Math 515  Assignment 2
due Friday 2018-02-09

1. (a) Find two sequences \( f_k, g_k \in L^2(0,1) \) such that \( f_k \rightharpoonup f \) and \( g_k \rightharpoonup g \) weakly in \( L^2(0,1) \), but
\[
\lim_{k \to \infty} \int_0^1 f_k g_k \, dx \neq \int_0^1 fg \, dx.
\]
(b) Suppose \( f_k \to f \) strongly in \( L^2(0,1) \) and \( g_k \rightharpoonup g \) weakly in \( L^2(0,1) \), show that
\[
\lim_{k \to \infty} \int_0^1 f_k g_k \, dx = \int_0^1 fg \, dx.
\]

2. Let \( B_1 \) be the unit ball in \( \mathbb{R}^3 \). By Sobolev imbedding theorem, \( W^{1,4}_0(B_1) \) is imbedded in \( C^{1/4}(B_1) \).
Find a sequence \( u_k \in W^{1,4}_0(B_1) \), \( k \in \mathbb{N} \), with \( \|\nabla u_k\|_{L^4(B_1)} \) uniformly bounded, but \( u_k \) has no subsequence that converges in \( C^{1/4}(B_1) \) as \( k \to \infty \).
Hint: Choose \( u_k(0) = 0 \).

3. Let \( u(x,t) \) be a bounded solution of the heat equation
\[
\partial_t u - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty); \quad u(x,0) = g(x),
\]
where \( g(x) \) is smooth and is compactly supported in \( B_1 \).
(a) Show that \( u(x,t) \) is positive for \( t > 0 \) if \( g(x) \) is nonnegative.
(b) Show that there exists a constant \( C \) such that
\[
|u(x,t)| \leq \frac{C}{t^{3/2}} \quad \text{for all} \quad x \in \mathbb{R}^3, \ t > 0.
\]

Remark. (Do NOT show the following.) Note that \( \int u(x,t) \, dx \) is constant in \( t \). Moreover, the majority of \( u \) at time \( t \) is supported in a disk whose diameter is of order \( \sqrt{t} \), as can be seen in the following estimate: For some \( c_1 > 0 \) and \( A > 0 \), we have
\[
\int_{|x| > L(\sqrt{t} + 1)} |u(x,t)| \, dx \leq Ae^{-c_1L}, \quad \forall L > 2, \forall t > 0.
\]
These are compatible with (b).

4. (Channel flow) (a) Find the stationary solutions of the incompressible Navier-Stokes equations with unit viscosity \( \nu = 1 \) and zero force in the strip \( (x,y) \in \mathbb{R} \times (0,1) \) of the form \( v = f(y)e_1 \), with \( v(x,0) = 0 \) and \( v(x,1) = e_1 \). Also find the pressure.
(b) Assume in addition that \( v(x, \frac{1}{2}) = e_1 \). Let \( X(\alpha, t) \) be the particle trajectory map of this flow and \( E = (0,1) \times (0,1) \). Find \( E(2) = X(E,2) \) and sketch it.