Stability in $H^{1/2}$ of the sum of K solitons for the Benjamin–Ono equation

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This note proves the orbital stability in the energy space $H^{1/2}$ of the sum of widely spaced 1-solitons for the Benjamin–Ono equation, with speeds arranged so as to avoid collisions. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3032578]

I. INTRODUCTION

In this article we study the stability problem of the sum of K solitons for the Benjamin–Ono (BO) equation for $u(t,x):\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$,

$$u_t = -(\mathcal{H}\partial_x u + u^2)_x, \tag{1.1}$$

where $\mathcal{H}$ is the Hilbert transform operator defined by

$$\mathcal{H}f(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(y)}{x-y}dy.$$

Alternatively, if we denote $D=\sqrt{-\partial_x^2}$, we have $\mathcal{H}\partial_x=-D$ and we can rewrite the Cauchy problem for (1.1) as[1]

$$\begin{aligned} u_t &= (Du - u^2)_x, \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.2}$$

This equation is a model for one-dimensional long waves in deep stratified fluids (Refs. 1 and 18).

The BO equation is completely integrable and has infinitely many conserved quantities (Refs. 11 and 12). Two of them are the L^2 mass,

$$N(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 dx,$$

and the energy,

$$E(u) = \int_{\mathbb{R}} \frac{1}{2}uDu - \frac{1}{3}u^3 dx.$$

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[1]The Fourier transform is given by $\hat{f}(\xi)=(1/\sqrt{2\pi})\int_{\mathbb{R}}f(x)e^{-ix\xi}dx$, $f(x)=(1/\sqrt{2\pi})\int_{\mathbb{R}}\hat{f}(\xi)e^{ix\xi}d\xi$, so that $\widehat{\partial_x f}(\xi)=i\xi\hat{f}(\xi)$, $\widehat{\mathcal{H}f}(\xi)=-i\,\text{sgn}(\xi)\hat{f}(\xi)$, and $\widehat{Df}(\xi)=|\xi|\hat{f}(\xi)$.

The energy space, where $E(u)$ is defined, is $H^{1/2}(\mathbb{R})$. The existence of global *weak* solutions $u \in C([0,\infty);H^{1/2}(\mathbb{R}))\cap C^1((0,\infty);H^{-3/2}(\mathbb{R}))$ to (1.2) with energy space initial data $u(0,x)=u_0(x) \in H^{1/2}(\mathbb{R})$ was shown by Saut[19] (see also the paper of Ginibre and Velo[7]). For the *strong* H^s-solution, Ionescu and Kenig[8] established global well posedness for $s \geq 0$ (see also the paper of Tao[20]). This solution conserves the functional $N(u)$ [and $E(u)$ when $s \geq 1/2$].

The BO equation admits "K-soliton" solutions.[9] The 1-solitons are of the form

$$u(t,x) = Q_c(x-ct-x_0) \quad (c>0, x_0 \in \mathbb{R}),$$

where

$$Q_c(x) = cQ(cx), \quad Q(x) = \frac{2}{1+x^2}. \tag{1.3}$$

They satisfy

$$\mathcal{H}\partial_x Q_c + Q_c^2 = cQ_c, \tag{1.4}$$

which can be verified by using $\hat{Q}(\xi)=\sqrt{2\pi}e^{-|\xi|}$. By the explicit form (1.3), we have

$$\int Q^2 = 2\pi, \quad \int Q^3 = 3\pi, \quad (Q,DQ) = \int Q(Q^2-Q) = \pi. \tag{1.5}$$

By rescaling,

$$N(Q_c) = cN(Q) = \pi c, \quad E(Q_c) = c^2 E(Q) = -\frac{\pi}{2}c^2.$$

The orbital (i.e., up to translations) stability of the 1-soliton in the energy norm ($H^{1/2}$) was established in Ref. 3. See Refs. 2 and 4 for earlier stability results. Here we address the stability of the sum of widely spaced 1-solitons, with speeds arranged so as to avoid collisions. Our main result is the following theorem.

Theorem 1.1: (Orbital stability of the sum of K solitons) *Let $0<c_1^0<\cdots<c_K^0$. There exist L_0, A_0, $\alpha_0>0$, and $\theta_0 \in (0,1)$ such that for any $u_0 \in H^{1/2}(\mathbb{R})$, $L>L_0$, and $0<\alpha<\alpha_0$, if*

$$\|u_0 - \sum_{k=1}^{K} Q_{c_k^0}(\cdot - x_k^0)\|_{H^{1/2}(\mathbb{R})} \leq \alpha \tag{1.6}$$

for some x_k^0 satisfying

$$x_{k+1}^0 - x_k^0 > L \quad (k=1,\ldots,K-1), \tag{1.7}$$

then there exist C^1-functions $x_k(t)$, $k=1,\ldots,K$, such that the solution of (1.2) satisfies

$$\|u(t) - \sum_{k=1}^{K} Q_{c_k^0}(\cdot - x_k(t))\|_{H^{1/2}(\mathbb{R})} \leq A_0(\alpha + L^{-\theta_0}), \quad \forall\, t>0.$$

Moreover,

$$|\dot{x}_k(t) - c_k^0| \leq A_0(\alpha + L^{-\theta_0}), \quad \forall\, t>0.$$

Integrable system techniques (in particular, higher conservation laws) have been used to establish the stability of exact K-soliton solutions (see Ref. 10 for Korteweg-de Vries (KdV) and Ref. 17 for BO 2-solitons) against perturbations which are small in (necessarily) higher Sobolev norms. Here we are considering a different problem: stability of sums of 1-solitons (configurations which are not themselves solutions) in the *energy space*. Results of this type were obtained for KdV-type equations and nonlinear Schrödinger (NLS) equations in Refs. 5, 6, 14, and 15, respectively. Our approach follows that in Ref. 14 for generalized KdV (gKdV), which adds to the energy method of Weinstein[21] for the 1-soliton case, the monotonicity property of the L^2-mass on the right of each soliton. Here we encounter two new difficulties. First, and most importantly, the operator $\mathcal{H}$ is nonlocal, necessitating commutator estimates. Second, the decay of the soliton $Q(x)$ is only algebraic, meaning that the error estimates are more delicate. In particular, we use cutoff functions whose supports expand sublinearly at the rate $O(t^\gamma)$, $2/3<\gamma<1$, similar to Ref. 15.

