} U(t, \mathcal{C}; \Lambda)}{\lambda_j}\Big) = 1,$$

or in terms of $\bar{\mathcal{T}} = (t, z, M)$ with $\Pi^\Lambda(\bar{\mathcal{T}}) = (K^\Lambda(\bar{\mathcal{T}}), L^\Lambda(\bar{\mathcal{T}}))$ to

$$(6.26)\qquad I_3^{U^j_{A^j}}\Big(t, \frac{\nabla_{\mathcal{C}} U(t, z + R^2(K^\Lambda(\bar{\mathcal{T}}), L^\Lambda(\bar{\mathcal{T}})), M - K^\Lambda(\bar{\mathcal{T}}), J^p - L^\Lambda(\bar{\mathcal{T}}); \Lambda)}{\lambda_j}\Big) = 1.$$

If U^j and the money supply are independent of time then so is U and we take $\bar{\mathcal{T}} := z$, i.e. $\hat{\mathcal{T}}(t)$ consists of only one semimartingale. Then the surface (6.26) would collapse to a point $z = b^j$ as in [19] and we could apply [22] Chapter 3, Theorem 6.22 and Problem 6.24, but in higher dimensions we have to rely on more exotic results which demand further technical assumptions.

In [28] Peskir gives a change of variable formula for the case $\bar{\mathcal{T}} = (t, z)$, i.e. the money supply $M(t)$ is constant, assuming the curve where ∇U is not differentiable has the form

$$z = b(t) \text{ where } b \text{ is continuous and of bounded variation.} \tag{6.27}$$

In [29] he generalizes to the case $\bar{\mathcal{T}} = (t, z, M)$, but now the surface (6.26) must have the form

$$z = b(t, M) \quad \text{where } b \text{ is continuous and } b(t, M(t)) \text{ is a semimartingale.} \tag{6.28}$$

$b \in C^{1,2}$ is sufficient but not necessary for (6.28). When there are several curves they must not intersect.

Example 6.8 We can examine these conditions in a very simple case. There are only two agents and they are in the production sector. Below we take $f_i^j \geq 0$ and continuously differentiable.

$$U^j(t, c, m, l) := f_1^j(t)(c+1)^{\frac{1}{2}} + f_2^j(t)m^{\frac{1}{2}} + f_3^j(t)l^{\frac{1}{2}}, \quad A^j := [0,\infty) \times [0,\infty) \times [0,1], \quad j = 1, 2.$$

Then $A = [0,\infty) \times (0,\infty) \times [0,2]$ and $\tilde{A} = [0,\infty) \times (0,\infty) \times (0,2]$, so the inaccessible boundaries are given by $\mathcal{I}_U^j = \{2, 3\}$ and the only binding constraints are $c^j \geq 0$ and $l^j \leq 1$, $j = 1, 2$. Without loss of generality we can scale Λ to $(1, \lambda)$. If $\mathcal{C} = (c, m, l)$ then

$$\begin{aligned} \mathcal{C}_1 &= \Big(\frac{k_1(t)}{1+k_1(t)}\, c + \frac{k_1(t)-1}{k_1(t)+1}, \frac{k_2(t)}{1+k_2(t)}\, m, \frac{k_3(t)}{1+k_3(t)}\, l\Big), \\ \mathcal{C}_2 &= \Big(\frac{1}{1+k_1(t)}\, c - \frac{k_1(t)-1}{k_1(t)+1}, \frac{1}{1+k_2(t)}\, m, \frac{1}{1+k_3(t)}\, l\Big), \end{aligned}$$

$$k_1(t) := \Big(\frac{f_1^1(t)}{\lambda f_1^2(t)}\Big)^2, \qquad k_2(t) := \Big(\frac{f_2^1(t)}{\lambda f_2^2(t)}\Big)^2, \qquad k_3(t) := \Big(\frac{f_3^1(t)}{\lambda f_3^2(t)}\Big)^2,$$

for $(c, m, l) \in [(\max\{k_1^{-1}(t) - 1, k_1(t) - 1\})^+, \infty) \times [0, \infty) \times [0, \min\{1 + k_3^{-1}(t), 1 + k_3(t)\}]$ where a^+ denotes $\max\{0, a\}$.

For $c \in [0, k_1^{-1}(t) - 1]$ we have $\hat{c}_1 = 1,\ \hat{c}_2 = c - 1$, for $c \in [0, k_1(t) - 1]$ we have $\hat{c}_1 = c - 1,\ \hat{c}_2 = 1$. So $\hat{c}_1 = 0 = \hat{c}_2$ only at times t when $k_1(t) = 1$, and then c must be 0. For other times only one of the two c constraints can be active. For $l \in [1 + k_3^{-1}(t), 2]$, we have $\hat{l}_1 = 1,\ \hat{l}_2 = l - 1$ and when $l \in [1 + k_3(t), 2]$, then $\hat{l}_1 = l - 1,\ \hat{l}_2 = 1$. Then $\hat{l}_1 = 1 = \hat{l}_2$ only at times t when $k_3(t) = 1$, and then $l = 2$. Again only one of the l constraints can be active at any time; for a different situation, see [6] Remark 2.8.

In the current case in the first region

$$
\begin{aligned}
U(t,c,m,l;\Lambda) &= \sqrt{(f_1^1(t))^2+(\lambda f_1^2(t))^2}\sqrt{c+2}+\sqrt{(f_2^1(t))^2+(\lambda f_2^2(t))^2}\sqrt{m}\\
&\quad+\sqrt{(f_3^1(t))^2+(\lambda f_3^2(t))^2}\sqrt{l}.
\end{aligned}\tag{6.29}
$$

If $c\in[0,k_1^{-1}(t)-1]$ the first term above becomes $f_1^1(t)+\lambda f_1^2(t)\sqrt{c+1}$; if $c\in[0,k_1(t)-1]$ then it becomes $f_1^1(t)\sqrt{c+1}+\lambda f_1^2(t)$. If $l\in[1+k_3^{-1}(t),2]$ then the last term in U above becomes $f_3^1(t)+\lambda f_3^2(t)\sqrt{l-1}$, whereas if $l\in[1+k_3(t),2]$ then it becomes $f_3^1(t)\sqrt{l-1}+\lambda f_3^2(t)$.

Next we find K^Λ, L^Λ when they are in $\mathrm{int}(Q(M))$; they are the solution of

$$
\begin{aligned}
&U_c(t,z+R^2(K,L),M-K,2-L;\Lambda)\nabla_{K,L}R^2(K,L)\\
&\quad=\ (U_m(t,z+R^2(K,L),M-K,2-L;\Lambda),\,U_l(t,z+R^2(K,L),M-K,2-L;\Lambda))^\top.
\end{aligned}
$$

We take $R^2(K,L):=\sqrt{K}+\sqrt{L}$. For

$$
z+R^2(K,L)\ge(\max\{k_1^{-1}(t)-1,k_1(t)-1\})^+,\quad 2\ge L\ge 2-\min\{1+k_3^{-1}(t),1+k_3(t)\},
$$

this reduces to

$$
z+2\sqrt{K}+2\sqrt{L}+2=\frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(f_2^1(t))^2+(\lambda f_2^2(t))^2}\Big(\frac{M-K}{K}\Big)=\frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(f_3^1(t))^2+(\lambda f_3^2(t))^2}\Big(\frac{2-L}{L}\Big).\tag{6.30}
$$

Solving the second equality in (6.30) for L in terms of K and substituting into the first gives

$$
\begin{aligned}
&z+2\sqrt{K}+2\\
&=\ \frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(f_2^1(t))^2+(\lambda f_2^2(t))^2}\Big(\frac{M-K}{K}\Big)-2\sqrt{2}\Big[1+\frac{(f_3^1(t))^2+(\lambda f_3^2(t))^2}{(f_2^1(t))^2+(\lambda f_2^2(t))^2}\Big(\frac{M-K}{K}\Big)\Big]^{-\frac12},
\end{aligned}\tag{6.31}
$$

and similarly

$$
\begin{aligned}
&z+2\sqrt{L}+2\\
&=\ \frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(f_3^1(t))^2+(\lambda f_3^2(t))^2}\Big(\frac{2-L}{L}\Big)-2\sqrt{M}\Big[1+\frac{(f_2^1(t))^2+(\lambda f_2^2(t))^2}{(f_3^1(t))^2+(\lambda f_3^2(t))^2}\Big(\frac{2-L}{L}\Big)\Big]^{-\frac12},
\end{aligned}\tag{6.32}
$$

which can be solved (in principle) for $K^\Lambda(t,z,M)$ and $L^\Lambda(t,z,M)$.

The constraint $\hat{l}_1=1$ becomes active when $k_3(t)\ge1$, (i.e. $f_3^1(t)\ge\lambda f_3^2(t)$) and $2-L=l=1+k_3^{-1}(t)$, i.e. $L=1-k_3^{-1}(t)$. Recall that this is part of the boundary of the region where U is given by (6.29). Then (6.32) implies

$$
z=-2\Big[1+\frac{(f_2^1(t))^2+(\lambda f_2^2(t))^2}{(f_3^1(t))^2-(\lambda f_3^2(t))^2}\Big]^{-\frac12}\sqrt{M}+\frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(f_3^1(t))^2-(\lambda f_3^2(t))^2}-2\frac{\sqrt{(f_3^1(t))^2-(\lambda f_3^2(t))^2}}{f_3^1(t)}-2.
$$

This is the surface in (t,z,M) space where L^Λ and K^Λ have discontinuous derivatives due to the $\hat{l}_1=1$ constraint. For $\hat{l}_2=1$, i.e. $f_3^1(t)\le\lambda f_3^2$ or $k_3\le1$, the surface is

$$
z=-2\Big[1+\frac{(f_2^1(t))^2+(\lambda f_2^2(t))^2}{(\lambda f_3^2(t))^2-(f_3^1(t))^2}\Big]^{-\frac12}\sqrt{M}+\frac{(f_1^1(t))^2+(\lambda f_1^2(t))^2}{(\lambda f_3^2(t))^2-(f_3^1(t))^2}-2\frac{\sqrt{(\lambda f_3^2(t))^2-(f_3^1(t))^2}}{\lambda f_3^2(t)}-2.
$$

With t fixed the z-intercept of both curves is negative for $(f_1^1(t))^2 + (\lambda f_1^2(t))^2$ sufficiently small (and $k_3(t) \neq 1$), and the intercept increases to ∞ as $k_3(t) \to 1$, i.e. when we switch from the one constraint to the other. We have $\hat{l}_j < 1$, i.e. the constraint is inactive, at points below the curve while above the curve $\hat{l}_j = 1$, $j = 1, 2$ depending on which of the two curves we are discussing.

