MATH 515 summary

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1 Initial meeting

We gave an introduction to the course. We discussed the topics. It seems a good idea to include boundary layer. I sent a request for a classroom for Mon Wed 10am-11:15am.

A few fluid references

1. Ladyzhenskaya [6]: classical book for PDE theory of NS
2. Temam [9]: good for both theoretical and numerical analysis (finite element)
4. Chorin and Marsden [1]: We will follow it for the derivation of Euler and NS.
5. Galdi [4]: comprehensive treatise on stationary Navier-Stokes equations
6. Tsai [10]

2 Lebesgue integral

1. Motivation: weak solutions are limits of $L^2$ functions, and we need limit properties in $L^2$. Lebesgue integral sets the framework for $L^p$.
2. review Riemann integral.
3. rough definition for Lebesgue integral.
4. Lebesgue measure, measurable sets and measurable functions.
5. basic properties of Lebesgue integral.

6. Three theorems on the limits of Lebesgue integrals of sequences of functions
   (a) Monotone convergence theorem
   (b) Fatou Lemma
   (c) Dominated convergence theorem
7. $L^p$ spaces, Hölder inequality and Cauchy-Schwartz inequality

References: This is usually taught in a real analysis course, e.g., MATH 507 with textbook [3] by Folland. I learned the material from [11] by Wheeden and Zygmund. See Wikipedia for a good summary.
3 Weak derivatives and Sobolev spaces

1. weak derivatives: motivation, definition and examples

2. Sobolev spaces: definition and examples

3. Sobolev inequalities and imbedding, best representative

4. dimension analysis and scaling

5. Gagliardo-Nirenberg inequality

References: Chapter 5 of Evans [2], or Chapter 7 of Gilbarg-Trudinger [5].

4 Weak limits and compactness

1. strong and weak limits.

2. oscillation, concentration, and escaping to infinity

3. weak compactness

4. compactness of Sobolev imbedding

References: Chapter 5 of Evans [2], or Chapter 7 of Gilbarg-Trudinger [5].

5 Heat equation

1. fundamental solution and Green function, semigroup form.

2. Galerkin method: approximation solutions in finite dimensional subspaces

3. local existence and a priori bounds of approximation solutions for the heat equation

4. more on the a priori bound

5. weak limit of the functions

6. weak limit of the equation

7. time-independent a priori bound

8. the semigroup method

References: §7.1 and §7.4 of Evans [2].
6 An introduction to incompressible fluid flows

1. Incompressible Euler and Navier-Stokes equations
2. zero normal velocity, no-slip, and Navier boundary conditions
3. Derivation of these equations from physical principles: assumptions, balance of mass and momentum, stress tensor for Newtonian fluid, equation of state

References: Chorin and Marsden [1, §1.1] and Tsai [10, §1.1-1.2].

7 Particle trajectory map, Lagrangian viewpoint

1. the particle trajectory map, material derivative
2. time derivative of the Jacobian of the particle trajectory map
3. Transport formula
4. volume preserving flows

References: Majda and Bertozzi [7, §1.3]

8 Symmetries of Euler and Navier-Stokes equations

translation, rotation, Galilean invariance, scaling
References: Majda and Bertozzi [7, §1.2]

9 Vorticity and deformation matrix, linear in $x$ velocity

1. vorticity and deformation matrix, examples
2. equations for vorticity and deformation matrix
3. solutions whose velocity fields are linear in $x$, examples

Announcement: We will have a make-up lecture on Thursday March 1, 1-3pm.

4. Example 1.4

References: mostly Majda and Bertozzi [7, §1.4] and something from Tsai [10, §1.4].

10 Shear layer flow: examples with diffusion

1. Shear layer flow
2. Examples 1.5 and 1.6 for $\gamma = 0.$
3. lemma on convergence of solution of heat equation to initial data
4. convergence of solutions of Ex 1.6 to that of Ex 1.5 as $\nu \to 0$ for $t < 1$.

5. Examples 1.7 and 1.8 for $\gamma > 0$.

6. convergence of solutions of Ex 1.8 to that of Ex 1.7 as $\nu \to 0$ for $t < 1$.

7. Convergence to Burgers shear layer as $t \to \infty$

References: Majda and Bertozzi [7, §1.5].