After the paper was completed, we learned that Kenig and Martel[13] have obtained an asymptotic stability result independently and simultaneously.

II. THE STABILITY PROOF

Here we proven Theorem 1.1 using a series of lemmas whose proofs are given in Sec. III.

So we begin by fixing speeds $0<c_1^0<\cdots<c_K^0$, and we suppose $u \in C([0,\infty);H^{1/2}(\mathbb{R}))\cap C^1((0,\infty);H^{-3/2}(\mathbb{R}))$ solves (1.2) with initial data satisfying (1.6) and (1.7) for $\alpha<\alpha_0$ and $L>L_0$, where $\alpha_0\ll 1$ and $L_0\gg 1$ will be determined (depending only on the speeds $\{c_k^0\}$) in the course of the proof.

A. Decomposition of the solution

Set

$$T = T(\alpha,L) \coloneqq \sup\left\{ t>0 \middle| \sup_{0\le s\le t}\ \inf_{y_j>y_{j-1}+L/2} \|u(s,\cdot) - \sum_{j=1}^{K} Q_{c_j^0}(\cdot - y_j)\|_{H^{1/2}} < \sqrt{\alpha} \right\}. \tag{2.1}$$

If we take $\alpha<1$, then since $u\in C([0,\infty);H^{1/2})$, we have $T>0$. In what follows, we will estimate on the time interval $[0,T]$, and in the end conclude (provided α sufficiently small, L sufficiently large) that $T=\infty$.

The first step is a decomposition of the solution.

Lemma 2.1: (Decomposition of the solution) *Let $0<c_1^0<\cdots<c_K^0$ be fixed. There exist $L_1>0$, $\alpha_1>0$, and $A_1>0$ such that if $u(t,x)$ is a solution of (1.2) with initial data satisfying (1.6) and (1.7) for some $\alpha<\alpha_1$ and $L>L_1$, then for $T>0$ defined by (2.1), there exist unique C^1-functions $c_j:[0,T]\to(0,+\infty)$ and $x_j:[0,T]\to\mathbb{R}$ such that*

$$u(t,x) = \sum_{j=1}^{K} R_j(t,x) + \varepsilon(t,x), \tag{2.2}$$

where

$$R_j(t,x) \coloneqq Q_{c_j(t)}(x-x_j(t)),$$

where $\varepsilon(t,x)$ satisfies the orthogonality conditions

$$\forall j,\ \forall t\in[0,T],\qquad \int R_j(t,\cdot)\varepsilon(t,\cdot) = \int (R_j(t,\cdot))_x\varepsilon(t,\cdot) = 0. \tag{2.3}$$

Moreover,

$$\|\varepsilon(0,\cdot)\|_{H^{1/2}} + \sum\nolimits_k |x_k(0)-x_k^0| + \sum\nolimits_k |c_k(0)-c_k^0| \le A_1\alpha, \tag{2.4}$$

and for all $t\in[0,T]$,

$$x_k(t)-x_{k-1}(t) \ge L/2, \quad k=2,\ldots,K, \tag{2.5}$$

$$\|\varepsilon(t,\cdot)\|_{H^{1/2}} + \sum_{j=1}^{K} |c_j(t)-c_j^0| \le A_1\sqrt{\alpha}, \tag{2.6}$$

$$\sum_{j=1}^{K} |\dot{x}_j(t) - c_j^0| + |c_j(t)| \le A_1(\sqrt{\alpha} + L^{-2}). \tag{2.7}$$

We will use $\|\varepsilon(t,\cdot)\|_{H^{1/2}} \le 1$ in the rest of the proof.

B. Almost monotonicity of local mass

The size of the remainder $\varepsilon(t,x)$ will be controlled by an "almost monotone" Lyapunov functional which we now construct. Fix

$$\gamma \in (2/3,1),$$

and a non-negative $\zeta(x) \in C^2(\mathbb{R})$ so that $\zeta(x)=1$ for $x>1$, $\zeta(x)=0$ for $x<0$, and $\sqrt{\zeta_x} \in C^1$. Set

$$\bar{x}_k^0 := \frac{x_{k-1}(0) + x_k(0)}{2}, \quad \sigma_k := \frac{c_{k-1}^0 + c_k^0}{2}, \quad k = 2, \ldots, K,$$

$\psi_1(t,x) \equiv 1$, and for $k=2,\ldots,K$,

$$\psi_k(t,x) := \zeta(y_k), \quad y_k := \frac{x - \bar{x}_k^0 - \sigma_k t}{(b+t)^\gamma}, \tag{2.8}$$

with $b := (L/16)^{1/\gamma}$, and, finally, set for $k=1,\ldots,K$,

$$\mathcal{I}_k(t) := \frac{1}{2}\int_{\mathbb{R}} \psi_k(t,x) u(t,x)^2 dx,$$

which, roughly speaking, measures the L^2 mass to the right of the kth soliton.

Setting

$$d_k := c_k(0) - c_{k-1}(0) \quad (k = 2, \ldots, K), \quad d_1 = c_1(0),$$

the Lyapunov function we will use is

$$\mathcal{G}(t) := E(u(t)) + \sum_{k=1}^{K} d_k \mathcal{I}_k(t). \tag{2.9}$$

Note that $E(u(t)) = E(u_0)$ by energy conservation. The "almost monotonicity" of this functional comes from the following key estimate.

Lemma 2.2: (Almost monotonicity of mass on the right of each soliton) *Under the decomposition in (2.2), there is $C_2 > 0$ such that*

$$\mathcal{I}_k(t) - \mathcal{I}_k(0) \le C_2 L^{1/\gamma - 3/2} + C_2 L^{1-1/\gamma} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2.$$

In light of (2.9), this lemma implies the estimate

$$\mathcal{G}(t) \le \mathcal{G}(0) + CL^{1/\gamma - 3/2} + CL^{1-1/\gamma} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2. \tag{2.10}$$

C. Decomposition of the energy

As above, set $R_k := Q_{c_k(t)}(x - x_k(t))$, $R := \sum_{k=1}^{K} R_k$, and define

$$\phi_k(t,x) := \psi_k(t,x) - \psi_{k+1}(t,x), \quad k = 1, \ldots, K-1, \quad \phi_K(t,x) = \psi_K(t,x)$$

(so ϕ_k is localized near the kth soliton), and the (time-dependent) operator

$$H_K := D - 2R + \sum_{k=1}^{K} c_k(t)\phi_k.$$

The functional $\mathcal{G}$ can be expanded as follows.