For t such that $k_3(t) = 1$, i.e. $\lambda f_3^2 = f_3^2$, we have $z = \infty$, but in fact we are only concerned with $(t, z, M) \in [0, T] \times [0, \kappa_R] \times [k_M, \kappa_M]$. If $b(t, M)$ denotes the right side of either of the last two equations then it follows that in the region of interest $b(t, M(t))$ is a semimartingale.

We now consider the constraint $\hat{c}_1 = 0$, so $k_1(t) \leq 1$. On the boundary of the region where U is given by (6.29) this means that $z + R^2(K, L) = k_1^{-1}(t) - 1$; substituting this into (6.30) gives

$$K = M \frac{(f_1^1(t))^2}{(f_1^1(t))^2 + (f_2^1(t))^2 + (\lambda f_2^2(t))^2}, \quad L = 2 \frac{(f_1^1(t))^2}{(f_1^1(t))^2 + (f_3^1(t))^2 + (\lambda f_3^2(t))^2}.$$

The latter two expressions can be substituted into $z + R^2(K, L) = k_1^{-1}(t) - 1$ to yield

$$\begin{aligned} z &= \frac{-2 f_1^1(t)}{\sqrt{(f_1^1(t))^2 + (f_2^1(t))^2 + (\lambda f_2^2(t))^2}} \sqrt{M} \\ &\quad + \frac{(\lambda f_1^2(t))^2 - (f_1^1(t))^2}{(f_1^1(t))^2} - \frac{2\sqrt{2} f_1^1(t)}{\sqrt{(f_1^1(t))^2 + (f_3^1(t))^2 + (\lambda f_3^2(t))^2}}. \end{aligned} \tag{6.33}$$

For t fixed the slope of this line in the $\sqrt{M} - z$ plane lies in $(-2, 0)$. The z intercept, hence z, increases to ∞ as $k_1(t) \downarrow 0$ and becomes negative as $k_1(t) \uparrow 1$. The c-constraint is inactive above the curve, i.e. $\hat{c}_1 > 0$. The case $\hat{c}_2 = 0$ gives (6.33) but with f_1^1 and λf_1^2 interchanged.

Again we see that $b(t, M(t))$ is a semimartingale where $b(t, M)$ is the right side of (6.33). It is clear that suitable restrictions on f_i^j will ensure that the surface from either c constraint does not intersect the surface from either l constraint. Hence [29] is applicable. □

For each agent in the production sector, i.e. $j \in \mathcal{J}^p$, the constraints $l^j \leq 1$ may become active, but so could the constraints $s \geq 0$, $s \notin \mathcal{I}_U^j$, cf. (4.11), (n.b. $s = c^j, m^j$ or l^j). Each such constraint will generate a surface in the (t, z, M)-space where $(K^\Lambda, \mathrm{L}^\Lambda)$ although continuous is not in $C^{1,2,2}$. We write these surfaces as $z = b^k(t, M)$, $k = 1, 2, \ldots, \bar{k}$. Then we make the **Assumption**

$$\left\{ \begin{array}{l} \text{For each } k,\ (t, M) \mapsto b^k(t, M) \text{ is continuous and } b^k(t, M(t)) \text{ is a semimartingale on} \\ \{(t, M) \in [0, T] \times [k_M, \kappa_M] : b^k(t, M) \in [0, \kappa_R]\}. \\ \text{Moreover the surfaces } z = b^k(t, M),\ k = 1, 2, \ldots, \bar{k}, \text{ do not intersect in the region} \\ (0, T) \times (0, \kappa_R) \times (k_M, \kappa_M). \end{array} \right. \tag{6.34}$$

If M is constant we are in the setting of [28] and make the alternate **Assumption**.

$$\left\{ \begin{array}{l} \text{For each } k,\ t \mapsto b^k(t) \text{ is continuous on } \{t \in [0, T] : b^k(t) \in [0, \kappa_R]\}. \text{ Moreover} \\ \text{the curves } z = b^k(t),\ k = 1, 2, \ldots, \bar{k}, \text{ intersect in the region } (0, T) \times (0, \kappa_R) \\ \text{at most at a finite number of times } t_0 < t_1 < t_2 < \cdots, t_n. \end{array} \right. \tag{6.35}$$

Then $b^k(t)$ is a semimartingale and the result of [28] applies on each interval $[t_{i-1}, t_i)$.

Of course we could eliminate all such surfaces by assuming that all the U^j satisfy the Inada conditions - in l we require $U^j_l(c, m, 1) = 0, j \in \mathcal{J}^p$.

At $z = b^k(t, M)$ we write

$$K^\Lambda_z(t, z, M) := \tfrac{1}{2}[K^\Lambda_z(t, b^k(t, M)+, M) + K^\Lambda_z(t, b^k(t, M)-, M)] \tag{6.36}$$

and similarly for $K^\Lambda_t,\ K^\Lambda_{z,z},\ K^\Lambda_{z,M},\ K^\Lambda_{M,M}$ and for the derivatives of L^Λ. The *local time* at time s of the process $(\cdot, z(\cdot), M(\cdot))$ at the surface $z = b^k(t, M)$ is

$$\ell^k(s) := P - \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^s \mathbb{1}_{\{|z(t)-b^k(t,M(t))|<\varepsilon\}}\, d[z(\cdot) - b^k(\cdot\,, M(\cdot))](t), \tag{6.37}$$

a continuous nondecreasing process. Here $[x]$ denotes the quadratic variation process of x. We write $\ell^K(\cdot)$ for the local time of the process $K^\Lambda(t, z(t), M(t))$ at 0, and similarly for ℓ^L. They are nondecreasing, continuous process, zero at $t = 0$, constant except when the corresponding process K^Λ or L^Λ is 0.

Proposition 6.9 *Assume (6.23) and one of (6.34),(6.35). The process* $\Pi^\Lambda(t, \hat{z}(t), M(t))$ *satisfies*

$$\begin{aligned} d\Pi^\Lambda(t, \hat{z}(t), M(t)) \;=\; & \left[\frac{\partial}{\partial t} + \mathcal{L}\right]\Pi^\Lambda(t, \hat{z}(t), M(t))\, dt \\ & +\nabla_{z,M}\Pi^\Lambda(t, \hat{z}(t), M(t))\ (\sigma_z(t), M(t)\sigma_M(t))^\top\, dW(t) \\ & +\nabla_{z,M}\Pi^\Lambda(t, \hat{z}(t), M(t))(\rho_z(t)\, d\hat{\nu}^+(t), M(t)\, d\beta_M(t))^\top + d\beta_\Pi(t) \\ & +\sum_k \tfrac{1}{2}[\Pi^\Lambda_z(t, \hat{z}(t)+, M(t)) - \Pi^\Lambda_z(t, \hat{z}(t)-, M(t))]\mathbb{1}_{\{\hat{z}(t)=b^k(t,M(t))\}}\, d\ell^k(t), \end{aligned} \tag{6.38}$$

with $\nabla_{z,M}\Pi^\Lambda = (\nabla_{z,M}K^\Lambda(t, z, M), \nabla_{z,M}L^\Lambda(t, z, M))^\top$ *and* $\beta_\Pi := (\ell^K, \ell^L)^\top$. *In addition* $\|\beta_\Pi\|_T < \infty$ *a.s.*

The technical proof is in the Appendix.

We define

$$U^{\Pi^\Lambda}_c(t, z, M) := U_c\Big(t, z + R^2(K^\Lambda(t, z, M), L^\Lambda(t, z, M)), M - K^\Lambda(t, z, M), J^p - L^\Lambda(t, z, M); \Lambda\Big)$$

and similarly for $U^{\Pi^\Lambda}_{c,c}, U^{\Pi^\Lambda}_{c,m}, U^{\Pi^\Lambda}_{c,l}$. Note that $U^{\Pi^\Lambda}_c \in C^{1,2}$ piecewise so all the integrands below are bounded. Then (6.16) implies

$$\begin{aligned} d\zeta(t) \;=\; & \left[\frac{\partial}{\partial t} + \mathcal{L}\right] U^{\Pi^\Lambda}_c(t, \hat{z}(t), M(t))\, dt \\ & +\nabla_{z,M} U^{\Pi^\Lambda}_c\Big(t, \hat{z}(t), M(t)\Big)^\top \Big(\rho_z(t)\, d\hat{\nu}^+(t), M(t)\, d\beta_M(t)\Big)^\top \\ & +\Big(U^{\Pi^\Lambda}_{c,c}(t, \hat{z}(t), M(t))R^2_K(0, L^\Lambda(t, \hat{z}(t), M(t))) - U^{\Pi^\Lambda}_{c,m}(t, \hat{z}(t), M(t))\Big)\, d\ell^K(t) + \end{aligned}$$

$$(6.39)\qquad +\Big(U_{c,c}^{\Pi^\Lambda}(t,\hat z(t),M(t))R_L^2(K^\Lambda(t,\hat z(t),M(t)),0) - U_{c,l}^{\Pi^\Lambda}(t,\hat z(t),M(t))\Big)\,d\ell^L(t)$$
$$+\sum_k \tfrac12\Big[\frac{\partial}{\partial z}U_c^{\Pi^\Lambda}(t,\hat z(t)+,M(t)) - \frac{\partial}{\partial z}U_c^{\Pi^\Lambda}(t,\hat z(t)-,M(t))\Big]\mathbb{1}_{\{\hat z(t)=b^k(t,M(t))\}}\,d\ell^k(t)$$
$$+\nabla_{z,M}U_c^{\Pi^\Lambda}\Big(t,\hat z(t),M(t)\Big)^\top\Big(\sigma_z(t)\,,\,M(t)\sigma_M(t)\Big)^\top\,dW(t).$$

