\[
\frac{Dw}{dt} = a' X_0 + b' X_\phi + a \frac{DX_0}{dt} + b \frac{DX_\phi}{dt}
\]
\[
= X_0 \left( a' + b \cot \phi \right)
+ X_\phi \left( b' - a \sin \phi \cos \phi \right)
\]

parallel \begin{align*}
& a' = -b \cot \phi \\
& b' = a \sin \phi \cos \phi
\end{align*}

let \( A = a \sin \phi \)
\begin{align*}
\begin{cases}
A' = -b \cos \phi \\
A(0) = \sin \phi
\end{cases}
& b' = A \cos \phi \\
b(0) = 0
\end{align*}

\[
A(t) = \sin \phi \cos(t \cos \phi)
\]
\[
\phi(t) = \sin \phi \sin(t \cos \phi)
\]

\[
w(t) = \cos(t \cos \phi) X_0 + \sin \phi \sin(t \cos \phi) X_\phi
\]
\[
|w(t)| = \sin \phi
\]

angle of \( w(t) \) with \( X_0 \): \( t \cos \phi \).

After one around, \( t = 2\pi \), angle = \( 2\pi \cos \phi \)

Alternatively, consider the tangent cone. By RK3,
\( \frac{Dw}{dt} \) on tangent cone is the same.

Now, because \( \frac{Dw}{dt} \) is intrinsic,
we can remove a generator \( L \) of the cone,
and flatten it to fit in a plane.
\( \beta = \? \)

After 1 round, \( \beta = 2\pi \cos \phi \)

\[
\frac{\beta}{2\pi \cos \phi} = \frac{t}{2\pi}, \quad \beta = t \cos \phi
\]

Arc length \( S = \beta \tan \phi = t \sin \phi \)

Can be also seen as \( \sin \phi \) is the radius of the circle centered at \( Q \), for which we compute arc length.
A regular parametrized curve
\( y: I \rightarrow S \) is a geodesic if its tangent \( t'(t) \)
is parallel along \( y \), \( \frac{Dy'}{dt} = 0 \)

- A curve on \( S \) is a geodesic if its parametrization by
  arclength is. \( \text{RK} \quad \alpha'' = k \kappa \parallel N \)

Ex 3(a) On a plane \( y: I \rightarrow \mathbb{R}^2 \),
\[ \frac{Dy'}{dt} = y'' = 0, \]
a straight line. (geodesic = straight line on surface.

- a circle of latitude

On a sphere, as shown in Ex 1, \( \phi = \text{const} + \phi_0 \)
satisfies \( \frac{Dy'}{dt} = 0 \) if and only if \( \phi_0 = \frac{\pi}{2} \).

It is the only great circle in this family, i.e., passing
the origin. Since we are free to choose the axis for
spherical coordinates,
A circle on a sphere is a geodesic if and only if
it is a great circle.

b). If \( L \subset S \) is a straight line, then \( L \) is a geodesic
of \( S \), as
\[ \frac{Dy'}{dt} = \pi y'' = \pi 0 = 0 \]

Given \( p, q \in S^2 \), how many geodesics connect them?

- Case 1: \( p \neq -q \), exactly 1.
  (generic)
- Case 2: \( p = -q \), \( \infty \) many.
  uncountable
Rk. If the trace of $f: I \rightarrow S$ is a geodesic, $x$ may not be a (param.) geodesic.

$$\dot{x}(t) = \dot{x}(s) \quad \text{arc length}$$

$$x'(t) = x'(s) \frac{ds}{dt}$$

$$x''(t) = x''(s) \left( \frac{ds}{dt} \right)^2 + x'(s) \frac{d^2{s}}{dt^2}$$

Being geodesic, $x''(s) = kn$

$$\frac{Dx'}{dt} = x'(s) \frac{d^2{s}}{dt^2} \neq 0 \quad \text{unless} \quad s = at + b$$

No contradiction to previous remark:

$$\frac{Dx'}{dt} = \frac{Dx'}{ds} \cdot \frac{ds}{dt} = 0$$

but $x'(s)$ is not $x' \frac{ds}{dt}$.
Ex4  Cylinder $S = x^2 + y^2 = 1$

- Any vertical line is a geodesic, by Ex3b.
- horizontal circles are geodesics,
  $$g'' = -\delta$$
  $$\therefore \pi \delta'' = 0$$

For other geodesics, we use the isometry $U = (0, 2\pi) \times \mathbb{R} \to S$

$$\phi(u, v) = (\cos u, \sin u, v)$$

All geodesics on $\mathbb{R}^2$ are straight lines

$$\begin{cases} u = a s \\ v = b s \end{cases}$$

$$a^2 + b^2 = 1 \quad \text{s.t.} \quad s \text{ is arc length.}$$

$a = 0$ line,  $b = 0$ circle

$a \to 0$,  $b \to ?$

On the cylinder,

$$d(s) = (\cos as, \sin as, bs)$$

a helix!

Given $p, q \in S$, how many geodesics connect them?

Case 1. $p_3 = q_3$, third component, exactly one, the horizontal circle

Case 2. $p_3 < q_3$ (countably) $\omega$ many (generic)
Straight lines are characterized by either
\[ \alpha'' = 0, \quad \text{or} \]
\[ k = 0 \]

Def. For a unit vector field \( W(t) \) along a parametric curve \( \alpha : I \rightarrow S \) on an oriented surface \( S \),
\[ \frac{dW}{dt} \perp W, N \]
\[ \frac{dW}{dt} = \lambda N \times W, \quad |N \times W| = 1 \]
\[ \chi(t) = \left[ \frac{dW}{dt} \right] \text{ represents the algebraic value of } \frac{dW}{dt}. \]
Its sign depends on the choice of orientation of \( C \) and \( S \).

Def. If \( C : \alpha : I \rightarrow S \) is parametrically defined by arc length \( s \), the geodesic curvature of \( C \) at \( P \) is
\[ k_g = \left[ D \frac{\alpha'}{ds} \right] \]
If \( C \) is a geodesic, then \( k_g = 0 \)

Recall \( \frac{d\alpha'}{ds} = \pi \alpha''(s) = \pi \kappa n \)
Orthogonal plane

\[ k_n = \text{normal curvature along } \phi(0) = II(\phi''(0)) \]

\[
\begin{align*}
k_n &= k_n N + k_g (N \times t) \\
k^2 &= k_n^2 + k_g^2
\end{align*}
\]

Ex1 again

\[ k = \frac{1}{\sin \phi}, \quad \sin \phi = \text{radius of circle} \]

\[ k_n = -1, \quad l = \text{radius of sphere} \]

\[ \frac{1}{\sin^2 \phi} = 1 + k_g^2 \]

\[ k_g = \cot^2 \phi \]

\[ k_g = 0 \iff \phi = \frac{\pi}{2} \]