Lemma 2.3: (Energy decomposition) *There is $C_3>0$ such that*

$$\left| \mathcal{G}(t) - \left\{ \sum_k [E(R_k) + c_k(0)N(R_k)] + \frac{1}{2}(\varepsilon(t), H_K\varepsilon(t)) \right\} \right| \leq C_3(L^{-2} + \|\varepsilon(t)\|_{L^2}\sum_k |c_k(0) - c_k(t)| + \|\varepsilon(t)\|_{H^{1/2}}^3).$$

We also need the following.

Lemma 2.4: *Let $F_c(u)=E(u)+cN(u)$. We have for some $C>0$ and c close to c^0 that*

$$0 \leq F_{c^0}(Q_{c^0}) - F_{c^0}(Q_c) \leq C(c - c^0)^2.$$

Combining Eq. (2.10) and Lemmas 2.3 and 2.4 yields

$$(\varepsilon(t), H_K\varepsilon(t)) \leq C\Big[\sum_k |c_k(0) - c_k(t)| \|\varepsilon(t)\|_{L^2} + \|\varepsilon(t)\|_{H^{1/2}}^3 + \sum_k |c_k(0) - c_k(t)|^2 + \|\varepsilon(0)\|_{H^{1/2}}^2 + L^{1/\gamma - 3/2} + L^{1-1/\gamma} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2 \Big]. \tag{2.11}$$

Next we need quadratic control of $c_k(t) - c_k(0)$.

Lemma 2.5: (Quadratic control of speed change)

$$\sum_k |c_k(t) - c_k(0)| \lesssim L^{1/\gamma - 3/2} + L^{1-1/\gamma} \sup_{0\leq\tau\leq t} \|\varepsilon(\tau)\|_{L^2}^2 + \|\varepsilon(t)\|_{H^{1/2}}^2 + \|\varepsilon(0)\|_{H^{1/2}}^2.$$

Combining this lemma with (2.11) and setting $\theta_0 = 1/2(3/2 - 1/\gamma)$ yields

$$(\varepsilon(t), H_K\varepsilon(t)) \lesssim \|\varepsilon(t)\|_{H^{1/2}}^3 + \|\varepsilon(0)\|_{H^{1/2}}^2 + L^{-2\theta_0} + L^{1-1/\gamma} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2. \tag{2.12}$$

D. Lower bound on quadratic form and completion of the proof

We want to use the quadratic form $(\varepsilon, H_K\varepsilon)$ to control $\|\varepsilon\|_{H^{1/2}}^2$, as is done for 1-soliton stability. Here we need a K-soliton version of this.

Lemma 2.6: (Positivity of the quadratic form) *There exist $L_2, \gamma_2 > 0$ such that if $L > L_2$, then*

$$\gamma_2 \|\varepsilon\|_{H^{1/2}}^2 \leq (\varepsilon, H_K\varepsilon).$$

Combining this lemma with (2.12) gives

$$\|\varepsilon(t)\|_{H^{1/2}}^2 \leq C[\|\varepsilon(t)\|_{H^{1/2}}^3 + \|\varepsilon(0)\|_{H^{1/2}}^2 + L^{-2\theta_0} + L^{1-1/\gamma} \sup_{t' \in [0,t]} \|\varepsilon(t')\|_{L^2}^2].$$

So using (2.4), this estimate implies, for α and $1/L$ sufficiently small, that there is $A_0>0$ such that

$$\sup_{t' \in [0,T]} \|\varepsilon(t')\|_{H^{1/2}} \leq A_0(\alpha + L^{-\theta_0}). \tag{2.13}$$

Hence for α and $1/L$ sufficiently small, we conclude that $T=\infty$, $x_k(t)$, and $c_k(t)$ exist for all time, and (2.13) gives the main estimate of the theorem. Finally, the last estimate of the theorem follows from (3.2) and (3.4) in the proof of Lemma 2.1. □

III. PROOFS OF LEMMAS

In this section, we shall prove lemmas mentioned in Sec. II.

A. Decomposition of the solution

Proof of Lemma 2.1: The existence of the functions $c_j(t)$ and $x_j(t)$ is established through the implicit function theorem applied to the map $F:H^{-3/2}(\mathbb{R})\times\mathbb{R}^K\times(\mathbb{R}^+)^K\to\mathbb{R}^{2K}$ defined by

$$F(u,\mathbf{y},\mathbf{c}) := ((\mathbf{R},u-R),(\mathbf{R}_x,u-R)),$$

where $R(x)=\Sigma_{j=1}^K R_j(x)$ with $R_j(x)=Q_{c_j}(x-y_j)$, and boldface denotes K-vectors, e.g., $\mathbf{y}=(y_1,\ldots,y_K)$ and $\mathbf{R}(x):=(R_1(x),\ldots,R_K(x))$. Here the inner product indicates $H^{3/2}-H^{-3/2}$ pairing. F is easily seen to be a C^1 map (note that it is affine in u). For any $\mathbf{y}$ and (bounded) $\mathbf{c}$, $F(R,\mathbf{y},\mathbf{c})=(\mathbf{0},\mathbf{0})$, and as a $2K\times 2K$ matrix,

$$D_{\mathbf{y},\mathbf{c}}F(R,\mathbf{y},\mathbf{c}) = \pi\begin{pmatrix} 0 & -Id \\ \operatorname{diag}(c_j^3) & 0 \end{pmatrix} + O((\min_{j\neq k}|y_j-y_k|)^{-2}), \tag{3.1}$$

is invertible, provided $\min_{j\neq k}|y_j-y_k|>L_1/2$ (L_1 a constant). Thus there is $\alpha_1>0$ such that for any $\mathbf{y}$ satisfying this condition, for u in an $H^{-3/2}$-ball about $\Sigma_{j=1}^K Q_{c_j^0}(x-y_j)$ of size $\beta\in(0,\sqrt{\alpha_1})$, there are unique $C^1(H^{-3/2};\mathbb{R}^K)$ functions $\mathbf{x}(u)$ and $\mathbf{c}(u)$ so that $F(u,\mathbf{x}(u),\mathbf{c}(u))=0$, with

$$|\mathbf{c}(u)-\mathbf{c}^0|+|\mathbf{x}(u)-\mathbf{y}|\lesssim\beta. \tag{3.2}$$