From this we can identify $r(t)$, $\theta(t)$ and $\beta(t)$, cf. (2.18). Recall that $\hat\nu^+$ is continuous so β defined by

$$\beta(t) \;=\; -\int_{[0,t)}\frac{1}{\zeta(s)}\Big[\nabla_{z,M}U_c^{\Pi^\Lambda}(s,\hat z(s),M(s))\Big](\rho_z(s)\,d\hat\nu^+(s),M(s)\,d\beta_M(s))^\top$$
$$-\int_{[0,t)}\frac{1}{\zeta(s)}\Big(U_{c,c}^{\Pi^\Lambda}(s,\hat z(s),M(s))R_K^2(0,L^\Lambda(s,\hat z(s),M(s))) - U_{c,m}^{\Pi^\Lambda}(s,\hat z(s),M(s))\Big)\,d\ell^K(s)$$
$$(6.40)\qquad -\int_{[0,t)}\frac{1}{\zeta(s)}\Big(U_{c,c}^{\Pi^\Lambda}(s,\hat z(s),M(s))R_L^2(K^\Lambda(s,\hat z(s),M(s)),0) - U_{c,l}^{\Pi^\Lambda}(s,\hat z(s),M(s))\Big)\,d\ell^L(s)$$
$$-\sum_k\int_0^t \tfrac12\Big[\frac{\partial}{\partial z}U_c^{\Pi^\Lambda}(s,\hat z(s)+,M(s)) - \frac{\partial}{\partial z}U_c^{\Pi^\Lambda}(s,\hat z(s)-,M(s))\Big]\mathbb{1}_{\{\hat z(s)=b^k(s,M(s))\}}\,d\ell^k(s)$$

is continuous with $\beta(0)=0$. As the integrands are bounded then $\|\beta\|_T<\infty$ a.s.

Remark 6.10 We pointed out in Remark 6.2 that $\tilde r$ is the marginal utility of holding money relative to that of consumption. What about r, or rather $r\,dt+d\beta$? This relates to the usual conclusions of the CAPM. We proceed as in [19], Remark 8.2, to define the growth "rate" of the marginal utility of consumption as the bounded variation term in $\frac{dU_c^{\Pi^\Lambda}}{U_c^{\Pi^\Lambda}}=\frac{d\zeta}{\zeta}$. From (6.39), this is $-[r(t)\,dt+d\beta(t)]=-\frac{d\mathcal{B}(t)}{\mathcal{B}(t)}$. Hence *the growth "rate" of the riskless asset is the negative of the growth "rate" of the marginal utility of consumption.* If $\beta\equiv0$ then these are true rates.

Moreover since the stochastic integrand in (6.39) is $-\zeta\theta^\top$, then

$$[\mu_S(t)\,dt+d\beta(t)]-[r(t)\,dt+d\beta(t)] = \mu_S(t)-r(t) = \theta(t)^\top\sigma_S(t)$$
$$=\;-\frac{1}{U_c^{\Pi^\Lambda}(t,\hat z(t),M(t))}\nabla_{z,M}U_c^{\Pi^\Lambda}(t,\hat z(t),M(t))^\top(\sigma_z(t),M(t)\sigma_M(t))^\top\sigma_S(t)$$
$$=\;\left[-\frac{\hat C(t)\frac{\partial}{\partial C}U_c^{\Pi^\Lambda}(t,R^1(\hat C(t)),M(t))}{U_c^{\Pi^\Lambda}(t,R^1(\hat C(t)),M(t))}\right]\sigma_C(t)^\top\sigma_S(t)$$
$$+\left[-\frac{M(t)\frac{\partial}{\partial M}U_c^{\Pi^\Lambda}(t,R^1(\hat C(t)),M(t))}{U_c^{\Pi^\Lambda}(t,R^1(\hat C(t)),M(t))}\right]\sigma_M(t)^\top\sigma_S(t).$$

In other words, *the excess rate of return on the stock, in real terms, can be decomposed into two terms, the first being proportional to the covariance between the stock price and the production capacity (or installed capital) and the second proportional to the covariance between the stock price and the money supply.* What about the constants of proportionality? Corollary 6.5

and (3.14) imply that we may think of the firm's manager as maximizing $\hat{U}(t, K, L; \Lambda)$, the utility of operating capital and labour, so we might call the constants of proportionality, s_C and s_M, the sensitivity to capacity risk and to monetary risk, respectively, of the manager. In fact, writing his utility at $(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}))$ as $\hat{U}(t, C, M) := \hat{U}(\mathcal{T}, K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T}); \Lambda)$, assuming $(K^\Lambda(\mathcal{T}), L^\Lambda(\mathcal{T})) \notin \mathrm{bdy}(Q(M))$, we obtain

$$s_C = -\frac{C\hat{U}_{C,C}}{\hat{U}_C} - C\frac{R_{C,C} + R_{C,K}K^\Lambda_C + R_{C,L}L^\Lambda_C}{R_C},$$

i.e. *the sensitivity is the coefficient of relative capacity risk aversion (a measure of the curvature of $\hat{U}$) adjusted by a factor depending on the production function.* Similarly

$$s_M = -\frac{M\hat{U}_{M,M}}{\hat{U}_M} - M\frac{R_{K,K}K^\Lambda_M + R_{K,L}L^\Lambda_M}{R_K}.$$

We can also investigate the nominal excess rate of return. From (2.16), (2.12) and Lemma 2.2,

$$\mu_{\tilde{S}}(t) - \tilde{r}(t) = \sigma_{\tilde{S}}(t)^\top(\theta(t) - \sigma_q(t)) = -\sigma_{\tilde{S}}(t)^\top\psi^q(t),$$

i.e. *the nominal excess rate of return is the negative of the covariance between the nominal return on the stock price $\frac{d\tilde{S}}{\tilde{S}}$ and the real return on the deflated nominal bond $\frac{d\mathcal{M}^q}{\mathcal{M}^q}$* since $\mathcal{M}^q(t) = \zeta(t)B(t)$, cf. Remark 2.3. Hence as expected, the more the stock price is positively correlated with the bond price, the lower the (nominal) excess rate of return. □

We can now find the market capitalization, $NS(t)$, from (3.9) using (6.4), i.e.

$$\begin{aligned} NS(t) &= \frac{1}{\epsilon(t)}E^\circ\Big\{\epsilon(T)G(\hat{C}(T)) \\ &\quad + \int_t^T \epsilon(s)[R(\hat{C}(s), \hat{K}(s), \hat{L}(s)) - \tilde{r}(s)\hat{K}(s) - w'(s)\hat{L}(s)]\,ds - \int_{[t,T)} \epsilon(s)\,d\hat{\nu}(s)\Big|\mathcal{F}_t\Big\}. \end{aligned} \tag{6.41}$$

It follows that $\epsilon(t)NS(t) + \int_0^t \epsilon[R - \tilde{r}\hat{K} - w'\hat{L}]\,ds - \int_{[0,t)} \epsilon\,d\hat{\nu} := \mathcal{M}^{NS}(t)$ is a P°-Brownian martingale, so $d\mathcal{M}^{NS}(t) = \epsilon(t)NS(t)\psi^{NS}(t)^\top\,dW^\circ(t)$ for some process ψ^{NS} such that $\epsilon NS\psi^{NS}$ is a.s. square integrable. Hence

$$NS\,d\epsilon + \epsilon S\,dN + \epsilon N\,dS + \epsilon[R - \tilde{r}\hat{K} - w'\hat{L}]\,dt - \epsilon\,d\hat{\nu} = \epsilon NS(\psi^{NS})^\top\,dW^\circ,$$

i.e., cf. (3.1),

$$dS(t) + \delta(t)\,dt = S(t)[r(t)\,dt + d\beta(t)] + S(t)\psi^{NS}(t)^\top\,dW^\circ(t).$$

It follows, cf. (2.20) and (2.16), that $\sigma_S = \psi^{NS}$ and $\mu_S = r + \theta^\top\sigma_S$.

Remark 6.11 Observe that S is continuous but NS is only continuous after $t = 0$. $NS(0+) = NS(0) + \hat{\nu}(0+)$, i.e. the capitalization after a possible initial share sale to attain the equilibrium value. □

To find $N(t)$ we observe that (cf. (3.1))

$$\frac{dN}{N} = \frac{d\hat{\nu}}{NS}$$

so

$$(6.42) \qquad \begin{aligned} N(t) &= N(0+) \exp \int_{[0,t)} \frac{1}{NS(s)} \, d\hat{\nu}^+(s), \\ N(0+) &= \begin{cases} N(0) \frac{NS(0+)}{NS(0)} & \text{if } N(0) \neq 0 \\ \frac{\hat{\nu}^+(0)}{S(0)} & \text{if } N(0) = 0. \end{cases} \end{aligned}$$

For the case $N(0) = 0$, an initial public offering (IPO) is made at time 0; $S(0)$ is the arbitrary share price at which the IPO is made. In any case we now have N as well as $S = (NS)/N$, $\delta = [R - \tilde{r}\hat{K} - w'\hat{L}]/N$.