11 Properties of vorticity

1. vorticity transport formula

2. proof of vorticity transport formula

3. remarks on nonuniqueness

4. vortex line, vortex sheet, and vortex tube

5. lemma on rate of circulation

6. Kelvin’s conservation of circulation, Helmholtz’s conservation of vorticity flux

References: Majda and Bertozzi [7, §1.6]. The proof of the circulation lemma is from Chorin and Marsden [1, page 22]

12 Conserved quantities for Euler equations

1. conserved quantities, importance of sign, divergence form

2. conservation of kinetic energy $\frac{1}{2} \int |v(x,t)|^2 dx$; its derivative for Navier-Stokes

3. conservation in 2D of $\int \omega(x,t) dx$ and $\int x^2 \omega(x,t) dx$

References: Majda and Bertozzi [7, §1.7].

Announcement:
The make-up lecture on Thursday March 1, 1-3pm will be held in Auditorium Annex room 142 (AUDX 142). https://learningspaces.ubc.ca/classrooms/audx-142

The room may not be easy to find if it is your first trip over to the inner courtyard/outside staircase of the AUDX and the specific room. You can enter AUDX through the entrance near MATH building, walk through the corridor to the inner courtyard, turn right and enter the next door labeled 140, and room 142 is immediately on your left.
13 Leray’s formulation and Helmholtz decomposition

1. Newtonian potential for Poisson equation
2. Leray’s formulation: elimination of pressure
3. Helmholtz decomposition and projection, orthogonality and uniqueness
4. Helmholtz decomposition: existence
5. Examples of Helmholtz decomposition

References: Majda and Bertozzi [7, §1.8] and Galdi [4, Chapter 3]. The pdf file of [4, Chapter 3] is in ownCloud. I made the last two examples.

14 2D vorticity and stream formulation

1. stream function and 2D Biot-Savart Law
2. steady inviscid eddies and time dependent viscous eddies
3. periodic flows
4. periodic flows continued
5. Kelvin-Stuart cat’s eye flow
6. brief summary of 3D exact solutions in [7, §2.3]

References: Majda and Bertozzi [7, §2.1-2.2].

15 3D vorticity and stream formulation

1. vector-stream function and 3D Biot-Savart Law
2. formula for velocity gradient
3. Riesz potentials and Calderon-Zygmund singular integrals
4. Estimates of velocity and velocity gradient in terms of vorticity

References: Majda and Bertozzi [7, §2.4]. For theorems on Riesz potential and singular integrals, see Stein [8, Chapter 2, §5.1], Gilbarg-Trudinger [5, §7.8, §9.4].
16 Energy method for the construction of solutions

1. Energy estimate for the difference of two solutions
2. continuous dependence on data and weak-strong uniqueness
3. Sobolev spaces review
4. product estimates in Sobolev spaces, interpolation inequalities, and commutator estimates
5. mollifiers

6. Existence proof part 1: Approximation solutions
   (a) short time existence by Picard existence theorem in Banach spaces
   (b) uniform \( L^2 \)-bound
   (c) \( \epsilon \)-dependent \( H^m \)-bound
   (d) global in time existence

7. Existence proof part 2: Limit of approximations
   (a) statement of existence theorem
   (b) short time uniform \( H^m \)-bound
   (c) convergence in \( C([0,T];L^2) \)
   (d) bound in \( L^\infty(0,T;H^m) \)
   (e) convergence in lower norms, \( C([0,T];H^k), k < m, \) and \( C([0,T];W^{1,\infty}) \).
   (f) convergence of equation
   (g) weak continuity, \( v \in C_W([0,T];H^m) \)
   (h) strong continuity, \( v \in C([0,T];H^m) \), Euler case
   (i) strong continuity, \( v \in C([0,T];H^m) \), Navier-Stokes case
   (j) remarks on Beal-Kato-Majda criterion

References: Majda and Bertozzi [7, §3.1.1, §3.2.1-§3.2.3].

17 Global in time weak solutions of Navier-Stokes equations

1. global in time a priori bound and energy class
2. weak form of N-S
3. definition of Leray-Hopf weak solution
4. lemmas on Banach space valued functions
5. existence of solutions of perturbed Stokes system: statement

6. existence of solutions of perturbed Stokes system

7. compactness lemma

8. existence of time-global Leray-Hopf weak solutions of Navier-Stokes equations

References: Tsai [10, Chapter 3], Temam [9, III.1-III.3].

References