So using the condition (2.1) for $0\le t\le T$, we take $\beta=\sqrt{\alpha}$ and set $\mathbf{c}(t):=\mathbf{c}(u(t))$ and $\mathbf{x}(t):=\mathbf{x}(u(t))$. Since $u\in C^1((0,\infty);H^{-3/2})$, $x_j(t)$ and $c_j(t)$ are C^1 functions of $t>0$. The equation $F(u,\mathbf{x}(t),\mathbf{c}(t))\equiv 0$ is equivalent to the orthogonality conditions (2.3). The estimates (2.6) follow from (2.1) and (3.2). An equation for $\varepsilon(t,x)$ can be derived using (1.2) and $(DR_k-R_k^2)_x-\partial_t R_k=(\dot{x}_k-c_k)\partial_x R_k-\dot{c}_k\partial_c R_k$,

$$\partial_t\varepsilon = \partial_x\left(D\varepsilon - 2R\varepsilon - \varepsilon^2 - \sum_{j\neq k} R_j R_k\right) + \sum_k (\dot{x}_k - c_k)\partial_x R_k - \dot{c}_k\partial_c R_k. \tag{3.3}$$

Computing $(d/dt)(R_k,\varepsilon)$ and $(d/dt)(\partial_x R_k,\varepsilon)$, in turn, and using (3.1) and (2.6) yield

$$|\dot{\mathbf{c}}| + |\dot{\mathbf{x}}-\mathbf{c}| \lesssim \|\varepsilon\|_{H^{1/2}} + L^{-2} \lesssim \sqrt{\alpha} + L^{-2}. \tag{3.4}$$

This implies that $c(t)$ and $x(t)$ are C^1 up to $t=0$ and, together with (3.2), it gives (2.7). Now α can be taken sufficiently small and L sufficiently large, so that (1.6) and (1.7), together with (3.2) with $\beta=\alpha$, imply (2.4), which in turn implies that $x_k(0)-x_{k-1}(0)\ge L/2$. Finally (2.5) follows from this and (2.7) via

$$\frac{d}{dt}(x_k(t)-x_{k-1}(t)) \ge c_k^0 - c_{k-1}^0 - A_1(\sqrt{\alpha}+L^{-2}) > 0$$

for α sufficiently small and L sufficiently large. □

B. Commutator estimates

We have to deduce several estimates for commutators. For two operators A and B, denote by $[A,B]=AB-BA$ their commutator.

Lemma 3.1:

(i) *Suppose $\chi\in C_c^1(\mathbb{R})$. We have*

$$\|[D^{1/2},\chi]u\|_{L^2(\mathbb{R})} \lesssim \||\xi|^{1/2}\hat{\chi}(\xi)\|_{L^1(d\xi)}\cdot\|u\|_{L^2}. \tag{3.5}$$

(ii) *Suppose $\phi\in B_{\infty,1}^{2-2\varepsilon}$ with $0<\varepsilon<1/2$, then*

$$\left| \int u_x[\mathcal{H},\phi]u_x \right| \lesssim \|\phi\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}} \|u\|^2_{H^{1/2}}. \tag{3.6}$$

Proof:

(i) One can show $||p|^{1/2}-|p-\xi|^{1/2}| \lesssim |\xi|^{1/2}$ by considering the two cases $|p|>3|\xi|$ and $|p|<3|\xi|$. Thus

$$|D^{1/2}(u\chi)-(D^{1/2}u)\chi|_{L^2(dx)} = \left|\int [|p|^{1/2}-|p-\xi|^{1/2}]\hat{u}(p-\xi)\hat{\chi}(\xi)d\xi\right|_{L^2(dp)} \lesssim \||\hat{u}|*|\xi|^{1/2}|\hat{\chi}|\|_{L^2}$$
$$\le |\hat{u}|_2 \times \||\xi|^{1/2}\hat{\chi}|_{L^1}.$$

(ii) First assume $\phi \in C^2_c(\mathbb{R})$. Let $\Gamma=\{\xi_1+\xi_2+\xi_3=0\}$. The integral is equal to

$$\int u_x[\mathcal{H},\phi]u_x = i\int_\Gamma \xi_1\xi_3 m(\xi)\hat{u}(\xi_1)\hat{\phi}(\xi_2)\hat{u}(\xi_3),$$

where

$$m(\xi) = \mathrm{sgn}(\xi_2+\xi_3) - \mathrm{sgn}(\xi_3).$$

Decompose the integral into a sum by Littlewood–Paley decomposition

$$\sum_{N_1,N_2,N_3} \int_\Gamma \xi_1\xi_3 m(\xi)\hat{u}_{N_1}(\xi_1)\hat{\phi}_{N_2}(\xi_2)\hat{u}_{N_3}(\xi_3),$$

where N_j are dyadic numbers, $N_j=2^k$ for $k\in\mathbb{Z}$.

If $N_3 \gg N_2$, then $m(\xi)=0$. If $N_3 \lesssim N_2$, then $N_1 \lesssim N_2$ on Γ. Thus we may assume $N_1,N_3 \lesssim N_2$. When $\xi_1+\xi_2+\xi_3=0$, $m(\xi)=m_1(\xi)+m_3(\xi)$, where $m_j(\xi)=-\mathrm{sgn}(\xi_j)$ is constant when $\xi_j\neq 0$. By multilinear estimates (Ref. 16, Theorem 1.1), we have

$$\left|\int_\Gamma m(\xi)\xi_1\hat{u}_{N_1}(\xi_1)\hat{\phi}_{N_2}(\xi_2)\xi_3\hat{u}_{N_3}(\xi_3)\right| \le C\|\nabla u_{N_1}\|_2\|\phi_{N_2}\|_\infty\|\nabla u_{N_3}\|_2.$$

Thus

$$\left|\int u_x[\mathcal{H},\phi]u_x\right| \lesssim \sum_{N_1,N_3\lesssim N_2} N_1N_3\|u_{N_1}\|_2\|\phi_{N_2}\|_\infty\|u_{N_3}\|_2 \lesssim \sum N_1^\varepsilon N_2^{2-2\varepsilon}N_3^\varepsilon\|u_{N_1}\|_2\|\phi_{N_2}\|_\infty\|u_{N_3}\|_2$$
$$= \|\phi\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}}\|u\|^2_{\dot{B}^\varepsilon_{2,1}}.$$

Since $\|u\|_{\dot{B}^\varepsilon_{2,1}} \lesssim \|u\|_{H^{1/2}}$ for $0<\varepsilon<\frac{1}{2}$, we have shown (3.6) for $\phi\in C^2_c$.