The total wealth in the economy is $X(t) := \sum_j X^j(t) = NS(t) + M(t)$, cf. (4.3), (6.7), (6.10) - (6.13).

Finally μ_D is defined from (2.15), so $\mu_D = r + \sigma_D^\top \theta$. Observe that the dynamics of D are the same as those of $\mathcal{B}$ whenever (σ_q, σ_S) is non-singular, so $D \equiv q(0)\mathcal{B}$ if (σ_q, σ_S) is non-singular for almost all (t, ω).

7 Existence

At this point we are able to provide a sufficient condition for existence of an equilibrium. We remind the reader that to this point we have made assumptions on the production and scrap value functions, cf. (3.4), (3.7), (6.22), (6.23) (this implies the continuity assumed in (3.18)), and on the utility functions, cf. (4.10), (4.12), (4.9) and either (6.34) or (6.35). The condition (3.7)(iii) is an assumption that the market is rational and defines $NS(T)$. In any case, *these assumptions are in force throughout this section.*

At the beginning, the number of shares outstanding, the operating capital of the firm (in nominal terms), the money held by the agents (in nominal terms) and the portfolio of the agents are fixed, so we **assume** that $N(0), \tilde{K}(0), \Phi^j(0), \tilde{m}^j(0)$ are given (exogenous) such that

$$(7.1) \qquad \begin{cases} \sum_j \Phi^j(0) = (0, \tilde{K}(0), 0, N(0)), \\ \sum_j \tilde{m}^j(0) = \tilde{M}(0) - \tilde{K}(0). \end{cases}$$

They determine the initial wealths of the agents. Note that (7.1) implies that for the initial holdings at time 0 five of the markets clear, i.e. (6.7) and (6.10) - (6.13). The other three markets deal in rates. At time 0 there may be a rebalancing of the moneyed portfolios and $\tilde{K}$ to bring them to their optimal values; this will not change the wealths. N is lcrl, so we may have $N(0) \neq N(0+)$. From (7.1) and (4.3) it follows that

$$(7.2) \qquad \sum_j X^j(0) = NS(0) + M(0).$$

Observe that X^j is continuous at 0 for $j \notin \mathcal{J}^c$, but for $j \in \mathcal{J}^c$ it is only right continuous, $X^j(0+) = X^j(0) + w^c(0+)$. In fact if $N(0) \neq N(0+)$ then there is an instantaneous capacity expansion at time 0 generating a jump in the wealth of the construction sector (in response to the capacity expansion provided at time 0).

We now construct the various endogenous processes; the sole exogenous process is the money supply, M (although μ_C and σ_C are also exogenous). Note that aside from $\hat{\nu}$ and $\hat{C}$ all processes depend on Λ used to define the representative agent's utility function! We will not show this in the notation.

(i) Define $\hat{\nu}$ from (3.15); this gives $\hat{C}(t) := C(t+;\hat{\nu})$.

(ii) Define $(K^\Lambda(t), L^\Lambda(t))^\top := \Pi^\Lambda(t, \hat{z}(t), M(t))$, the solution of (P_Λ), cf. Lemma 6.4 and Definition 6.6, with $\hat{z}(t) := R^1(\hat{C}(t))$.

(iii) Define $\zeta,\ \tilde{r},\ w'$ from (6.16), i.e.

$$\nabla U(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t); \Lambda) = \zeta(t)(1, \tilde{r}(t), w'(t))^\top$$

and define

$$w^c(t) := \int_{[0,t)} \Big(J^c - I^u(s, \zeta(s); \Lambda)\Big)^{-1} d\hat{\nu}(s).$$

(iv) Define $r,\ \beta,\ \theta$ from (6.39) and (2.18), i.e. with $\mathcal{L}$ as in (6.25), $\sigma_z(t) := \hat{C}(t)R^1_C(\hat{C}(t))\sigma_C$ and $\rho_z(t) := R^1_C(\hat{C}(t))f_C$,

$$\begin{aligned} r(t) &:= -\frac{1}{\zeta(t)}\left[\frac{\partial}{\partial t} + \mathcal{L}\right] U_c^{\Pi^\Lambda}(t, \hat{z}(t), M(t)), \\ \theta(t)^\top &:= -\frac{1}{\zeta(t)}(\nabla_{z,M} U_c^{\Pi^\Lambda}(t, \hat{z}(t), M(t))^\top (\sigma_z(t)\,,\, M(t)\sigma_M(t))^\top, \end{aligned}$$

(with average values used at discontinuities of any derivatives, cf. (6.36)) and β given by (6.40).

(v) Define $q,\ \mu_q, \sigma_q$ from Lemma 2.2.

(vi) Define $N\!S$ from (6.41) and μ_S, σ_S correspondingly.

(vii) Define N from (6.42).

(viii) Define $S(t) := N\!S(t)/N(t)$, $\delta(t) := [R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)) - \tilde{r}(t)K^\Lambda(t) - w'(t)L^\Lambda(t)]/N(t)$. In case $N(0) = 0$, define $S(0) = N\!S(0+)/N(0+)$.

Set

$$g(t) := \int_{[0,t)} q(s)\, d\tilde{M}(s) + [q, \tilde{M}](t) = M(t) - M(0) - \int_{[0,t)} M(s)\, \frac{dq(s)}{q(s)}. \tag{7.3}$$

(ix) Define $w^j(t) := \int_0^t w'(s)\,ds$ if $j \in \mathcal{J}^p$, $w^j(t) = w^c(t)$ if $j \in \mathcal{J}^c$ and $w^j(t) = 0$ if $j \in \mathcal{J}^w$. Then set (cf. (7.1))

$$\begin{aligned} X^j(0) &:= \phi^j_{\mathcal{B}}(0) + \phi^j_B(0)q(0) + \phi^j_D(0)q(0) + \phi^j_S(0)S(0) + \tilde{m}^j(0)q(0), \\ g^j(t) &:= \frac{X^j(0) + E\int_{[0,T)} \zeta(s)\,dw^j(s)}{M(0) + N\!\!S(0) + \sum_j E\int_{[0,T)} \zeta(s)\,dw^j(s)]} g(t). \end{aligned}$$

(x) Define

$$\xi^j := X^j(0) + E\int_{[0,T)} \zeta(s)\,[dw^j(s) + dg^j(s)],$$

η^j by (4.17), $\hat{\mathcal{C}}^j$ by (4.18) and $\hat{\Phi}^j$ by Theorem 4.4 with $\bar{x} = I^{V^j}(\eta^j\zeta(T))$. Recall that σ_D was already defined in terms of $\ker(\Sigma')$. Also define $\mu_D = r + \sigma_D^\top\theta$.

We require another assumption. g, as defined in (7.3), is the change in the nominal money supply, valued in real terms, over the time interval $[0,t)$. As this money is passed to the agents via g^j we have losses when g decreases, but these should not be too great.
Assumption: The Central Bank acts so that

$$\xi := N\!\!S(0) + M(0) + E\int_{[0,T)} \zeta(t)\Big[\sum_j dw^j(t) + dg(t)\Big] > 0, \tag{7.4}$$

i.e. the expected total initial wealth plus expected earnings are not all lost due to fiscal and monetary policies. Without this there would be no motivation to work, no consumption and in fact no economy.

Remark 7.1 When will (7.4) hold? By using (6.41), (7.3) and Lemma 2.2, we can find the equivalent condition

$$\begin{aligned} &E\{\zeta(T)[M(T) + G(\hat{C}(T))]\} \;+\; E\int_0^T \zeta(t)\Big[\pi(t)^\top I^U(t, \eta^\Lambda\zeta(t)\pi(t); \Lambda)\,dt + I^u(t, \eta^\Lambda\zeta(t), \Lambda)\,dw^c(t)\Big] \\ &\qquad\qquad > \;\; -E\int_0^T \zeta(t)q(t)[\tilde{M}(t)\sigma_{\tilde{M}}^\top]\theta(t)\,dt. \end{aligned} \tag{7.5}$$

In other words, the expected terminal wealth in the economy (= money supply plus scrap value) plus expected value of consumption must exceed the expected price of risk times the risk in the nominal money supply valued in real terms, summed over time, i.e. *the expected benefit of the economy must exceed the expected cost of risk in the nominal money supply.* But, cf. **(iv)** above, $\zeta\theta$ is proportional to $(\hat{C}R^1(\hat{C})\sigma_C, M\sigma_M)$, so (7.4) holds if $\sigma_C, \sigma_{\tilde{M}}$ are sufficiently small. Hence two things must happen to guarantee (7.4) (which will eventually lead to equilibrium): the volatility of the money supply must not be too large (it may be that the monetary authorities can manage this) but also the volatility of the productive capacity must not be too large. Perhaps the government can encourage this by following consistent policies, but there are other extraneous factors, e.g. natural disasters, contract disputes, which can play a destructive role because they increase the volatility. □

In our setup it is possible that with positive probability the firm will go bankrupt at some time $t > 0$ i.e. $N\!\!S(t) = 0$. We will say more about this in Remark 7.3. From now on we **assume** that for each $t > 0$, $N\!\!S(t) > 0$ a.s., i.e. the economy does not collapse almost surely.

Theorem 7.2 *In addition to the standing assumptions, assume (7.4). If there exists* $\Lambda = (\lambda_1, \ldots, \lambda_J)$ *such that* $\lambda_j \eta^j = 1$*, then with this* Λ *the above choice of parameters in* **(i)** - **(x)** *provides an equilibrium.*

Proof: Observe that (5.2), the fact that $I^V(y) = \sum_j I^{V^j}(y/\lambda_j)$ and our assumption about λ_j imply that η^Λ, the solution of (5.9), is 1.

Let us show that (2.2), (2.4), (2.6), (2.10), (2.16), (2.17), (2.19), (2.21), (3.5), (3.8), (4.1) and (4.6) hold. Let $\kappa_R := \kappa_{R^1} + R^2(\kappa_M, 1)$.

In the existence theorem, Theorem 7.4, we will take Λ in a compact set S_Λ. Consider (5.6) with $\bar{\mathcal{C}} = (R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t)) \in [0, \kappa_R] \times [0, \kappa_M] \times [0, J^p]$. As in Lemma 4.3 of [6] using $(\Lambda, t, R^1) \in S_\Lambda \times [0, T] \times [0\kappa_{R^1}]$ rather than just Λ, $z = (K, L)^\top$, $Z = (M(t), J^p)^\top := Z(t) \leq (\kappa_M, J^p)^\top$, $v(z) = R^2(z)$, $\tilde{g}(z) = (v(z), (Z(t) - z)^\top)^\top := \tilde{g}(z, t)$, $g^j(x, \Lambda, t) = I^{U^j_{A^j}}(t, \frac{\nabla U(t,x;\Lambda)}{\lambda_j})$, $\tilde{g}^j(z, \Lambda, t, R^1) = g^j((R^1, 0, 0)^\top + \tilde{g}(z, t), \Lambda, t)$, $u(\tilde{g}(z, t); \Lambda, t, R^1) := U(t, R^1 + v(z), Z(t) - z; \Lambda)$, $u^j(t, \tilde{g}^j(z, \Lambda, t, R^1)) = U^j(t, \tilde{g}^j(z, \Lambda, t, R^1))$ it follows that $(R^1 + R^2(K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t))$ lies in a compact subset of $\tilde{A}$ for $(\Lambda, t, R^1) \in S_\Lambda \times [0, T] \times [0, \kappa_M]$. Moreover the proof of [6], Lemma 4.4, shows that $\nabla U(\cdot, \cdot; \cdot)$ is continuous, so for $s = c, m, l$

$$\begin{aligned} 0 &< \min_{t,\omega} U_s(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t); \Lambda), \\ &\leq \max_{t,\omega} U_s(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t); \Lambda) \\ &= \kappa_s. \end{aligned}$$

This yields (2.4), (2.19) and (3.5).

Equation (2.16) follows from the definition of μ_q and μ_S, cf. **(v), (vi)**. We will now show that θ as defined in **(iv)** is bounded, so (2.17) holds. Note that $M\sigma_M$ is bounded by assumption and, cf. (6.24), that σ_z is also bounded. It suffices to show that $\{(U^{\Pi^\Lambda}_{cc} R^2_K - U^{\Pi^\Lambda}_{cm}, U^{\Pi^\Lambda}_{cc} R^2_L - U^{\Pi^\Lambda}_{cl}) \nabla \Pi^\Lambda + (U^{\Pi^\Lambda}_{cc}, U^{\Pi^\Lambda}_{cm})\}$ is bounded. The compactness used above and the fact that $U_c \in C^{1,2}([0, T] \times \tilde{A})$ yields this; hence θ is bounded.

Similarly $|r|$ as defined in **(iv)** is integrable. Proposition 6.9 and following imply that for β as defined in **(iv)**, $\|\beta\|_T < \infty$ a.s., so (2.2) holds.

Now Lemma 2.2 implies (2.21) and (2.10).

Let us establish (2.6). The boundedness of G and R together with (6.41) imply that $N\!\!S(t)$ is bounded above by a constant $\kappa_{N\!\!S}$, hence $S(t) \leq \kappa_{N\!\!S}(1 + \hat{C}(t))/N(0+)$ and hence $\tilde{S}$ is bounded a.s. ($q > 0$ a.s.). Since $N\delta \leq R$ is bounded, this also establishes (3.8).

To obtain the integrability in (2.6), we need $\zeta(t) N\!\!S(t) > 0$ a.s. for $t > 0$. This follows from the assumption that the firm does not go bankrupt since $\zeta(t) > 0$. Continuity of $\zeta N\!\!S$ on $(0, T]$ implies $\inf_{0 \leq t \leq T} \zeta(t) N(t+) S(t) > 0$ a.s. Since $\zeta N\!\!S \psi^{N\!\!S}$ is a.s. square integrable, then so is ψ^{NS}, hence σ_S, hence $\tilde{S}\sigma_{\tilde{S}}$ ($\tilde{S}$ is a.s. bounded). As μ_S is integrable a.s. then so is $\tilde{S}\mu_{\tilde{S}}$. This establishes (2.6).

Let us establish (4.1). Preceding Remark 6.2 we established that $J^c - I^u(t, \zeta(t); \Lambda) > 0$. Since the I^j are continuous, then so is $I^u(\cdot, \cdot; \cdot)$ hence is bounded away from zero on the compact set $[0,T] \times [k_\zeta, \kappa_\zeta] \times \{\text{compact}\}$ if $\Lambda \in \{\text{compact}\}$, the set used in the existence theorem to follow. This implies (4.1) since $E\hat{\nu}(T) < \infty$.

We now turn to (4.6). From (7.3)

$$dg = M[(\mu_M - \mu_q)\, dt + (\sigma_M - \sigma_q)^\top\, dW - d\beta]$$

so $E\int_{[0,T)} \zeta\, dg < \infty$ is well defined thanks to (2.2), (2.10) and (2.19) and the bounds on M, μ_M, σ_M. Hence the same is true for $E\int_{[0,T)} \zeta\, dg^j$. Define

$$\varsigma^j_\Lambda := \frac{X^j(0) + E\int_{[0,T)} \zeta(t)\, dw^j(t)}{M(0) + NS(0) + E\sum_j \int_{[0,T)} \zeta(t)\, dw^j(t)}, \tag{7.6}$$

so, cf. **(ix)**, $g^j = \varsigma^j_\Lambda g$. From (7.2) we have $\sum_j \varsigma^j_\Lambda = 1$. Then (7.4) implies that (cf. **(x)**)

$$\begin{aligned} \xi^j &= X^j(0) + E\int_{[0,T)} \zeta(t)\,[dw^j(t) + dg^j(t)] \\ &= \varsigma^j_\Lambda \Big\{ M(0) + NS(0) + E\sum_j \int_{[0,T)} \zeta(t)\, dw^j(t) + E\int_{[0,T)} \zeta\, dg(t) \Big\} = \varsigma^j_\Lambda \xi > 0, \end{aligned} \tag{7.7}$$

i.e. (4.6) holds.

The relations (6.1)-(6.5) and (6.9) hold by construction. Now (6.6)-(6.8) follow from (5.2) and (6.16), cf. **(iii)**.

It remains to show that the financial markets clear, i.e. that (6.10) - (6.13) hold. To do so, we consider the total wealth, $X(t) := \sum_j X^j(t)$. Applying (4.8) to each agent, adding, using (5.8), (6.4), (6.6), (6.7), (6.8), (6.9), we obtain

$$\begin{aligned} d[\epsilon(t)X(t)] &= \epsilon(t)[dg(t) - \tilde{r}(t)M(t)\, dt] + \epsilon(t)[d\hat{\nu}(t) - \delta(t)N(t)\, dt] \\ &\quad + \epsilon(t)\Big(\sum_j \hat{\phi}^j_B(t)B(t) + M(t) - K^\Lambda(t), \sum_j \hat{\phi}^j_D(t)D(t), \sum_j \hat{\phi}^j_S(t)S(t) \Big)\Sigma(t)\, dW^\circ(t). \end{aligned}$$

However

$$\begin{aligned} d[\epsilon(t)M(t)] &= \epsilon(t)M(t)\{[\mu_M(t) - r(t) - \sigma_M^\top(t)\theta(t)]\, dt + \sigma_M^\top(t)\, dW^\circ(t) - d\beta(t)\} \\ &= \epsilon(t)M(t)[-\tilde{r}(t)\, dt + \sigma_q^\top(t)\, dW^\circ(t)] \\ &\quad + \epsilon(t)M(t)\{[\tilde{r}(t) + \mu_M(t) - r(t) - \sigma_M^\top(t)\theta(t)]\, dt \\ &\qquad + [\sigma_M(t) - \sigma_q(t)]^\top\, dW^\circ(t) - d\beta(t)\} \\ &= \epsilon(t)M(t)[-\tilde{r}(t)\, dt + \sigma_q^\top(t)\, dW^\circ(t)] \\ &\quad + \epsilon(t)M(t)\{[\mu_M(t) - \mu_q(t)]\, dt + [\sigma_M(t) - \sigma_q(t)]^\top\, dW(t) - d\beta(t)\} \\ &= \epsilon(t)M(t)[-\tilde{r}(t)\, dt + \sigma_q^\top(t)\, dW^\circ(t)] + \epsilon(t)\, dg(t) \end{aligned}$$

thanks to (2.16) and (7.3). From (6.41) and following we conclude that

$$d[\epsilon(t)NS(t)] = \epsilon(t)[d\hat{\nu}(t) - \delta(t)N(t)\,dt] + \epsilon(t)NS(t)\sigma_S^\top(t)\,dW^\circ(t).$$

Combining the last three results we obtain

$$\begin{aligned}(7.8)\ &\epsilon(t)[X(t) - NS(t) - M(t)]\\ &= \int_0^t \epsilon(s)\Big(\sum_j \hat{\phi}_B^j(s)B(s) - K^\Lambda(s),\ \sum_j \hat{\phi}_D^j(s)D(s), (\sum_j \hat{\phi}_S^j(s) - N(s))S(s)\Big)\Sigma(s)\,dW^\circ(s).\end{aligned}$$