For general $\phi \in B^{2-2\varepsilon}_{\infty,1}$, take $\eta_R(x)=\eta(x/R)$, where $\eta(x)$ is a fixed smooth function which equals 1 for $|x|<1$ and 0 for $|x|>2$. We have

$$\|\phi\eta_R\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}} \lesssim \|\phi\|_{L^\infty}\|\eta_R\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}} + \|\eta_R\|_{L^\infty}\|\phi\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}}.$$

Sending R to infinity in (3.6) with the above estimate, we get (3.6) for $\phi\in B^{2-2\varepsilon}_{\infty,1}$. □

Lemma 3.2: *Suppose $\chi(x)\in C^1_c(\mathbb{R})$ and $\widehat{D^{1/2}\chi}\in L^1$. For all $u\in H^{1/2}$,*

$$\int_{\mathbb{R}} u^3\chi^2 dx \le C\left(\int_{\mathrm{supp}\,\chi} u^2 dx\right)^{1/2}\int (|D^{1/2}u|^2\chi^2 + u^2\chi^2 + u^2\|\widehat{D^{1/2}\chi}\|^2_{L^1_\xi})dx.$$

Here C is a constant independent of u and χ.

Proof: First note the Gagliardo–Nirenberg inequality

$$\int u^4 dx \lesssim \int |D^{1/2}u|^2 dx \int |u|^2 dx. \tag{3.7}$$

This can be proven by first noting

$$\|u\|_4 \lesssim \|\hat{u}\|_{4/3} \le \|\langle\xi\rangle^{1/2}\hat{u}\|_2 \|\langle\xi\rangle^{-1/2}\|_4 = C(\|D^{1/2}u\|_2 + \|u\|_2),$$

and then rescaling with a minimizing scaling parameter. By Hölder inequality and the above inequality,

$$\left(\int u^3\chi^2 dx\right)^2 \le \int_{\text{supp }\chi} u^2 dx \int (u\chi)^4 dx \le \int_{\text{supp }\chi} u^2 dx \int |D^{1/2}(u\chi)|^2 dx \int u^2\chi^2 dx.$$

By Eq. (3.5), we conclude

$$\left(\int u^3\chi^2 dx\right)^2 \lesssim \int_{\text{supp }\chi} u^2 dx \int u^2\chi^2 dx \left(\int |D^{1/2}u|^2\chi^2 dx + \int u^2 dx \| |\xi|^{1/2}\hat{\chi}\|_{L^1}^2\right),$$

from which the lemma follows. □

C. Almost monotonicity

Proof of Lemma 2.2: We may assume u is smooth since the general case follows from approximation. We may assume $k \ge 2$ since $\mathcal{I}_1(t)$ is constant. Denote $\psi = \psi_k(t,x) = \zeta(y_k)$ for simplicity of notation. Note $\psi \in B^{2-}_{\infty,1}$ and

$$\psi_x = (b+t)^{-\gamma}\zeta'(y_k), \quad \text{supp }\psi_x \subset \bar{x}_k^0 + \sigma_k t + [0,(b+t)^\gamma]. \tag{3.8}$$

Consider

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_k(t) &= \int -\psi u[\mathcal{H}u_x + u^2]_x + \frac{1}{2}u^2\partial_t\psi dx \\ &= \int (\psi_x u + \psi u_x)\mathcal{H}u_x + \frac{2}{3}u^3\psi_x - \frac{1}{2}u^2\left(\sigma_k\psi_x + \zeta'(y_k)\frac{\gamma}{b+t}y_k\right)dx.\end{aligned}$$

By $\mathcal{H}\partial_x = -D$ and by Lemma 3.1 part (i) with $\chi = \psi_x$, we have

$$\int \psi_x u\mathcal{H}u_x = -\int \psi_x |D^{1/2}u|^2 - \int (D^{1/2}u)[D^{1/2},\psi_x]u = -\int \psi_x|D^{1/2}u|^2 + O(\|u\|_{H^{1/2}}^2 \| |\xi|^{1/2}\widehat{\psi_x}\|_{L^1}).$$

Since $\int \psi u_x \mathcal{H}u_x = -\int u_x\mathcal{H}(\psi u_x) = -\int \psi u_x\mathcal{H}u_x - \int u_x[\mathcal{H},\psi]u_x$, by Lemma 3.1 part (ii),

$$\int \psi u_x\mathcal{H}u_x = -\frac{1}{2}\int u_x[\mathcal{H},\psi]u_x = O(\|u\|_{H^{1/2}}^2 \|\psi\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}}).$$

Here we choose $\varepsilon \in (0,\frac{1}{4})$. By Lemma 3.2 with $\psi_x = \chi^2$,

$$\int \frac{2}{3}u^3\psi_x \lesssim \|u\|_{L^2(\text{supp }\psi_x)} \int (|D^{1/2}u|^2 + u^2)\psi_x + u^2 \| |\xi|^{1/2}\mathcal{F}(\sqrt{\psi_x})\|_{L^1_\xi}^2.$$

Now by (2.4) and (2.6) and the definition of σ_k, we have for all k,

$$\text{dist}(x_k(t), \text{ supp }\psi_x) \ge \tfrac{1}{3}(L + \sigma_0 t),$$

where

$$\sigma_0 := \tfrac{1}{2}\min_{k=2,\ldots,K}(c_1^0, c_k^0 - c_{k-1}^0) > 0,$$

and so

$$\|u(t)\|_{L^2(\text{supp }\psi_x)} \le C(L+\sigma_0 t)^{-2} + \|\varepsilon(t)\|_{H^{1/2}(\mathbb{R})} \ll 1.$$

The formula $\widehat{\psi_x}(\xi) = e^{-i(x_0+\sigma t)\xi}\widehat{\zeta'}((b+t)^\gamma \xi)$ gives us

$$D^s\psi_x(x) = \frac{1}{(b+t)^{\gamma(1+s)}}\int e^{i(x-x_0-\sigma t/(b+t)^\gamma)\eta}|\eta|^s\widehat{\zeta'}(\eta)d\eta.$$

Thus

$$\||\xi|^{1/2}\widehat{\psi_x}\|_{L^1_\xi} \lesssim (b+t)^{-3\gamma/2}, \quad \|\psi\|_{\dot{B}^{2-2\varepsilon}_{\infty,1}} \lesssim (b+t)^{-2\gamma(1-\varepsilon)}.$$

Similarly,

$$\||\xi|^{1/2}\mathcal{F}(\sqrt{\psi_x})\|^2_{L^1_\xi} \lesssim (b+t)^{-2\gamma}.$$

We can also found

$$\frac{\gamma}{b+t}|y_k| \le \frac{\sigma_k}{4(b+t)^\gamma} + \frac{C\gamma^2 y_k^2}{\sigma_k (b+t)^{2-\gamma}}$$

and

$$\int u^2\zeta'(y_k)\frac{\gamma^2 y_k^2}{\sigma_k(b+t)^{2-\gamma}} \le C(b+t)^{-2+\gamma}[\|\varepsilon\|^2_{L^2} + (L+\sigma_0 t)^{-2}].$$

Summing the estimates, we get

$$\frac{d}{dt}\mathcal{I}_k(t) \le -\frac{1}{2}\int \psi_x|D^{1/2}u|^2 - \frac{\sigma_k}{4}\int \psi_x u^2 + C(b+t)^{-3\gamma/2}\|u\|^2_{H^{1/2}} \quad + C(b+t)^{-2+\gamma}[\|\varepsilon\|^2_{L^2} + (L+\sigma_0 t)^{-2}].$$

Integrating in time and noting $2/3 < \gamma < 1$, we get the lemma. □

D. Energy decomposition

Proof of Lemma 2.3: Note $c_k(0) = d_1 + \cdots + d_k$ and

$$\sum_{k=1}^K d_k\psi_k = \sum_{k=1}^K d_k[\phi_k + \cdots + \phi_K] = \sum_{k=1}^K c_k(0)\phi_k.$$

So

$$\mathcal{G}(t) = E(u(t)) + \sum_{k=1}^K d_k\mathcal{I}_k(t) = E(u(t)) + \int_{\mathbb{R}}\sum_{k=1}^K \frac{1}{2}c_k(0)\phi_k u^2 dx.$$

Using the decomposition $u = R + \varepsilon$ and $R = \sum_{k=1}^K R_k$, we can decompose $\mathcal{G}(t)$ according to orders in ε,

$$\mathcal{G}(t) = G_0 + G_1 + \tfrac{1}{2}(\varepsilon(t),\ H_K\varepsilon(t)) + \tfrac{1}{2}(\varepsilon(t),\ \textstyle\sum_k (c_k(0) - c_k(t))\phi_k\varepsilon(t)) - \tfrac{1}{3}\int_{\mathbb{R}}\varepsilon(t)^3,$$

where G_0 denotes terms without ε,

$$G_0 = E(R) + \frac{1}{2}\int_{\mathbb{R}} \sum_{k=1}^{K} c_k(0)\phi_k R^2,$$

G_1 denotes terms linear in ε,

$$G_1 = \int_{\mathbb{R}} \varepsilon\left[DR - R^2 + \sum_{k=1}^{K} c_k(0)\phi_k R \right],$$

and H_K denotes the linear operator

$$H_K = D - 2R + \sum_{k=1}^{K} c_k(t)\phi_k.$$

We can further decompose

$$G_0 = \sum_{k=1}^{K} E(R_k) + \int \sum_{j<k} R_j D R_k - \frac{1}{3}\left(R^3 - \sum_{k=1}^{K} R_k^3\right) + \frac{1}{2}\sum_{k=1}^{K} c_k(0) R_k^2 + \frac{1}{2}\sum_{k=1}^{K} c_k(0)(\phi_k R^2 - R_k^2).$$

Using $DR_k - R_k^2 + c_k(t)R_k = 0$, we have

$$G_1 = \int \varepsilon\left\{ \left[\left(\sum_{k=1}^{K} R_k^2\right) - R^2 \right] + \sum_{k=1}^{K} [c_k(0)R_k(\phi_k - 1) + c_k(0)\phi_k(R - R_k) + (c_k(0) - c_k(t))R_k] \right\}.$$

Note

$$\|R^m - \sum_{k=1}^{K} R_k^m\|_{L^1\cap L^\infty(\mathbb{R})} \le CL^{-2} \quad (m = 2,3),$$

$$\|R_k(\phi_k - 1)\|_{L^2\cap L^\infty(\mathbb{R})} + \|\phi_k(R - R_k)\|_{L^2\cap L^\infty(\mathbb{R})} \le CL^{-2}.$$

Thus

$$|G_1(t)| \le CL^{-2} + C\sum_k |c_k(0) - c_k(t)|\|\varepsilon\|_{L^2}. \tag{3.9}$$

Also, since $R_j DR_k = R_j[c_k(t)R_k - R_k^2]$,

$$\|R_j DR_k\|_{L^1\cap L^\infty(\mathbb{R})} \le CL^{-2} \quad (j \neq k).$$

We have

$$|G_0(t) - \sum_k E(R_k) - \sum_k c_k(0)N(R_k)| \le CL^{-2}. \tag{3.10}$$

Finally,

$$\left| \frac{1}{2}\left(\varepsilon, \sum_k (c_k(0) - c_k(t))\phi_k\varepsilon\right) - \frac{1}{3}\int \varepsilon^3 \right| \le C\sum_k |c_k(0) - c_k(t)|\|\varepsilon\|_{L^2}^2 + C\|\varepsilon\|_{H^{1/2}}^3, \tag{3.11}$$

completing the proof of Lemma 2.3. □

Proof of Lemma 2.4: First proof: By energy decomposition around Q_{c^0}, we have for real-valued η small in $H^{1/2}$ that

$$F_{c^0}(Q_{c^0}+\eta)=F_{c^0}(Q_{c^0})+\tfrac{1}{2}(\eta,\ H^{c^0}\eta)+O(\|\eta\|_{H^{1/2}}^3).$$

In particular, for $\eta=Q_c-Q_{c^0}$ we get the lemma. In fact, $\eta\sim(c-c^0)\eta_0$ with $\eta_0=\partial_c|_{c=c^0}Q_c$ and $\frac{1}{2}(\eta_0,H^{c^0}\eta_0)=\frac{1}{2}(\eta_0,-Q_{c^0})=-\frac{1}{4}\partial_c\int Q_c^2=-\pi/2$.