Moreover (6.23)(vi), (5.6) and Proposition 5.2 imply that

$$\begin{aligned} V_x(X(T);\Lambda) &= \zeta(T) = U_c(T, R(\hat{C}(T), K^\Lambda(T), L^\Lambda(T)), M(T) - K^\Lambda(T), J^p - L^\Lambda(T);\Lambda) \\ (7.9) \qquad &\leq V_x(G(\hat{C}(T)) + M(T);\Lambda) \end{aligned}$$

or

$$X(T) \geq G(\hat{C}(T)) + M(T).$$

So the zero-mean martingale in (7.8) is non-negative at time T, hence is zero for all time. The same conclusion follows from (6.23)(v) using $u_l(T, J^c - \frac{d\hat{\nu}}{dw^c};\Lambda)$, but the inequalities are reversed, i.e. the martingale is non-positive at time T. It follows that

$$(7.10) \qquad \left(\sum_j \hat{\phi}_B^j(t)B(t) - K^\Lambda(t),\ \sum_j \hat{\phi}_D^j(t)D(t),\ (\sum_j \hat{\phi}_S^j(t) - N(t))S(t)\right)\Sigma(s) = (0,0).$$

If $\Sigma'(t)$ is invertible then $\sigma_D(t) = 0$ (recall that $\sigma_D \in \ker\Sigma'$) so (7.10) becomes

$$(7.11) \qquad \left(\sum_j \hat{\phi}_B^j(t)B(t) - K^\Lambda(t), (\sum_j \hat{\phi}_S^j(t) - N(t))S(t)\right)\Sigma'(t) = (0,0).$$

Then (6.11), (6.13) hold; moreover (7.10), (6.11) and (6.13) imply

$$\begin{aligned} 0 &= X(t) - NS(t) - M(t) \\ (7.12) \qquad &= (\sum_j \hat{\phi}_S^j(t) - N(t))S(t) + \sum_j \hat{\phi}_{\mathcal{B}}^j(t)\mathcal{B}(t) + \sum_j \hat{\phi}_B^j(t)B(t) + \sum_j \hat{\phi}_D^j(t)D(t) - K^\Lambda(t) \\ &= \sum_j \hat{\phi}_{\mathcal{B}}^j(t)\mathcal{B}(t), \end{aligned}$$

i.e. (6.10) also holds.

If $\Sigma'(t)$ is not invertible then $\sigma_S(t) = \alpha(t)\sigma_q(t)$ for some α and (7.10) becomes

$$\sum_j \hat{\phi}_D^j(t)D(t)\sigma_D^\top(t) + \Big[\alpha(t)\sum_j(\hat{\phi}_S^j(t) - N(t))S(t) + \sum_j \hat{\phi}_B^j(t)B(t) - K^\Lambda(t)\Big]\sigma_q^\top(t) = (0,0),$$

with $\sigma_D(t)$ and $\sigma_q(t)$ independent so $\sum_j \hat{\phi}^j_D(t) = 0$ and

$$\alpha(t) = -\frac{\sum_j \hat{\phi}^j_B(t)B(t) - K^\Lambda(t)}{\sum_j (\hat{\phi}^j_S(t) - N(t))S(t)}.$$

Since

$$\begin{aligned} dS(t) + \delta(t)\,dt &= S(t)[r(t)\,dt + \alpha(t)\sigma_q^\top(t)\,dW^\circ(t) + d\beta(t)] \\ &= (1-\alpha(t))S(t)\frac{d\mathcal{B}(t)}{\mathcal{B}(t)} + \alpha(t)S(t)\frac{dB(t)}{B(t)} \end{aligned}$$

then we can replicate one share of the stock by holding $\phi^{\text{syn}}_B = \alpha S B^{-1}$ units of the nominal bond and $\phi^{\text{syn}}_{\mathcal{B}} = (1-\alpha)S\mathcal{B}^{-1}$ units of the real bond. Since by definition $N^{\text{syn}}(t) = \sum_j \hat{\phi}^j_S(t) - N(t)$, then

$$\sum_j \hat{\phi}^j_S(t) = N(t) + N^{\text{syn}}(t), \tag{7.13}$$

i.e. the market in shares clears.

Moreover (7.10) implies

$$K^\Lambda(t) = \sum_j \hat{\phi}^j_B(t)B(t) + \alpha(t)N^{\text{syn}}(t)S(t) = [\sum_j \hat{\phi}^j_B(t) + \phi^{\text{syn}}_B(t)N^{\text{syn}}(t)]B(t). \tag{7.14}$$

This states that the number of units of the nominal bond held as such, plus the number of units used to form the synthetic stock shares, equals $\hat{K}B^{-1}$. Hence the market in nominal bonds clears.

Returning to the first two equalities in (7.12), which hold irrespective of the invertibility of Σ', using $\sum_j \phi^j_D = 0$ and (7.13), (7.14), we obtain

$$0 = N^{\text{syn}}(t)S(t) + \sum_j \phi^j_{\mathcal{B}}(t)\mathcal{B}(t) + (\hat{K}(t) - \alpha(t)N^{\text{syn}}(t)S(t)) - \hat{K}(t),$$

i.e.

$$\sum_j \phi^j_{\mathcal{B}}(t)\mathcal{B}(t) = -(1-\alpha(t))N^{\text{syn}}(t)S(t) = -\phi^{\text{syn}}_{\mathcal{B}}(t)N^{\text{syn}}(t)\mathcal{B}(t),$$

or $\sum_j \phi^j_{\mathcal{B}}(t) + \phi^{\text{syn}}_{\mathcal{B}}(t)N^{\text{syn}}(t) = 0$. This states that the number of units of the real bond held as such, plus the number of units used to form the synthetic stock shares, equals zero. Thus all the markets clear. □

Remark 7.3 The issue of the firm's bankruptcy, i.e. the economy's collapse, is avoided in the alternative problem where the manager's discount $e^{-\mu_F}$ is replaced by ζ. We need to show that $\zeta(t)NS(t) > 0$ a.s. Indeed the dynamic programming principle tells us that $\hat{\nu}$ also maximizes

$$\begin{aligned} \mathcal{J}_{t,C(t;\hat{\nu})}(\nu) &:= \frac{1}{\zeta(t)}E\Big\{\zeta(T)G(C(T;\nu)) + \int_t^T \zeta(s)\tilde{R}(C(s;\nu),\tilde{r}(s),w'(s))\,ds \\ &\qquad - \int_{[t,T)} \zeta(s)\,d\nu(s)\,\Big|\,\mathcal{F}_t\Big\} \end{aligned}$$

over $\{\nu \in \mathcal{S}, \nu(s) \equiv \hat{\nu}(s) \text{ for } s \le t\}$. $\tilde{R}$ is defined in (3.10). Hence $\mathcal{J}_{t,C(t;\hat{\nu})}(\hat{\nu}) \ge \mathcal{J}_{t,C(t;\hat{\nu})}(\nu^t)$ where $\nu^t(s) = \hat{\nu}(t \wedge s)$. But

$$\mathcal{J}_{t,C(t;\hat{\nu})}(\nu^t) = \frac{1}{\zeta(t)} E\Big\{\zeta(T)G(C(T;\nu^t)) + \int_t^T \zeta(s)\tilde{R}(C(s;\nu^t), \tilde{r}(s), w'(s))\, ds \,\Big|\, \mathcal{F}_t\Big\} > 0 \text{ a.s.}$$

since $\hat{C}(0) > 0$ so $\hat{C}(t) > 0$ a.s. and $C(s;\nu^t) = \hat{C}(t)C^o(s)/C^o(t) > 0$ a.s. It follows that

$$\zeta(t)N\!S(t) = \mathcal{J}_{t,C(t;\hat{\nu})}(\hat{\nu}) \ge \mathcal{J}_{t,C(t;\hat{\nu})}(\nu^t) > 0 \text{ a.s.}$$

The difficulty with this approach is that step **(i)** now provides $\hat{\nu}$ as a function of ζ. Similarly $\hat{K}$ and $\hat{L}$ will depend on ζ, and for ζ we must solve a difficult fixed point problem:

$$\zeta(t) = U_c(t, R(\hat{C}(t), \hat{K}(t), \hat{L}(t)), M(t) - \hat{K}(t), J^p - \hat{L}(t); \Lambda).$$

□

We can now show that the above selection of parameters provides an equilibrium for at least one choice of Λ. Note that all the processes defined above, with the exception of $\hat{\nu}$, depend on Λ.

Theorem 7.4 *In addition to the standing assumptions, assume (7.4). Then there exists* $\Lambda = (\lambda_1, \ldots, \lambda_J) \in \Re^J_{++}$ *such that an equilibrium exists.*

Proof: According to Theorem 7.2 it suffices to show that $\eta^j \lambda_j = 1$, where η_j are defined by (4.17). We shall use a fixed point argument as in [18], Theorem 11.1, to establish the result.

We begin by finding a new representation for ξ^j. Using

$$d(\epsilon M) = \epsilon M[-r\, dt - d\beta] + \epsilon\, dM,$$

and (7.3), we find

$$\begin{aligned} \epsilon\, dg &= \epsilon\, dM - \epsilon M[(r - \tilde{r})\, dt + \sigma_q^\top\, dW^o + d\beta] \\ &= d(\epsilon M) + \epsilon M \tilde{r}\, dt - \epsilon M \sigma_q^\top\, dW^o, \\ E\int_0^T \zeta(t)\, dg(t) &= E\zeta(T)M(T) - M(0) + E\int_0^T \zeta(t)M(t)\tilde{r}(t)\, dt. \end{aligned}$$

This, (7.7) and the definition of w^j yield

$$\begin{aligned} \xi^j \;=\; \varsigma^j_\Lambda\Big[& N\!S(0) + EU_c\Big(T, R(\hat{C}(T), K^\Lambda(T), L^\Lambda(T)), M(T) - K^\Lambda(T), J^p - L^\Lambda(T); \Lambda\Big)M(T) \\ & + E\int_0^T U_m(\Big(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t); \Lambda\Big)M(t)\, dt \\ & + J^p E\int_0^T (U_l\Big(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t) - K^\Lambda(t), J^p - L^\Lambda(t); \Lambda\Big)\, dt \end{aligned}$$

$$
\begin{aligned}
&\quad +J^c E \int_{[0,T)} \zeta(t)\, dw^c(t)\Big] \\
&= \varsigma^j_\Lambda E\Big[\int_0^T \nabla U\Big(t, R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t)-K^\Lambda(t), J^p - L^\Lambda(t);\Lambda\Big)\cdot \\
&\qquad\qquad \Big(R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t)-K^\Lambda(t), J^p - L^\Lambda(t)\Big)^\top dt \\
&\quad + \int_{[0,T)} U_c\Big(T, R(\hat{C}(T), K^\Lambda(T), L^\Lambda(T)), M(T)-K^\Lambda(T), J^p - L^\Lambda(T);\Lambda\Big) I^u(t\zeta(t);\Lambda)\, dw^c(t) \\
&\quad + U_c\Big(T, R(\hat{C}(T), K^\Lambda(T), L^\Lambda(T)), M(T)-K^\Lambda(T), J^p - L^\Lambda(T);\Lambda\Big)\Big(G(\hat{C}(T)) + M(T)\Big)\Big]
\end{aligned}
$$

using (6.41) to represent $\mathcal{NS}(0)$ and the definition of w^c.

With this ξ^j, suppressing the middle arguments in ∇U and in U_c, (4.17) becomes

$$
\begin{aligned}
&E\Big\{\int_0^T \nabla U(t;\Lambda)^\top \Big[I^{U^j_{Aj}}(t, \tfrac{1}{\lambda_j}\nabla U(t;\Lambda)) - \varsigma^j_\Lambda\Big(R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t)-K^\Lambda(t), J^j - L^\Lambda(t)\Big)^\top\Big]\, dt \\
&\qquad + \int_{[0,T)} U_c(T;\Lambda)\Big[I^{u^j_{aj}}(t, \tfrac{1}{\lambda_j}U_c(t;\Lambda)) - \varsigma^j_\Lambda I^u(t, U_c(t;\Lambda);\Lambda)\Big]\, dw^c(t) \\
&\qquad + U_c(T;\Lambda)\Big[I^{V^j}(\tfrac{1}{\lambda_j}U_c(T;\Lambda)) - \varsigma^j_\Lambda\Big(G(\hat{C}(T)) + M(T)\Big)\Big]\Big\} = 0
\end{aligned} \tag{7.15}
$$

We denote the left side of (7.15) by $\mathcal{K}^j(\Lambda)$. It is defined on $\Re^J_{++}$. We normalize Λ to lie in $\mathcal{U}_+ := \{\Lambda \in \Re^J_{++} : \sum_j \lambda_j = 1\}$; scaling Λ does not affect the choice of (K^Λ, L^Λ) or any of the parameters in the model, cf. [18] for a discussion. We extend the definition of $\mathcal{K}^j$ to $\mathcal{U} := \{\Lambda \in \Re^J_+ : \sum_j \lambda_j = 1\}$ by adopting the convention that $I^{U^j_{Aj}}(t, \frac{1}{\lambda_j}y) = 0$, $I^{u^j_{aj}}(t, \frac{1}{\lambda_j}y) = 0$, $I^{V^j}(\frac{1}{\lambda_j}x) = 0$, if $\lambda_j = 0$, for $j = 1, \ldots, J$, $y \in \Re^3_+$, $x \in \Re_{++}$. Then $\lambda_i \mapsto I^{U^j}(t, \frac{1}{\lambda_j}y)$ is continuous on $[0,\infty)$ for each t, y, and similarly for the others.

We claim that $\mathcal{K}^j$ is continuous on $\mathcal{U}$. Certainly $I^U(t, y;\Lambda)$ as defined in (5.2), $I^V(x;\Lambda)$, $U(t, \mathcal{C};\Lambda)$ and $V(x;\Lambda)$ are continuous in all arguments including $\Lambda \in \mathcal{U}$. The continuity of Π^Λ established in Proposition 6.7 extends to $\Lambda \in \mathcal{U}$. It follows that ∇U, hence $\mathcal{K}^j$ is continuous in $\Lambda \in \mathcal{U}$.

Any solution of (7.15) in $\mathcal{U}$ must lie in $\mathcal{U}_+$. In fact, if say $\lambda_j = 0$, then

$$
\begin{aligned}
\mathcal{K}^j(\Lambda) &= -\varsigma^j_\Lambda E\Big\{\int_0^T \nabla U(t;\Lambda)^\top \Big(R(\hat{C}(t), K^\Lambda(t), L^\Lambda(t)), M(t)-K^\Lambda(t), J^p - L^\Lambda(t)\Big)^\top dt \\
&\qquad + \int_{[0,T)} I^u(t, U_c(t;\Lambda);\Lambda)\, dw^c(t) + U_c(T;\Lambda)\Big(G(\hat{C}(T)) + M(T)\Big)\Big\} < 0
\end{aligned}
$$

because $\nabla U(t;\Lambda) \in \Re^3_{++}$ and (3.10), (3.4)(ii),(iv) imply that at least one of $R, M-K, J-L$ is positive (they are all non-negative). We take $\varsigma^j_\Lambda > 0$ for otherwise the corresponding agent would be absent from the economy.

Recall that $\sum_j \varsigma^j_\Lambda = 1$, so

$$\begin{aligned}
&\sum_j \mathcal{K}^j(\Lambda)\\
&= E\Big\{\int_0^T \nabla U(t;\Lambda)^\top \Big[I^U(t,\nabla U(t;\Lambda);\Lambda) - \Big(R(\hat{C}(t),K^\Lambda(t),L^\Lambda(t)), M(t)-K^\Lambda(t), J^p - L^\Lambda(t)\Big)^\top\Big]\,dt\\
&\qquad + \int_{[0,T)} U_c(T;\Lambda)\Big[I^u(t,U_c(t;\Lambda);\Lambda) - I^u(t,U_c(t;\Lambda);\Lambda)\Big]\,dw^c(t)\\
&\qquad + U_c(T;\Lambda)\Big[I^V(U_c(T;\Lambda)) - \Big(G(\hat{C}(T)) + M(T)\Big)\Big]\Big\}\\
&= 0
\end{aligned}$$

since $I^U(t,\nabla U(t;\Lambda);\Lambda) = \Big(R(\hat{C}(t),K^\Lambda(t),L^\Lambda(t)), M(t)-K^\Lambda(t), J^p - L^\Lambda(t)\Big)^\top$ and $I^V(U_c(T;\Lambda)) = X(T) = G(\hat{C}(T)) + M(T)$ according to (7.12). Now the proof of [18], Lemma 12.1, shows that there exists Λ such that $\mathcal{K}^j(\Lambda) = 0$. □

We make no claim about uniqueness, but we expect it to hold under mild conditions as in [18].

8 Appendix: proof of Proposition 6.9

Recall that $0 \le K^\Lambda(\bar{\mathcal{T}}(t)) < M(t)$ and $0 \le L^\Lambda(\bar{\mathcal{T}}(t)) < J^p$, cf (4.10)(x). To begin with we do not know that say $K^\Lambda(\bar{\mathcal{T}}(t))$ is a semimartingale, but if we take a smooth function $f : \Re \mapsto \Re$ with support in $(0,\infty)$ to eliminate the boundary behaviour of K^Λ at 0, then $f(K^\Lambda(\bar{\mathcal{T}}(t)))$ is a semimartingale since $f(K^\Lambda(\bar{\mathcal{T}}))$ is a piecewise smooth function, cf. Proposition 6.7 and [29], Theorem 2.1.

We define and calculate

$$\begin{aligned}
f_n(x) &:= x\mathbb{1}_{\{0\le x<\infty\}} - \tfrac{1}{2}\Big(x + \frac{\sin(n\pi x)}{n\pi}\Big)\mathbb{1}_{\{0<x<\frac{1}{n}\}},\\
f'_n(x) &= \mathbb{1}_{\{0<x<\infty\}} - \tfrac{1}{2}\Big(1+\cos(n\pi x)\Big)\mathbb{1}_{\{0<x<\frac{1}{n}\}},\\
f''_n(x) &= \tfrac{n\pi}{2}\sin(n\pi x)\mathbb{1}_{\{0<x<\frac{1}{n}\}},
\end{aligned} \tag{8.1}$$

Then $|f_n(x)-x|\mathbb{1}_{\{0<x<\infty\}} \le \frac{1}{2n}$ and $\int_0^\infty |f'_n(x)-1|\,dx = \frac{1}{n\pi}$. Also $f''_n(\cdot)$ has support on $(0,\frac{1}{n})$. Now abbreviate $K^\Lambda(t) := K^\Lambda(\bar{\mathcal{T}}(t)) = K^\Lambda(t,\hat{z}(t),M(t))$, $K^\Lambda_z(t) := K^\Lambda_z(t,\hat{z}(t),M(t))$, etc. Define $K^n(t) := f_n(K^\Lambda(t))$; it is a semimartingale since $K^\Lambda(t) < M(t)$. From (8.1) it follows that

$$\begin{aligned}
K^n_z(t) &= \mathbb{1}_{\{0<K^\Lambda(t)<M(t)\}}K^\Lambda_z(t) - \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(t))\Big)K^\Lambda_z(t),\\
K^n_{z,z}(t) &= \mathbb{1}_{\{0<K^\Lambda(t)<M(t)\}}K^\Lambda_{z,z}(t) - \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(t))\Big)K^\Lambda_{z,z}(t)\\
&\quad + \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}n\pi\sin(n\pi K^\Lambda(t))|K^\Lambda_z(t)|^2,
\end{aligned}$$