Second proof: By the scaling property and (1.5),

$$F_{c^0}(Q_c)=c^2E(Q)+cc^0N(Q)=-\frac{\pi}{2}c^2+cc^0\pi,\quad F_{c^0}(Q_{c^0})=(c^0)^2\frac{\pi}{2}.$$

Thus $F_{c^0}(Q_{c^0})-F_{c^0}(Q_c)=(\pi/2)(c-c^0)^2$. □

E. Quadratic control of $c_k(t)-c_k(0)$

Proof of Lemma 2.5: As in the energy expansion above, with $u=R+\varepsilon$, $R=\Sigma_jR_j$, and using $(\varepsilon,R_j)\equiv 0$ and $|x_j(t)-x_k(t)|\geq L/2$ for $j\neq k$, we have

$$\Big|E(u)-\sum_j E(R_j)\Big|\lesssim L^{-2}+\|\varepsilon\|_{H^{1/2}}^2+\|\varepsilon\|_{H^{1/2}}^3.$$

Now using the conservation of energy, the fact $E(R_j)=ac_j^2$, and $\|\varepsilon(t)\|_{H^{1/2}}\leq 1$, we get

$$\Big|\textstyle\sum_k[(c_k(t))^2-(c_k(0))^2]\Big|\lesssim L^{-2}+\|\varepsilon(t)\|_{H^{1/2}}^2+\|\varepsilon(0)\|_{H^{1/2}}^2. \tag{3.12}$$

Since $\phi_k=\psi_k-\psi_{k+1}$, we have

$$\mathcal{I}_j(t)=\frac{1}{2}\int u^2\psi_j dx=\frac{1}{2}\int u^2\sum_{k=j}^K\phi_k dx=\sum_{k=j}^K\int_{\mathbb{R}}\frac{1}{2}\phi_k u^2dx.$$

Again using $(\varepsilon,R_j)\equiv 0$ and $|x_j(t)-x_k(t)|\geq L/2$ for $j\neq k$, we see easily that

$$\left|\frac{1}{2}\int\phi_k u^2dx-N(R_k)\right|\lesssim L^{-2}+\|\varepsilon\|_{H^{1/2}}^2.$$

So using $N(R_k)=c_kN(Q_1)=c_k\pi$ and the local monotonicity Lemma 2.2, we get

$$\delta_k(t):=\sum_{j=k}^K[c_j(t)-c_j(0)]\lesssim g(t),\quad k=1,\ldots,K, \tag{3.13}$$

where

$$g(t)=L^{1/\gamma-3/2}+L^{1-1/\gamma}\sup_{0\leq\tau\leq t}\|\varepsilon(\tau)\|_{L^2}^2+\|\varepsilon(t)\|_{H^{1/2}}^2+\|\varepsilon(0)\|_{H^{1/2}}^2.$$

Denote $\delta_{K+1}=0$ and $c_0(0)=0$. Using $|\delta|\leq-\delta+2\delta_+$ for any $\delta\in\mathbb{R}$ and (3.13), we get

$$\sum_{k=1}^K|\delta_k(t)|\lesssim\sum_{k=1}^K[c_k(0)-c_{k-1}(0)]|\delta_k(t)|\leq\sum_{k=1}^K[c_k(0)-c_{k-1}(0)][-\delta_k(t)+Cg]. \tag{3.14}$$

By Abel resummation,

$$-\sum_{k=1}^K[c_k(0)-c_{k-1}(0)]\delta_k(t)=-\sum_{k=1}^K c_k(0)[\delta_k(t)-\delta_{k+1}(t)]=\sum_{k=1}^K c_k(0)[c_k(0)-c_k(t)]$$
$$=\frac{1}{2}\textstyle\sum_k[(c_k(0))^2-(c_k(t))^2]+\frac{1}{2}\sum_k|c_k(t)-c_k(0)|^2.$$

Using (3.14), the above equality, and (3.12), we arrive at

$$\sum_k |\delta_k(t)| \lesssim g(t) + \sum_k |c_k(t) - c_k(0)|^2.$$

Since $|c_k(t)-c_k(0)| \le |\delta_k(t)| + |\delta_{k+1}(t)|$, we have

$$\sum_k |c_k(t) - c_k(0)| \lesssim g(t) + \sum_k |c_k(t) - c_k(0)|^2.$$

By the continuity of $c_k(t)$ and the smallness of $g(t)$, we get Lemma 2.5. □

F. Lower bound for the quadratic form

We first recall the 1-soliton case. Suppose a function $u(x)$ is a perturbation of $Q_c(x-a)$ of the form

$$u(x) = Q_c(x-a) + \varepsilon(x),$$

where $\varepsilon(x)$ is small in some sense. Then

$$(E + cN)(u) = (E + cN)(Q_c) + \frac{1}{2}(\varepsilon, H^{c,a}\varepsilon) - \frac{1}{3}\int \varepsilon^3.$$

Here $H^{c,a} = D + c - 2Q_c(x-a)$.

Lemma 3.3: (Reference 3) *Let $H = D + 1 - 2Q$ with $Q(x) = 2/(1+x^2)$. Its continuous spectrum is $[1,\infty)$. Its eigenvalues are 0, 1, and $\lambda_\pm = \frac{1}{2}(-1 \pm \sqrt{5})$, with corresponding normalized eigenfunctions*

$$\phi_0 = \frac{-4}{\sqrt{\pi}} \frac{x}{(1+x^2)^2} = \frac{1}{\sqrt{\pi}} Q_x,$$

$$\phi_1 = \frac{2}{\sqrt{\pi}} \frac{x(x^2-1)}{(1+x^2)^2} = \frac{1}{\sqrt{\pi}} (xQ + Q_x),$$

$$\phi_\pm = N_\pm \left(\frac{1 \pm \sqrt{5}}{1+x^2} - \frac{4}{(1+x^2)^2} \right) = N_\pm ((1 + \lambda_\pm) Q - Q^2).$$

Here $N_\pm = (1/\sqrt{\pi})(1 \pm (2/\sqrt{5}))^{1/2}$. Moreover, there is $\gamma_0 \in (0,1)$ so that if $\varepsilon \in H^{1/2}$ satisfies $(\varepsilon, Q) = (\varepsilon, Q_x) = 0$, then

$$\gamma_0 \|\varepsilon\|^2_{H^{1/2}} \le (\varepsilon, H\varepsilon). \tag{3.15}$$

This lemma, except (3.15), is due to Ref. 3. We have reformulated it in a form convenient to us. To prove Eq. (3.15), decompose $\varepsilon = a\phi_- + h$ with $h \perp \phi_-, Q_x$. Thus

$$(\varepsilon, H\varepsilon) = \lambda_- a^2 + (h, Hh) \ge \lambda_- a^2 + \lambda_+ (h,h) = \lambda_+ (\varepsilon,\varepsilon) - (\lambda_+ - \lambda_-) a^2.$$

Now decompose $\phi_- = bQ + k$ with $k \perp Q$ and hence

$$a^2 = (\varepsilon, \phi_-)^2 = (\varepsilon, k)^2 \le (\varepsilon, \varepsilon)(k,k).$$

Thus

$$(\varepsilon, H\varepsilon) \ge \gamma(\varepsilon,\varepsilon), \quad \gamma = \lambda_+ - (\lambda_+ - \lambda_-)(k,k).$$

One can compute $(k,k) = 1/2 - 1/\sqrt{5}$ and $\gamma = 1/2$, and Eq. (3.15) follows with $\gamma_0 = 1/9$.

We can rescale (3.15) and get the following: Let $R(x) = Q_c(x-a)$. If $\varepsilon \in H^{1/2}(\mathbb{R})$ satisfies $(\varepsilon, R) = (\varepsilon, R_x) = 0$, then

$$\gamma_0(\varepsilon,(D+c)\varepsilon) \le (\varepsilon,(D+c-2R)\varepsilon). \tag{3.16}$$

Proof of Lemma 2.6: This is a time-independent statement and everything is evaluated at t, e.g., $c_k=c_k(t)$. Let $\chi(x)$ be a non-negative smooth function supported in $|x|\le 2$, $\chi(x)=1$ for $|x|\le 1$, and $\chi^2(x)\le 1/2$ if and only if $|x|\ge 3/2$. Let $\chi_k(x)=\chi((x-x_k)/L_2/16)$. In particular, $\phi_k(x)=1$ when $\chi_k(x)\ne 0$ and $\phi_k(x)\ge 2\chi_k^2(x)$ when $\chi_k^2(x)\le 1/2$. Decompose

$$(\varepsilon,H_K\varepsilon) = \sum_k (\chi_k\varepsilon,(D+c_k-2R_k)(\chi_k\varepsilon)) + (\varepsilon,D\varepsilon) - \sum_k (\chi_k\varepsilon,D(\chi_k\varepsilon)) + \sum_k c_k(\varepsilon,(\phi_k-\chi_k^2)\varepsilon)$$
$$+ \left(\varepsilon,-\sum_k 2R_k(1-\chi_k^2)\varepsilon\right) =: I_1+I_2+I_3+I_4.$$

It follows from Lemma 3.3 that

$$I_1 \ge \sum_k \gamma_0(\chi_k\varepsilon,(D+c_1)(\chi_k\varepsilon)). \tag{3.17}$$

By Lemma 3.1,

$$|(\chi_k\varepsilon,D(\chi_k\varepsilon))^{1/2} - \|\chi_k D^{1/2}\varepsilon\|_{L^2}| \le \|[D^{1/2},\chi_k]\varepsilon\|_{L^2} \le \||\xi|^{1/2}\hat{\chi}_k(\xi)\|_{L^1(d\xi)}\|\varepsilon\|_{L^2}.$$

By definition of χ_k,

$$\||\xi|^{1/2}\hat{\chi}_k(\xi)\|_{L^1(d\xi)} \le CL_2^{-1/2}.$$

Thus

$$I_1 \ge \text{RHS of } (3.17) \ge \frac{\gamma_0}{2}\sum_k \|\chi_k D^{1/2}\varepsilon\|_{L^2}^2 - CL_2^{-1}\|\varepsilon\|_{L^2}^2 + \gamma_0 c_1\sum_k \|\chi_k\varepsilon\|_{L^2}^2,$$

and

$$I_2 \ge (\varepsilon,D\varepsilon) - \left(1+\frac{\gamma_0}{4}\right)\sum_k \|\chi_k D^{1/2}\varepsilon\|_{L^2}^2 - CL_2^{-1}\|\varepsilon\|_{L^2}^2$$
$$= \int \left[1-\left(1+\frac{\gamma_0}{4}\right)\sum_k \chi_k^2\right]|D^{1/2}\varepsilon|^2 - CL_2^{-1}\|\varepsilon\|_{L^2}^2.$$

We also have

$$I_3 \ge \sum_k c_1(\varepsilon,\phi_k 1(\chi_k^2\le 1/2)\varepsilon) = c_1\int_{\Sigma_k\chi_k^2\le 1/2}\varepsilon^2,$$

$$|I_4| \le CL_2^{-2}(\varepsilon,\varepsilon).$$

Summing up, we have

$$(\varepsilon,H_K\varepsilon) \ge \frac{\gamma_0}{4}(\varepsilon,D\varepsilon) - CL_2^{-1}\|\varepsilon\|_2^2 + \gamma_0 c_1\sum_k \|\chi_k\varepsilon\|_{L^2}^2 + c_1\int_{\Sigma_k\chi_k^2\le 1/2}\varepsilon^2$$

which is greater than $(\gamma_0/4)(\varepsilon,(D+c_1)\varepsilon)$ if L_2 is sufficiently large. □

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