$$\begin{aligned} K^n_M(t) &= \mathbb{1}_{\{0<K^\Lambda(t)<M(t)\}}K^\Lambda_M(t) - \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(t))\Big)K^\Lambda_M(t),\\ K^n_{M,M}(t) &= \mathbb{1}_{\{0<K^\Lambda(t)<M(t)\}}K^\Lambda_{M,M}(t) - \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(t))\Big)K^\Lambda_{M,M}(t)\\ &\quad +\tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}n\pi\sin(n\pi K^\Lambda(t))|K^\Lambda_M(t)|^2,\\ K^n_{z,M}(t) &= \mathbb{1}_{\{0<K^\Lambda(t)<M(t)\}}K^\Lambda_{z,M}(t) - \tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(t))\Big)K^\Lambda_{z,M}(t)\\ &\quad +\tfrac{1}{2}\mathbb{1}_{\{0<K^\Lambda(t)<\frac{1}{n}\}}n\pi\sin(n\pi K^\Lambda(t))K^\Lambda_z(t)\,K^\Lambda_M(t), \end{aligned}$$

and similarly for K^n_t. Recall that the derivatives of K^Λ at $(t, b^k(t,M), M)$ are the averages of the values at $(t, b^k(t,M)+, M)$ and $(t, b^k(t,M)-, M)$, cf. (6.36). According to the change of variable formula, cf. [28], [29]

$$\begin{aligned} K^n(t) &= K^n(0) + \int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<M(s)\}}\Big[\frac{\partial}{\partial s}+\mathcal{L}\Big]K^\Lambda(s)\,ds\\ &\quad + \int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<M(s)\}}\nabla_{z,M}K^\Lambda(s)^\top(\sigma_z(s), M(s)\sigma_M(s))^\top\,dW(s)\\ &\quad + \int_{[0,t)} \mathbb{1}_{\{0<K^\Lambda(s)<M(s)\}}\nabla_{z,M}K^\Lambda(s)^\top(\rho_z(s)\,d\hat{\nu}^+(s), M(s)\,d\beta_M(s))^\top\\ &\quad + \sum_k \int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<M(s)\}}\tfrac{1}{2}[K^\Lambda_z(s,\hat{z}(s)+, M(s))\\ &\qquad\qquad -K^\Lambda_z(s,\hat{z}(s)-, M(s))]\mathbb{1}_{\{\hat{z}(s)=b^k(s,M(s))\}}\,d\ell^k(s) + \beta_{K^n}(t), \end{aligned} \tag{8.2}$$

where $\nabla_{z,M}K^\Lambda(t) := (K^\Lambda_z(t,\hat{z}(t),M(t)),\, K^\Lambda_M(t,\hat{z}(t),M(t)))^\top$ and

$$\begin{aligned} \beta_{K^n}(t) &= \tfrac{1}{4}\int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}}n\pi\sin(n\pi K^\Lambda(s))\times\\ &\qquad \Big[\|K^\Lambda_z(s)\sigma_z\|^2 + 2K^\Lambda_z(s)K^\Lambda_M(s)M(s)\sigma_z^\top(s)\sigma_M(s) + \|K^\Lambda_M(s)M(s)\sigma_M(s)\|^2\Big]\,ds\\ &\quad -\tfrac{1}{2}\int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}}\Big(1+\cos(n\pi K^\Lambda(s))\Big)\Big\{\Big[\frac{\partial}{\partial s}+\mathcal{L}\Big]K^\Lambda(s)\,ds + \nabla_{z,M}K^\Lambda(s)^\top\times\\ &\qquad \Big[(\sigma_z(s), M(s)\sigma_M(s))^\top\,dW(s) + (\rho_z(s)\,d\hat{\nu}^+(s), M(s)\,d\beta_M(s))^\top\Big]\\ &\qquad +\tfrac{1}{2}\sum_k[K^\Lambda_z(s,\hat{z}(s)+,M(s)) - K^\Lambda_z(s,\hat{z}(s)-,M(s))]\mathbb{1}_{\{\hat{z}(s)=b^k(s,M(s))\}}\,d\ell^k(s)\Big\} \end{aligned} \tag{8.3}$$

remembering that at any surface or curve of discontinuity, $z = b^k(t,M)$,

$$\nabla K^\Lambda(t,z,M) := \tfrac{1}{2}[\nabla K^\Lambda(t,z+,M) + \nabla K^\Lambda(t,z-,M)].$$

To pass to the limit as $n\to\infty$ in (8.2), we need K^Λ, K^Λ_t and the derivatives in z, M up to order two of K^Λ to be bounded on $\{(t,\hat{z}(t),M(t)) : t\in[0,T]\}$ a.s..

Certainly $\Pi^\Lambda(\bar{\mathcal{T}}(t))$ is bounded since it lies in $[0,M(t)]\times[0,J^p]$ and $M(t)\le\kappa_M$ (cf. Definition 6.6 for $\bar{\mathcal{T}}$ and $\bar{U}$). We now show that $\Pi^\Lambda_t(\bar{\mathcal{T}}(t)), \Pi^\Lambda_z(\bar{\mathcal{T}}(t)), \Pi^\Lambda_M(\bar{\mathcal{T}}(t)), \Pi^\Lambda_{zz}(\bar{\mathcal{T}}(t)), \Pi^\Lambda_{zM}(\bar{\mathcal{T}}(t))$ and $\Pi^\Lambda_{MM}(\bar{\mathcal{T}}(t))$ are bounded on $[0,T]\times\Omega$.

For $(\bar{\mathcal{T}}, \Lambda)$ in the compact set $[0,T] \times [0,\kappa_R] \times [0,\kappa_M] \times \{\Lambda\}$ it follows from [6], Lemma 4.3, that $(z + R^2(K^\Lambda, L^\Lambda), M - K^\Lambda, J^p - L^\Lambda)$ lies in a compact subset of $\tilde{A}$ (where ∇U is finite, cf. Lemma 5.1 (iii)). Hence $(z + R^2(K^\Lambda, L^\Lambda), M - K^\Lambda, J^p - L^\Lambda)$ stays uniformly away from the coordinate planes identified by $\cup_j \mathcal{I}_U^j$ where $\|\nabla_{c,m,l} U\|$ blows up. It follows that Π^Λ remains in a compact subset of $\mathrm{dom}(\nabla_\Pi \bar{U}(\bar{\mathcal{T}}, \cdot\,; \Lambda))$. Since the derivatives of Π^Λ of concern are piecewise continuous, cf. Proposition 6.7, the required boundedness follows.

We can now pass to the limit as $n \to \infty$ in (8.2) to obtain

$$
\begin{aligned}
K^\Lambda(t) \;&=\; K^\Lambda(0) + \int_0^t \left[\frac{\partial}{\partial s} + \mathcal{L}\right] K^\Lambda(s)\, ds + \int_0^t \nabla_{z,M} K^\Lambda(s)^\top (\sigma_z(s), M(s)\sigma_M(s))^\top\, dW(s) \\
&\quad + \int_{[0,t)} \nabla_{z,M} K^\Lambda(s)^\top (\rho_z(s)\, d\hat{\nu}^+(s), M(s)\, d\beta_M(s))^\top + \beta_K(t) \\
&\quad + \sum_k \int_0^t \tfrac{1}{2}[K_z^\Lambda(s, z(s)+, M(s)) - K_z^\Lambda(s, z(s)-, M(s))] \mathbb{1}_{\{z(s)=b^k(s,M(s))\}}\, d\ell^k(s),
\end{aligned} \tag{8.4}
$$

where

$$
\beta_K(t) := \lim_{n\to\infty} \tfrac{1}{4} \int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}} n\pi \sin(n\pi K^\Lambda(s)) \|K_z^\Lambda(s)\sigma_z(s) + K_M^\Lambda(s) M(s)\sigma_M(s)\|^2\, ds. \tag{8.5}
$$

Note that the last integral on the right side of (8.3) converges to 0 since the integrands are bounded and $\cap_n \{s : 0 < K^\Lambda(s) < \frac{1}{n}\} = \emptyset$, and for the stochastic integral

$$
\lim_{n\to\infty} \int_0^t \left| \tfrac{1}{2} \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}} \Big(1 + \cos(n\pi K^\Lambda(s))\Big) \right|^2 ds \le \lim_{n\to\infty} \int_0^T \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}}\, ds = 0
$$

by the bounded convergence theorem. The limit in (8.5) exists since the other terms in (8.4) are finite. Moreover the integrand in (8.5) is non-negative so the integral defines an increasing process and this property is inherited by the limit β_K.

Then β_K is a bounded variation process, so $K^\Lambda(t)$ is a continuous semimartingale (cf. Proposition 6.7 for the continuity). If $[K^\Lambda](t)$ denotes its quadratic variation process, then [22], Chapter 3, Theorem 7.1 implies that K^Λ has a local time at a, $\ell^K(t,a)$, such that

$$
\begin{aligned}
\lim_{n\to\infty} \tfrac{1}{4} \int_0^t \mathbb{1}_{\{0<K^\Lambda(s)<\frac{1}{n}\}} n\pi \sin(n\pi K^\Lambda(s))\, d[K^\Lambda](s) \;&=\; \lim_{n\to\infty} \tfrac{1}{2} \int_{-\infty}^{\infty} \mathbb{1}_{\{0<a<\frac{1}{n}\}} n\pi \sin(n\pi a) \ell^K(t,a)\, da \\
&=\; \lim_{n\to\infty} \tfrac{1}{2} \int_0^\pi \sin(y) \ell^K(t, \tfrac{y}{n\pi})\, dy \\
&=\; \ell^K(t,0) := \ell^K(t)
\end{aligned}
$$

since $\ell^K(t,a)$ is right continuous in a and $\int_0^\pi \sin(y)\, dy = 2$. Hence $\beta_K(t) = \ell^K(t)$. As a local time, ℓ^K is continuous, nondecreasing, adapted processes, so $\|\beta_K\|_T = \ell^K(T) < \infty$ a.s.

The case of L^Λ is treated similarly. □

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